MASTER'S THESIS

TOPOLOGICAL COMPLEXITY

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Preface

Topological complexity is a numerical homotopy invariant which arises from the problem of designing motion planning algorithms. The algorithmic motion planning problem is central in robotics and requires the application of tools of algebraic topology. A motion planning algorithm of a given mechanical system with configuration space of states X is a function which associates to a pair $(x, y) \in X \times X$ a continuous motion from x to y or in other words a continuous path in X with initial point x and terminal point y. Stated more precisely, let PX denote the space of all continuous paths in X endowed with the compact-open topology. The endpoints map $\pi : PX \to X \times X$ given by $\pi(\gamma) = (\gamma(0), \gamma(1))$ is a fibration. A motion planning algorithm is a (not necessarily continuous) section of π. The discontinuities of motion planning algorithms provide a measure of the complexity of robot navigation. On the other hand, a continuous section of π exists if and only if X is contractible. Thus, the discontinuities of motion planning algorithms may reflect homotopy properties of X. Thus, outside robotics topological complexity is an interesting numerical homotopy invariant which may help to understand the nature of some geometric problems.

The topological complexity $TC(X)$ of a path-connected space X is a positive integer (or infinity) and is defined in an analogous way as its Lusternik-Schnirelmann category $catX$. They are both special cases of the more general notion of the genus of a fibration introduced and studied by A.S. Schwarz in [20]. The Schwarz genus (or sectional category) of a fibration $p : E \to B$ is the smallest positive integer k such that B can be covered by k open sets U_1, U_2,...,U_k for which there are continuous sections $s_i : U_i \to E$, $1 \leq i \leq k$, for p. The topological complexity $TC(X)$ is the genus of the endpoints fibration $\pi : PX \to X \times X$ and depends only on the homotopy type of X.

The study of the notion of topological complexity was initiated by M. Farber in [8] and [9]. It is a new active area of research. In this work we present some basic parts of the research that has been done during the last twelve years giving emphasis to the computation of the topological complexity of a large number of spaces. In the first chapter we give the basic prperties of the notion and its relation to the Lusternik-Schnirelmann category. In the second chapter we give cohomological lower bounds using the notion of sectional category weight of a cohomology class with respect to a fibration. The last chapter is devoted to the still open problem of the calculation of the topological complexity of the real projective spaces and its relation to the immersion problem.

Contents

Chapter 1

Introduction to topological complexity

1.1 Basic properties of topological complexity

For a topological space X we consider the space PX of paths in X, i.e. continuous maps $\gamma : I \to X$, equiped with the compact-open topology. Also, we consider the continuous map $\pi : PX \to X \times X$ defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$, which is called the endpoints fibration of X because of the following:

Lemma 1.1.1 *The map π is a fibration for any space X.*

Proof Let $f : Y \to PX$ be a continuous map of a space Y into the space PX, and let $F : Y \times I \to X \times X$ be a homotopy such that $F(y, 0) = \pi f(y)$ for all $y \in Y$. We write $F = (F_1, F_2)$ for the components of F. Let $\tilde{H} : Y \times I \times I \to X$ be defined by

$$\tilde{H}(y,t,s) = \begin{cases} F_1(y, t-3s), & \text{if } 0 \leq s \leq \frac{t}{3} \\ f(y)\left(\dfrac{s-\frac{t}{3}}{1-\frac{2t}{3}}\right), & \text{if } \frac{t}{3} \leq s \leq 1-\frac{t}{3} \\ F_2(y, 3s+(t-3)), & \text{if } 1-\frac{t}{3} \leq s \leq 1 \end{cases}$$

The map $\tilde{H}$ is continuous and induces a continuous map $H : Y \times I \to PX$ with $H(y,t)(s) = \tilde{H}(y,t,s)$. Also, $H(y, 0) = f(y)$ for $y \in Y$ and $\pi \circ H = F$, i.e. H lifts F. $\blacksquare$

The endpoints fibration π rarely admits a continuous section. Actually, the following holds.

Proposition 1.1.2 *Given a nonempty space X the fibration $\pi : PX \to X \times X$ has a continuous section $s : X \times X \to PX$ if and only if the space X is contractible.*

Proof Suppose that there is a continuous section $s : X \times X \to PX$ of π. This means that $s(x, y)$ is a path in X starting at x and ending at y for all $x, y \in X$. We fix a point $x_0 \in X$, and define a map $F : X \times I \to X$ by $F(x,t) = s(x, x_0)(t)$ for $x \in X$ and $t \in I$. Then F is a homotopy of 1_X to the constant map with value x_0, thus X is contractible.

Conversely, suppose that the space X is contractible. Then there is a point $x_0 \in X$ and a homotopy $F : X \times I \to X$ satisfying $F(x, 0) = x$ and $F(x, 1) = x_0$ for all $x \in X$. Let $s : X \times X \to PX$ be the map defined by $s(x, y) = F(\cdot, x) * F(\cdot, y)^{-1}$, where $*$ denotes the concatenation. The map s is a continuous section of π. $\blacksquare$

Remark: The fact that there is always a continuous section of π for a contractible space X is a special case of the fact that every fibration over a contractible base space has a continuous section.

Let $p : E \to B$ be a fibration with B contractible and E non-empty. If $\tilde{x}_0$ is any point of E, $x_0 = p(\tilde{x}_0)$ and $F : B \times I \to B$ is a homotopy such that $F(x, 0) = x_0$ and $F(x, 1) = x$ for $x \in X$ then the homotopy lifting property gives a homotopy $\tilde{F} : B \times I \to E$ with $\tilde{F}(x, 0) = \tilde{x}_0$ for $x \in B$ and $p \circ \tilde{F} = F$, i.e. $\tilde{F}$ lifts F. Since $p(\tilde{F}(1, x)) = x$ for all $x \in B$, the map $s = \tilde{F}(1, \cdot) : B \to E$ is a continuous section.

$$\begin{tikzcd}
\{0\} \times B \arrow[r, "\tilde{x}_0"] \arrow[d, hook, "i"] & E \arrow[d, "p"] \\
{[0,1]} \times B \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & B
\end{tikzcd}$$

Definition 1.1.3 *If X is a path-connected space we define the topological complexity $TC(X)$ of X as the least positive integer k such that there is a covering of $X \times X$ by k open subsets $U_1, U_2, \ldots, U_k \subset X \times X$ and there are continuous maps $s_i : U_i \to PX$, $1 \leq i \leq k$ with $\pi \circ s_i = i_{U_i}$, where $i_{U_i} : U_i \hookrightarrow X \times X$ is the inclusion map. If no such k exists we define $TC(X) = \infty$.*

The statement $\pi \circ s_i = i_{U_i}$ in this definition means that s_i is a section of $\pi|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$.

$$\begin{tikzcd}
U_i \arrow[r, "s_i"] \arrow[dr, hook] & PX \arrow[d, "\pi"] \\
& X \times X
\end{tikzcd}$$

The Proposition 1.1.2 says that $TC(X) = 1$ if and only if the space X is contractible.

The topological complexity of a path-connected space is a special case of the more general notion of Schwarz genus of a fibration. The Schwarz genus (or sectional category) of a fibration $p : E \to B$ is defined to be the minimal cardinality of open coverings of the base space B consisting of sets on each of which there exist a continuous section.

Remark: For a subspace $G \subset X \times X$, there is a continuous section $s : G \to PX$ of π if and only if there is a homotopy of maps $s_t : G \to X$, $0 \leq t \leq 1$ such that s_0, s_1 are the projections of G onto the first and the second coordinates, respectively.

Example: Let us show that $TC(S^n) = 2$ for n odd. Since S^n is not contractible, by Proposition 1.1.2, $TC(S^n) > 1$. Thus, it suffices to show that $TC(S^n) \leq 2$. To do this, we consider the sets $U_1 = \{(A,B) \in S^n \times S^n : A \neq -B\}$ and $U_2 = \{(A,B) \in S^n \times S^n : A \neq B\}$ and define $s_1 : U_1 \to PS^n$, $s_2 : U_2 \to PS^n$ as follows:

For $(A,B) \in U_1$, $s_1(A,B)$ is the unique shortest arc of S^n connecting A and B passed with velocity of constant length.

For $(A,B) \in U_2$, $s_2(A,B)$ is the concatenation of the shortest arc from A to $-B$ with constant velocity and a path moving from $-B$ to B. For the constraction of the path from $-B$ to B, we consider a tangent vector field v on S^n which is nonzero at each point. Such a tangent vector field exists, since n is odd. Then we consider the spherical arc from $-B$ to B:

$$-\cos \pi t \cdot B + \sin \pi t \cdot \frac{v(B)}{|v(B)|}, \quad 0 \leq t \leq 1.$$

Example: Assuming that n is even, we will show that $TC(S^n) \leq 3$. We define U_1, s_1 as in the odd dimensional case. Then we consider a tangent vector field v on S^n, which vanishes at a point $B_0 \in S^n$ and is nonzero at all points $B \in S^n$, $B \neq B_0$. Setting $U_2 = \{(A,B) \in S^n \times S^n : A \neq B \text{ and } B \neq B_0\}$ we define $s_2 : U_2 \to PS^n$ as in the odd dimensional case. If we choose a point $C \in S^n$, $C \neq B_0, -B_0$ then the set $Y = S^n - C$ is homeomorphic with $\mathbb{R}^n$, hence there is a continuous section $s : U_3 = Y \times Y \to PS^n$. Since $S^n \times S^n - (U_1 \cup U_2) = \{(-B_0, B_0)\}$, the sets U_1, U_2, U_3 cover $S^n \times S^n$ and therefore $TC(S^n) \leq 3$. We shall prove later that actually $TC(S^n) = 3$ when n is even.

The following Theorem shows that the topological complexity depends only on the homotopy type of X.

Theorem 1.1.4 *If there are continuous maps $f : X \to Y$ and $g : Y \to X$ between topological spaces X and Y such that $f \circ g \simeq 1_Y$, then $TC(Y) \leq TC(X)$.*

Proof It suffices to show that if U is an open set in $X \times X$ which admits a continuous section of the endpoints fibration π_X of X then the set $V = (g \times g)^{-1}(U) \subset Y \times Y$ admits a continuous section of π_Y. For then, if $k = TC(X)$ and $U_1 \cup U_2 \cup \ldots \cup U_k = X \times X$ is an open covering of $X \times X$ such that each U_i admits a continuous section of π_X, then the sets $V_i = (g \times g)^{-1}(U)$, $i = 1, 2, \ldots, k$, form an open covering of $Y \times Y$ on each member of which there is a continuous section of π_Y, thus $TC(Y) \leq TC(X)$. Suppose that U is an open set in $X \times X$ that admits a continuous section $s : U \to PX$ of π_X. We define a continuous section $\sigma : V \to PY$ of π_Y for the set $V = (g \times g)^{-1}(U)$ as follows. We consider a homotopy $h_t : Y \to Y$, $0 \leq t \leq 1$, with $h_0 = 1_Y$ and $h_1 = f \circ g$. For $(A,B) \in V$ we define the path $\sigma(A,B) : I \to V$ by

$$\sigma(A,B)(t) = \begin{cases} h_{3t}(A), & \text{for } 0 \leq t \leq \frac{1}{3} \\ f(s(gA, gB)(3t-1)), & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3} \\ h_{3(1-t)}(B), & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases} \quad \blacksquare$$

Corollary 1.1.5 *(Homotopy Invariance) If two spaces X and Y have the same homotopy type then $TC(X) = TC(Y)$.*

Corollary 1.1.6 *If a space X retracts to a subspace $A \subset X$ then $TC(X) \geq TC(A)$.*

Equivalent characterizations of topological complexity can be given for special classes of spaces.

Definition 1.1.7 *Let A be a subspace of a topological space X. We say that A is a neighborhood retract in X if A is a retract of some open neighborhood of itself.*

Definition 1.1.8 *A topological space X is called a Euclidean Neighborhood Retract (ENR) if it is homeomorphic to a neighborhood retract in $\mathbb{R}^n$ for some n.*

A subspace $X \subset \mathbb{R}^n$ is an ENR if and only if it is locally compact and locally contractible. (see [5], chapter 4, section 8)

Proposition 1.1.9 *Let X be an ENR. We define the following numbers:*

- *$k = k(X)$ is the least positive integer k with the property that there is a sequence $U_1 \subset U_2 \subset \ldots \subset U_k = X \times X$ of k open subsets and a section $s : X \times X \to PX$ of the fibration π such that the restrictions $s|_{U_{i+1}-U_i}$, $i = 1, 2, \ldots, k-1$ are continuous.*
- *$l = l(X)$ is the least positive integer l with the property that there is a sequence $F_1 \subset F_2 \subset \ldots \subset F_l = X \times X$ of l closed subsets and a section $s : X \times X \to PX$ of π such that the restrictions $s|_{F_{i+1}-F_i}$, $i = 1, 2, \ldots, l-1$ are continuous.*
- *$r = r(X)$ is the least positive integer r with the property that there is a splitting $G_1 \cup G_2 \cup \ldots \cup G_r = X \times X$ of $X \times X$ consisting of r pairwise disjoint, locally compact subspaces of $X \times X$ each of which admits a continuous section of π.*
- *$q = q(X)$ is the least positive integer q with the property that there is a splitting $G_1 \cup G_2 \cup \ldots \cup G_q = X \times X$ of $X \times X$ consisting of q locally compact subspaces of $X \times X$ each of which admits a continuous section of π.*

Then these numbers are equal to $TC(X)$, i.e., $TC(X) = k = l = r = q$.

We shall use the following elementary Lemmas.

Lemma 1.1.10 *If X is a normal space and $X = U_1 \cup U_2 \cup \ldots \cup U_n$, where $U_1, U_2, \ldots, U_n$ are open subsets of X then there are closed sets $F_1, F_2, \ldots, F_n$ such that $F_i \subset U_i$ for all i and $X = F_1 \cup F_2 \cup \ldots \cup F_n$.*

Proof We proove the lemma by induction on n. First, we suppose that $X = U_1 \cup U_2$ where X is normal and U_1, U_2 are open sets. Then, the sets $X - U_1, X - U_2$ are closed and disjoint, thus there are disjoint open sets $O_1 \supset X - U_1$ and $O_2 \supset X - U_2$, since X is normal. The sets $F_1 = X - O_1, F_2 = X - O_2$ are closed, $F_i \subset U_i$ and $X = F_1 \cup F_2$. This proves the lemma in the case $n = 2$.

Now, let $n > 2$ and assume that the conclusion is true for $n-1$ sets $U_1, U_2, \ldots, U_{n-1}$. Suppose that $X = U_1 \cup \ldots \cup U_n$ where X is normal and $U_1, \ldots, U_n$ are open sets. There are closed sets $F_1 \subset U_1$ and $G \subset U_2 \cup \ldots \cup U_n$ with $X = F_1 \cup G$. Since G is normal and $G = (U_2 \cap G) \cup \ldots \cup (U_n \cap G)$, by our assumption, there are closed sets $F_2 \subset U_2 \cup G, \ldots, F_n \subset U_n \cup G$ in G such that $G = F_2 \cup \ldots \cup F_n$. Hence, we have constructed a finite sequence $F_1, \ldots, F_n$ with the required properties. ∎

The next lemma is Exercise 2 at the end of Charter 4 in [5].

Lemma 1.1.11 *Let Y be an ANR and X be a binormal space, i.e., $X \times I$ is a normal space. Let $A \subset X$ be a closed set and $f, g : X \to Y$ be continuous maps such that $f|_A \simeq g|_A$. Then there exists an open set $A \subset V \subset X$ such that $f|_V \simeq g|_V$.*

Proof Let $M = X \times \{0\} \cup X \times \{1\} \cup A \times I$ and $H : f|_A \simeq g|_A$. We set $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ for $x \in X$ and $F(a, t) = H(a, t)$ for $a \in A$, $t \in I$ to get a well defined continuous map $F : M \to Y$. Since Y is an ANR and M is a closed subset of the normal space $X \times I$, there exist an open set $M \subset U \subset X \times I$ and a continuous extension $\tilde{F} : U \to Y$ of F. Since each $a \in A$ has an open neighborhood N_a in X such that $N_a \times I \subset U$, there exists an open set $A \subset V \subset X$ such that $A \times I \subset V \times I \subset U$. Obviously, $\tilde{F}|_{V \times I} : f|_V \simeq g|_V$. ∎

Proof of Proposition 1.1.9 Let $TC(X) = s$. The proposition will be prooved, showing the inequalities $s \geq k, s \geq l, k \geq r, l \geq r, r \geq q, q \geq s$.

For the first inequality, we consider an open covering of $X \times X$ consisting of s open sets $W_1, W_2, \ldots, W_s$ such that each W_i admits a continuous section s_i of π. We put $U_i = W_1 \cup \ldots \cup W_i$ for $i = 1, \ldots, s$ and define a section $s : X \times X \to PX$ by $s(x, y) = s_i(x, y)$ where i is the smallest index with the property $(x, y) \in W_i$. Hence $U_1 \subset U_2 \subset \ldots \subset U_s = X \times X$ and, since $U_{i+1} - U_i = W_{i+1} - (W_1 \cup \ldots \cup W_i)$ and $s = s_i$ in $U_{i+1} - U_i$, it follows that $s|_{U_{i+1} - U_i}$ is a continuous map.

For the second inequality, using Lemma 1.1.10 for the metrizable, hence normal space $X \times X$, we obtain closed sets $V_1, \ldots, V_s$ in $X \times X$ such that each V_i is contained in W_i and $V_1 \cup \ldots \cup V_s = X \times X$. Applying the same argument as in the first inequality, using the sets V_i in place of U_i we conclude the second inequality.

For the inequality $k \geq r$, we consider a sequence $U_1 \subset \ldots \subset U_k = X \times X$ of k open sets in $X \times X$ such that each of the sets $U_{i+1} - U_i$. $i = 0, 1, \ldots, k-1$ admits a continuous section of π. We set $G_i = U_i - (U_1 \cup \ldots \cup U_{i-1})$. Then the sets G_i are locally compact. (see [6], Theorem 6.5, p. 239) Also, since the sets G_i cover $X \times X$ and they are pairwise disjoint, we have $k \geq r$.

Similarly, it follows that $l \geq r$.

The inequality $r \geq q$ is trivial.

For the last inequality $q \geq s$, we consider a covering of $X \times X$ consisting of s locally compact subsets $G_1, G_2, \ldots, G_s$ such that each G_i admits a continuous section $s_i : G_i \to PX$. The map $s_i : G_i \to PX$ corresponds to a homotopy $h^i_t : G_i \to X$ between the projections h^i_0, h^i_1 of G_i onto the first and the second coordinates, respectively. Since G_i is locally compact, there is an open set $W_i \subset X \times X$ such that $G_i = \overline{G}_i \cap W_i$. (see [6], Theorem 6.5, p. 239) It follows that there is an open set U_i with $G_i \subset U_i \subset W_i$ such that the projections of U_i onto the first and the second coordinates are homotopic by a homotopy $H^i_t : U_i \to X$ by Lemma 1.1.11. This homotopy H^i_t, $0 \leq t \leq 1$, corresponds to a continuous section $S_i : U_i \to PX$. Since the sets U_i cover $X \times X$ and each U_i admits a continuous section $S_i : U_i \to PX$, we conclude that $q \geq s$. ∎

If we only assume that X is a locally compact metrizable space then $TC(X) \geq \max\{k(X), l(X)\}$ and $\min\{k(X), l(X)\} \geq r(X)$. If in addition X is ANR, then Proposition 1.1.9 above remains still true. This follows from the arguments of the proof. Recall that a space X is called absolute neighborhood retract (ANR) if for every normal space Y, closed subspace $A \subset Y$, and every continuous map $f : A \to X$ there exists a continuous extension of f to a neighborhood of A (in Y).

Example: Let Y be an ENR space for which the suspension $X = \Sigma Y$ is ENR. The space X is the quotient of $Y \times I$ identifying the points of the subspaces $Y \times 0$ and $Y \times 1$ into two single points p and q, respectively. Also, all points $x \in X - p$ are assosiated with continuous paths σ_x in $X - p$ starting at q and ending at x that depend continuously on $X - p$. We will show that $TC(X) \leq 3$. We consider the sequence $F_1 \subset F_2 \subset F_3 = X \times X$ of closed subsets with $F_1 = \{(p, p)\}$ and $F_2 = p \times X \cup X \times p$. We define a section $s : X \times X \to PX$ of π as follows. We set $s(p, p)$ to be the constant loop at the point p. For $x \in X - p$ we define $s(p, x) = \gamma_0 * \sigma_x$, and $s(x, p) = s(p, x)^{-1}$ = the inverse path of $s(p, x)$, where γ_0 is a fixed path in X from p to q. Since $F_2 - F_1 = \{p\} \times (X - \{p\}) \cup (X - \{p\}) \times \{p\}$, we have defined the map s in F_2. For $(x, y) \in F_3 - F_2 = (X - p) \times (X - p)$ we define $s(x, y) = \sigma_x^{-1} * \sigma_y$. From the construction of s, the maps $s|_{F_i - F_{i-1}}$ for $i = 1, 2, 3$ are continuous and the inequality $TC(X) \leq 3$ follows from Proposition 1.1.9.

1.2 LS category and topological complexity

The definition of topological complexity is inspired by the notion of Lusternik-Schnirelmann category.

Definition 1.2.1 *Let X be a topological space. A set $A \subset X$ is called categorical if the inclusion $i : A \hookrightarrow X$ is nullhomotopic.*

Definition 1.2.2 *The category $catX$ of a space X is defined to be the least positive integer k such that there are k open categorical subsets $U_1, \ldots, U_k$ of X that cover X. If no such integer k exists, we put $catX = \infty$.*

We observe that a non-empty space X is contractible if and only if $catX = 1$. Also, if X is a suspension (of some space) then $catX \leq 2$. In particular $catS^n = 2$.

Like topological complexity, the Lusternik-Schnirelmann category of a space X can be thought of as the Schwarz genus of a particular fibration. Let X be path-connected and fix a point $x_0 \in X$. Let $P_0X = \{\gamma | \gamma : I \to X \text{ with } \gamma(0) = x_0\}$ be the space of paths in X with initial point x_0 equiped with the compact-open topology. The continuous map $\pi_0 : P_0X \to X$ sending each path $\gamma \in P_0X$ to $\gamma(1)$ is a fibration. Actually it is the fibration induced from the endpoints fibration $\pi : PX \to X \times X$ by the map $f : X \to X \times X$ defined by $f(x) = (x_0, x)$. (Identifying each path $\gamma \in P_0X$ to $(\gamma, \gamma(1))$ we take $P_0X = \{(\gamma, x) \in PX \times X | \pi(\gamma) = f(x)\}$) (see [19], corollary 8, p. 99)

$$\begin{tikzcd} P_0X = f^*(PX) \arrow[r, "pr_1"] \arrow[d, "\pi_0 = pr_2"] & PX \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X \times X \end{tikzcd}$$

Since there is a continuous section of π_0 over an open set $U \subset X$ if and only if U is categorical, the Schwarz genus of the fibration π_0 is the Lusternik-Schnirelmann category of X.

The next proposition shows that this number is a topological invariant.

Proposition 1.2.3 *If X and Y are topological spaces and $f : X \to Y$ and $g : Y \to X$ are continuous maps such that $f \circ g \simeq 1_Y$ then $catX \geq catY$.*

Proof Let $U \subset X$ be open and categorical. Then $g^{-1}(U)$ is open and categorical (in Y). In fact, the inclusion map $j : g^{-1}(U) \hookrightarrow Y$ is nullhomotopic, since $j = 1_Y \circ j \simeq f \circ g|_{g^{-1}(U)}$ and the map $f \circ g|_{g^{-1}(U)}$ is nullhomotopic because so is the map $f|_U$. Thus, a collection of open categorical subsets of X that cover X pulls back to a covering of Y consisting of open categorical sets and the inequality $catX \geq catY$ follows. ∎

Corollary 1.2.4 *If two spaces X and Y have the same homotopy type then $catX = catY$.*

Proposition 1.2.5 *Let R be a commutative ring with a unity and let X be a topological space. If $u_1, \ldots, u_k \in H^*(X; R)$ are non-zero cohomology classes of positive degree and $u_1 \smile \cdots \smile u_k \neq 0$ then $catX > k$.*

Proof Let $catX \leq k$ and let $X = U_1 \cup \ldots \cup U_k$, where each U_i is open and categorical. We consider the cohomology sequence of the pair (X, U_i)

$$\cdots \longrightarrow H^q(X, U_i; R) \xrightarrow{j_i^*} H^q(X; R) \longrightarrow H^q(U_i; R) \longrightarrow \cdots$$

Since U_i is categorical, the induced by the inclusion homomorphism $H^q(X; R) \to H^q(U_i; R)$ in the above exact sequence is trivial for $q > 0$. Thus, by exactness, the homomorphisms $j_i^* : H^q(X, U_i; R) \to H^q(X; R)$ for $q > 0$ are epimorphisms. Hence $u_i = j_i^*(\overline{u}_i)$ for some $\overline{u}_i \in H^*(X, U_i; R)$.

The commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "\Delta"] & (X, A \cup B) \arrow[d, "\Delta"] \\ X \times X \arrow[r, "j_1 \times j_2"] & (X, A) \times (X, B) \end{tikzcd}$$

shows that $j^*\Delta^* = \Delta^*(j_1^* \times j_2^*)$, where Δ is the diagonal map and j, j_1, j_2 are the inclusions. Thus $j^*(a \smile b) = j^*\Delta^*(a \times b) = \Delta^*(j_1^* \times j_2^*)(a \times b) = \Delta^*(j_1^*(a) \times j_2^*(b)) = j_1^*(a) \smile j_2^*(b)$. This means that

$$\begin{aligned} j^*(\overline{u}_1 \smile \cdots \smile \overline{u}_k) &= j_1^*(\overline{u}_1) \smile \cdots \smile j_k^*(\overline{u}_k) \\ &= u_1 \smile \cdots \smile u_k \neq 0, \end{aligned}$$

where $j : X \hookrightarrow (X, U_1 \cup \ldots \cup U_k)$, $j_i : X \hookrightarrow (X, U_i)$. Since $H^*(X, U_1 \cup \ldots \cup U_k; R) = H^*(X, X; R) = 0$, we see that $\overline{u}_1 \smile \cdots \smile \overline{u}_k = 0$, and so $j^*(\overline{u}_1 \smile \cdots \smile \overline{u}_k) = 0$, a contradiction. ∎

Example: We will show that $cat\mathbb{R}P^n = cat\mathbb{C}P^n = n+1$. The cohomology ring of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}_2$ is $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[a]/ < a^{n+1} >$, where a is an element of degree 1 and $a^n \neq 0$. Proposition 1.2.5 says that $cat\mathbb{R}P^n > n$. In the case of the complex projective space $\mathbb{C}P^n$, we have

$$H^q(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0,2,\ldots,2n \\ 0 & \text{for } q \neq 0,2,\ldots,2n \end{cases}$$

and

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[a]/ < a^{n+1} >,$$

where a is an element of degree 2 and $a^n \neq 0$. Hence $cat\mathbb{C}P^n > n$. Writing points of $\mathbb{C}P^n$ in homogeneous coordinates $[z_0,\ldots,z_n]$, the open subsets $U_i = \{[z_0,\ldots,z_n] \in \mathbb{C}P^n : z_i \neq 0\}$, $0 \leq i \leq n$, which define the standard complex manifold structure on $\mathbb{C}P^n$, form an open covering consisting of categorical sets, thus $cat\mathbb{C}P^n \leq n+1$. We obtain that $cat\mathbb{C}P^n = n+1$ and similarly $cat\mathbb{R}P^n = n+1$.

In the sequel we shall repeatedly use the following simple observation.

Lemma 1.2.6 *If X is path-connected and $\{A_j\}_{j\in J}$ is a collection of open categorical subsets of X such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then the union $\bigcup_{j\in J} A_j$ is also categorical.*

Proof Since X is path-connected, all constant maps from a subspace of X to X are homotopic. Thus, we may choose a collection of homotopies $F_j : A_j \times I \to X$, $j \in J$, such that $F_j(\cdot,0) = i_j : A_j \hookrightarrow X$ and $F_j(\cdot,1) = b$ for some point $b \in X$. If $A = \bigcup_{j\in J} A_j$, we may define $F : A \times I \to X$ by $F(y,t) = F_j(y,t)$ for $y \in A_j$, $0 \leq t \leq 1$. Since $F|_{A_j\times I} = F_j$ and the sets $A_j \times I$ are open in $A \times I$ and disjoint and the maps F_j are continuous, it follows that the map F is a well defined continuous homotopy from the inclusion map $A \hookrightarrow X$ to a constant map. ∎

Definition 1.2.7 *Let $m \geq 1$ be an integer. We say that the (covering) dimension of a Hausdorff space X is at most m, denoted $dimX \leq m$, if every collection of open sets in X that covers X has an open refinement such that each point is contained at most in $m+1$ elements of this refinement. We write $dimX = m$ if $dimX \leq m$ and $dimX \nleq m-1$.*

Proposition 1.2.8 *If X is a path-connected, paracompact and locally contractible space and $dimX \leq m$, then $catX \leq m+1$. In other words, $catX \leq dimX + 1$.*

Proof Let $\{U_j\}_{j\in J}$ be a finite open covering of X consisting of categorical sets. Since X is paracompact, there is a partition of unity $\{\pi_j\}_{j\in J}$ subordinated to $\{U_j\}_{j\in J}$. For $x \in X$, we define $S(x) = \{j \in J | \pi_j(x) > 0\}$. The set $S(x)$ is finite since $x \in supp\pi_j$ for finitely many $j \in J$. Also, for a finite set $S \subset J$, we define

$$W(S) = \{x \in X | \pi_i(x) < \pi_j(x) \text{ for all } i \in J - S \text{ and } \pi_j(x) > 0 \text{ for all } j \in S\}.$$

We will proove that $W(S)$ is open. Since $W(S) = \bigcap_{j\in J} K_j$ where

$$K_j = \{x \in X | \pi_i(x) < \pi_j(x) \text{ for all } i \in J - S \text{ and } \pi_j(x) > 0\},$$

it suffices to proove that each set K_j is open. In fact, a point $x \in K_j$ has an open neighborhood U_x such that the set $J_x = \{i \in J | U_x \cap supp\pi_i \neq \emptyset\}$ is finite. We see that

$$\begin{aligned}
U_x \cap K_j &= \{y \in U_x | \pi_i(y) < \pi_j(y) \text{ for all } i \in J - S \text{ and } \pi_j(y) > 0\} \\
&= \{y \in U_x | \pi_i(y) < \pi_j(y) \text{ for all } i \in J_x - S \text{ and } \pi_j(y) > 0\} \\
&= \Big(\bigcap_{i \in J_x - S} K_j^i\Big) \cap \{y \in U_x | \pi_j(y) > 0\}
\end{aligned}$$

where $K_j^i = \{y \in U_x | \pi_i(y) < \pi_j(y)\}$. Therefore, $U_x \cap K_j$ is open and hence so is $K_j = \bigcup_{x \in K_j}(U_x \cap K_j)$.

Also, if $S \not\subseteq S'$ and $S' \not\subseteq S$ then $W(S) \cap W(S') = \emptyset$, because if $x \in W(S) \cap W(S')$ then $\pi_i(x) < \pi_j(x) < \pi_i(x)$ for $i \in S' - S, j \in S - S'$. We set

$$W_k = \bigcup \{W(S(x)) | x \in X, S(x) \text{ has } k \text{ elements}\}$$

for $k = 1, 2, \ldots$. The sets W_k are open and since $W(S) \subset U_j$ for all $j \in S$, Lemma 1.2.6 implies that the sets W_k are categorical.

If there is a number n such that $U_{j_1} \cap U_{j_2} \cap \ldots \cap U_{j_{n+1}} = \emptyset$ for all distinct $j_1, j_2, \ldots, j_{n+1} \in J$, then $W_k = \emptyset$ for $k \geq n+1$. Thus, we obtain an open covering of X consisting of n categorical sets.

Using this argument for an open refinement of a covering consisting of open categorical subsets of X such that each point contained at most in $m+1$ elements of the refinement, we obtain $catX \leq m+1$. ∎

Since each n-dimensional manifold has covering dimension at most n (see [18], Theorem 2.15, p. 24) we obtain the following.

Corollary 1.2.9 *If M is a path-connected n-manifold then $catM \leq n+1$.*

Example: We will show that $catT^n = n+1$. The cohomology ring $H^*(T^n;\mathbb{Q})$ of the n-torus T^n with coefficients in $\mathbb{Q}$ is an exterior algebra on n generators, hence $catT^n > n$ by Proposition 1.2.5. Also, by Corollary 1.2.9, $catT^n \leq n+1$, and therefore $catT^n = n+1$.

Example: The above Proposition 1.2.8 is false without the hypothesis that X is locally contractible. We consider the space $X = \bigcup_{n=1}^{\infty} C_n$, where C_n is the circle in the plane $\mathbb{R}^2$ with center at $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$. We will show that the point $(0,0)$ has no neighborhoods which are categorical. If it is not true then some circle C_n is categorical with respect to X, hence with respect to $\mathbb{R}^2 - \{x\}$, where x is any point in the interior of C_n. But this is false, since C_n is a deformation retract of $\mathbb{R}^2 - \{x\}$ and the inclusion map $C_n \hookrightarrow \mathbb{R}^2 - \{x\}$ induces an isomorphism on the fundamental groups. This means that $catX = \infty$. Also, $dimX = 1$. In fact, if we take an open covering of X then we can construct an open refinement of this covering as follows. We take an open ball B with center at $(0,0)$ such that the set $B \cap X$ is contained in an element of the covering. Then the set $X - B$ consists of a finite number of disjoint arcs. We cover these arcs by smaller open arcs such that no point of X is contained in three of them. We choose the diameter of these small arcs to be very small (i.e. smaller than a Lebesgue number of the covering) so that each small arc is contained in a set of the covering. All small arcs together with $B \cap X$ form an open refinement of the original covering such that no point of X belongs to more than two sets of the refinement.

Remark: If X is a Hausdorff space then $catX \leq n$ if and only if there is a sequence $V_1 \subset V_2 \subset \ldots \subset V_n = X$ of n open sets such that each of the differences $V_i - V_{i-1}$ is contained in an open categorical subset of X.(Here $V_0 = \emptyset$) If there exists such a sequence, the corresponding differences cover the space X, hence we obtain a covering of X consisting of n open categorical subsets, and $catX \leq n$. Conversely, if $catX \leq n$ then there are open categorical sets $U_1, \ldots, U_n$ in X that cover X, hence the sets $V_i = U_1 \cup \ldots \cup U_i$ for $i = 1, \ldots, n$ form a sequence as above.

The proof of the following proposition is taken from [3] and is included here for the sake of completeness.

Proposition 1.2.10 *If X and Y are path-connected spaces such that the space $X \times Y$ is completely normal then*

$$cat(X \times Y) \leq catX + catY - 1.$$

Proof Let $catX = n$ and $catY = m$. Then there are sequences $U_1 \subset \ldots \subset U_n = X$ and $V_1 \subset \ldots \subset V_m = Y$ of open sets for X and Y, respectively, such that there exist categorical open sets $Z_1, \ldots, Z_n \subset X$ and $W_1, \ldots, W_m \subset Y$ with the property $Z_i \supset U_i - U_{i-1}$ and $W_j \supset V_j - V_{j-1}$. (Here $U_0 = V_0 = 0$)

Setting $C_t = \bigcup_{i=1}^{t} U_i \times V_{t+1-i}$ we define a sequence $C_1 \subset \ldots \subset C_{n+m-1} = X \times Y$. (Here $U_i = X$ for $i > n$ and $V_j = Y$ for $j > m$)

We see that

$$\begin{aligned}
C_{j+1} - C_j &= \bigcup_{k=1}^{j+1} U_k \times V_{j+2-k} - \bigcup_{k=1}^{j+1} U_k \times V_{j+1-k} \\
&= \bigcup_{k=1}^{j+1} \bigcap_{l=1}^{j+1} (U_k \times V_{j+2-k}) \cap (U_l \times V_{j+1-l})^{\complement} \\
&= \bigcup_{k=1}^{j+1} \bigcap_{l=1}^{j+1} \big((U_k - U_l) \times V_{j+2-k}\big) \cup \big(U_k \times (V_{j+2-k} - V_{j+1-l})\big) \\
&= \bigcup_{k=1}^{j+1} (U_k - U_{k-1}) \times (V_{j+2-k} - V_{j+1-k}) \\
&= \bigcup_{k=1}^{j+1} A_k^{j+1}
\end{aligned}$$

where $A_k^{j+1} = (U_k - U_{k-1}) \times (V_{j+2-k} - V_{j+1-k})$.

In addition, $A_k^{j+1} \subset Z_k \times W_{j+2-k}$ and the set $Z_k \times W_{j+2-k}$ is open and categorical with respect to $X \times Y$. Also, if $k > l$ then $\overline{(U_k - U_{k-1})} \subset X - U_{k-1}$, so $\overline{(U_k - U_{k-1})} \cap (U_l - U_{l-1}) = \emptyset$ and

$$\begin{aligned}
\overline{A_k^{j+1}} \cap A_l^{j+1} &= \overline{(U_k - U_{k-1}) \times (V_{j+2-k} - V_{j+1-k})} \cap (U_l - U_{l-1}) \times (V_{j+2-l} - V_{j+1-l}) \\
&= \overline{(U_k - U_{k-1})} \cap (U_l - U_{l-1}) \times \overline{(V_{j+2-k} - V_{j+1-k})} \cap (V_{j+2-l} - V_{j+1-l}) \\
&= \emptyset
\end{aligned}$$

Thus, for all $k \neq l$, we have $\overline{A_k^{j+1}} \cap A_l^{j+1} = A_k^{j+1} \cap \overline{A_l^{j+1}} = \emptyset$. Because the space $X \times Y$ is completely normal and $A_k^{j+1} \subset Z_k \times W_{j+2-k}$, there are disjoint open categorical neighborhoods of the sets A_k^{j+1} and A_l^{j+1} for $k \neq l$. Since these categorical neighborhoods are disjoint and the space $X \times Y$ is path-connected, their union is also categorical, hence the union of the sets A_k^{j+1} and A_l^{j+1} has a categorical open neighborhood.

Also, $\overline{(A_k^{j+1} \cup A_l^{j+1})} \cap A_r^{j+1} = (\overline{A_k^{j+1}} \cap A_r^{j+1}) \cup (\overline{A_k^{j+1}} \cap A_r^{j+1}) = \emptyset$ and $(A_k^{j+1} \cup A_l^{j+1}) \cap \overline{A_r^{j+1}} = \emptyset$ for distinct k, l, r. Therefore the sets $A_k^{j+1} \cup A_l^{j+1}$ and A_r^{j+1} are separated by disjoint categorical open neighborhoods, and the union $A_k^{j+1} \cup A_l^{j+1} \cup A_r^{j+1}$ is contained in a categorical open set. We continue the process and we obtain that the set $C_{j+1} - C_j$ is contained in a categorical open set. Therefore $cat(X \times Y) \leq n + m - 1$. ∎

Remark: Note that the product of two completely normal spaces may not be a normal space. For example the space $\mathbb{R}_u$, which is the set of real numbers with the topology having basis the intervals $(a, b]$, $a < b$, is completely normal, but $\mathbb{R}_u \times \mathbb{R}_u$ is not normal. (see [6], p. 144)

Proposition 1.2.11 *If X is any path-connected metrizable space then*

$$catX \leq TC(X) \leq 2catX - 1.$$

Proof Suppose that there is a continuous section $s : U \to PX$ of the endpoints fibration π over an open set $U \subset X \times X$. Then the set $V = \{B \in X | (A_0, B) \in U\}$ is open and categorical, where A_0 is a fixed point of X. In fact, the inclusion map $i_V : V \hookrightarrow X$ is homotopic to the constant map with value A_0 by the homotopy $V \times I \to X$, $(B, t) \mapsto s(A_0, B)(t)$. Thus, if $\{U_i\}$ is an open covering of $X \times X$ such that each U_i admits a continuous section of π then the sets $V_i = \{B \in X | (A_0, B) \in U_i\}$ form an open covering of X by categorical sets, hence $TC(X) \geq catX$. Since by Proposition 1.2.10, $cat(X \times X) \leq 2catX - 1$, it suffices to proove that $TC(X) \leq cat(X \times X)$. Let $U \subset X \times X$ be an open categorical (in $X \times X$) subset. Then, taking a homotopy $h_t : U \to X \times X$ with $h_0 = i_U$ and $h_1 = (A_0, B_0)$ for some $(A_0, B_0) \in X \times X$, we construct a continuous section $s : U \to PX$ of π by sending each $(A, B) \in U$ to the path $s(A, B) = pr_1 h_t(A, B) * \gamma * pr_2 h_{1-t}(A, B)$, where $*$ denotes the concatenation of paths, pr_1, pr_2 are the projections of $X \times X$ onto the first and the second factor and γ is a path from A_0 to B_0. ∎

Corollary 1.2.12 *Let X be a path-connected metrizable locally contractible space. Then*

$$TC(X) \leq 2dimX + 1.$$

Proof This is immediate consequence of the right inequality of Proposition 1.2.11 and the inequality of Proposition 1.2.8. ∎

Corollary 1.2.13 *If G is a connected Lie group then $TC(G) = catG$.*

Proof ¿From Proposition 1.2.11, we have $TC(G) \geq catG$. Let $U \subset G$ be an open categorical set. We will show that over the open set $W = \{(A, B) \in G \times G | A \cdot B^{-1} \in U\}$ there is a continuous section $s : W \to PG$ of the endpoints fibration π. Since G is connected, there is a homotopy $h_t : U \to G$ such that $h_0 = i_U$ and $h_1 = e$, where e is the identity element of G. Then we can define $s : W \to PG$ by $s(A, B) = h_t(A \cdot B^{-1})\cdot B$ for $(A, B) \in W$. This argument shows that $TC(G) \leq catG$, bacause if $catG = k$ and $U_1 \cup \ldots \cup U_k = G$ is an open covering consisting of categorical (in G) sets then the sets $W_i = \{(A, B) \in G \times G | A \cdot B^{-1} \in U_i\}$ for $i = 1, \ldots, k$ form an open covering of $G \times G$ such that each set of this covering admits a continuous section of π. ∎

Example: We have shown that $catT^n = cat\mathbb{R}P^n = n + 1$. Since T^n and $\mathbb{R}P^3$ are connected Lie groups ($\mathbb{R}P^3$ is homeomorphic to the 3-dimensional rotation group $SO(3)$), $TC(T^n) = catT^n = n + 1$ and $TC(\mathbb{R}P^3) = cat\mathbb{R}P^3 = 4$.

Remark: Proposition 1.2.10 is actually true if we only assume that X,Y are path-connected normal spaces. Therefore Proposition 1.2.11 is true if X is normal and Corollary 1.2.12 if X is paracompact, path-connected and locally contractible.

There is a product formula for topological complexity analogous to Proposition 1.2.10.

Theorem 1.2.14 *For any path-connected metrizable spaces X and Y,*

$$TC(X \times Y) \leq TC(X) + TC(Y) - 1.$$

Proof There are open coverings $U_1 \cup \ldots \cup U_n = X \times X$ and $V_1 \cup \ldots \cup V_m = Y \times Y$ for $X \times X$ and $Y \times Y$, respectively, such that there is a continuous section $s_i : U_i \to PX$ of π_X for $i = 1, \ldots, n$ and there is a continuous section $\sigma_j : V_j \to PY$ of π_Y for $j = 1, \ldots, m$, where $n = TC(X)$ and $m = TC(Y)$. Since $X \times X$ is paracompact, there is a partition of unity $f_i : X \times X \to \mathbb{R}$ for $i = 1, \ldots, n$ subordinated to the covering $\{U_i\}$. Similarly, there is a partition of unity $g_j : Y \times Y \to \mathbb{R}$ for $j = 1, \ldots, m$ subordinated to the covering $\{V_j\}$. For non-empty sets $S \subset \{1, \ldots, n\}$ and $T \subset \{1, \ldots, m\}$ we define $W(S, T) \subset (X \times Y) \times (X \times Y)$ to be the set consisting of all 4-tuples $(A, B, C, D) \in (X \times Y) \times (X \times Y)$ such that $f_i(A, C)\cdot g_j(B, D) > f_{i'}(A, C)\cdot g_{j'}(B, D)$ for all $(i, j) \in S \times T$ and for all $(i', j') \notin S \times T$. The sets $W(S, T)$ have the following properties:

1. Each set $W(S, T)$ is open in $(X \times Y) \times (X \times Y)$.
2. $W(S, T) \cap W(S', T') = \emptyset$ whenever $S \times T \not\subseteq S' \times T'$ and $S' \times T' \not\subseteq S \times T$.
3. The set $W(S, T)$ is contained in $U_i \times V_j$ for all $(i, j) \in S \times T$.
4. On each $W(S, T)$ there exists a continuous section $W(S, T) \to P(X \times Y)$.
5. The sets $W(S, T)$ cover $(X \times Y) \times (X \times Y)$.

The set $W(S, T)$ is the finite intersection of the open sets $W^{(i',j')}_{(i,j)}$ for $(i, j) \in S \times T$ and $(i', j') \notin S' \times T'$, where $W^{(i',j')}_{(i,j)}$ is the set consisting of $(A, B, C, D) \in (X \times Y) \times (X \times Y)$ such that $f_i(A, C)\cdot g_j(B, D) > f_{i'}(A, C)\cdot g_{j'}(B, D)$. Hence, each $W(S, T)$ is open in $(X \times Y) \times (X \times Y)$. The property 2 follows since $f_i(A, C)\cdot g_j(B, D) > f_{i'}(A, C)\cdot g_{j'}(B, D) > f_i(A, C)\cdot g_j(B, D)$ for $(A, B, C, D) \in W(S, T) \cap W(S', T')$, $(i, j) \in (S \times T) - (S' \times T')$ and $(i', j') \in (S' \times T') - (S \times T)$. The property 3 follows from the fact that $W(S, T) \subset (suppf_i) \times (suppg_j) \subset U_i \times V_j$ for $(i, j) \in S \times T$. The set $W(S, T)$ admits the section $W(S, T) \to P(X \times Y)$, $(A, B, C, D) \mapsto (s_i(A, C), \sigma_j(B, D))$, so the property 4 follows.

For property 5, we choose $(A, B, C, D) \in (X \times Y) \times (X \times Y)$. Let S be the set of indices $i \leq n$ such that $f_i(A, C) = \max_{k \leq n} f_k(A, C)$ and let T be the set of $j \leq m$ such that $g_j(B, D) = \max_{l \leq m} g_l(B, D)$. Then $(A, B, C, D) \in W(S, T)$, hence the property 5 follows.

We define the sets

$$W_k = \bigcup_{|S|+|T|=k} W(S, T), \quad k = 2, 3, \ldots, n + m.$$

These sets form an open covering of $(X \times Y) \times (X \times Y)$. By property 2, if $|S| + |T| = |S'| + |T'| = k$ then the sets $W(S, T)$ and $W(S', T')$ either coincide (when $S = S'$ and $T = T'$) or are disjoint (otherwise). Therefore, there is a continuous section $W_k \to P(X \times Y)$ over each W_k, and the inequality follows. ∎

The method of proofs of Proposition 1.2.8 and Theorem 1.2.14 is a modification of Milnor' s procedure.

There is an upper bound for the topological complexity of the total space of a fibration in terms of the topological complexity of the fiber and the Lusternik-Schnirelmann category of the cartesian product of the base space with itself. We will use it later in Chapter 2.

Proposition 1.2.15 *Let $p : E \to B$ be a fibration with B path-connected and let $F = p^{-1}(x_0)$ be the fiber of p over a point $x_0 \in B$. Then $TC(E) \leq TC(F) \cdot cat(B \times B)$.*

Proof Let $B \times B = U_1 \cup \ldots \cup U_k$, $F \times F = V_1 \cup \ldots \cup V_l$ be open coverings such that each U_j is categorical with respect to $B \times B$ and there is a continuous section $s_i : V_i \to PF$ of the endpoints fibration π_F of F over each V_i, where $k = cat(B \times B)$, $l = TC(F)$. A homotopy of the inclusion map $U_j \hookrightarrow B \times B$ to the constant map $U_j \to B \times B$ with constant value (x_0, x_0) corresponds to a continuous map $h_j : U_j \to PB \times PB = P(B \times B)$. If $(x, y) \in U_j$ then, setting $h_j(x, y) = (a_{x,y}, b_{x,y})$, we have $a_{x,y}(0) = x$, $a_{x,y}(1) = x_0$, $b_{x,y}(0) = y$, $b_{x,y}(1) = x_0$.

We define $\overline{B} = \{(e, \omega) \in E \times PB : \omega(0) = p(e)\}$. Since p is a fibration, there is a continuous map $\lambda : \overline{B} \to PE$ such that for $(e, \omega) \in \overline{B}$, $\lambda(e, \omega)(0) = e$ and $p \circ \lambda(e, \omega) = \omega$. (see [19], Theorem 8, chapter 2, section 7, p. 92)

We define a continuous map $k_j : (p \times p)^{-1}(U_j) \to F \times F$ by sending each $(e, e') \in (p \times p)^{-1}(U_j)$ to $(\lambda(e, a_{x,y})(1), \lambda(e', b_{x,y})(1))$, where $x = p(e)$, $y = p(e')$. The sets $k_j^{-1}(V_i)$ for $j = 1, \ldots, k$ and for $i = 1 \ldots, l$ clearly form an open covering of $E \times E$. We define a continuous section of the endpoints fibration $\pi_E : PE \to E \times E$ over $k_j^{-1}(V_i)$ by sending each $(e, e') \in k_j^{-1}(V_i)$ to $\lambda(e, a_{x,y}) * s_i(\lambda(e, a_{x,y})(1), \lambda(e', b_{x,y})(1)) * \lambda(e', b_{x,y})^{-1}$, where $x = p(e)$, $y = p(e')$. We obtain that $TC(X) \leq kl$. ∎

1.3 Relative topological complexity

Definition 1.3.1 *We define the relative topological complexity $TC_X(A)$ of a space X with respect to a subspace A of $X \times X$ to be the Schwarz genus of the fibration $\pi|_{\pi^{-1}(A)} : \pi^{-1}(A) \to A$, where π denotes the usual fibration $\pi : PX \to X \times X$. Equivalently, $TC_X(A)$ is the least integer $k \geq 1$ such that there exist k open subsets $U_1, \ldots, U_k$ of A*

that cover A with the propery that the projections $U_i \to X$ of each U_i on the first and the second factors are homotopic.

Lemma 1.3.2 *Let X be a space and $A \subset X \times X$. The following properties are equivalent:*

- $TC_X(A) = 1$
- *The projections $A \to X$ of A on the first and the second factors are homotopic. (Equivalently, there is a continuous section $A \to PX$ of the fibration $\pi : PX \to X \times X$ over A)*
- *The inclusion map $i : A \hookrightarrow X \times X$ is homotopic to a continuous map $g : A \to X \times X$ with $g(A) \subset \Delta_X = \{(x, x) : x \in X\}$.*

Proof The equivalence of the first two properties follows immediately from the definition of $TC_X(A)$. We shall proove the equivalence of the second and the third assertion. If the projections pr_1, $pr_2 : A \to X$ are homotopic then the map $(pr_1, pr_2) : A \to X \times X$ is homotopic to the map $(pr_2, pr_2) : A \to X \times X$, whose values are in Δ_X. Conversely, if the third property is true then there is a homotopy $s_t = (s_{1t}, s_{2t}) : A \to X \times X$, $0 \le t \le 1$, such that the maps $s_{10}, s_{20} : A \to X$ are the projections of A on the first and the second factors, respectively, and $s_{11} = s_{21}$. Hence, we have the continuous section $s : A \to PX$ of $\pi : PX \to X \times X$ defined by $s(a, b) = s_{1t}(a, b) * s_{2t}(a, b)^{-1}$ for $(a, b) \in A$. ∎

Note that $TC_X(A) \le TC_X(B)$ if $A \subset B \subset X \times X$, and in the case $B = X \times X$, we obtain $TC_X(A) \le TC(X)$.

Lemma 1.3.3 *Let $A \subset B \subset X \times X$ and suppose that the inclusion map $B \hookrightarrow X \times X$ is homotopic to a map $B \to X \times X$, whose values are in A. Then $TC_X(A) = TC_X(B)$.*

Proof Since $TC_X(A) \le TC_X(B)$, it suffices to proove that $TC_X(A) \ge TC_X(B)$. Let $A = U_1 \cup \ldots \cup U_k$ be an open covering of A such that the projections $U_i \to X$ of each U_i on the first and the second factors are homotopic, where $k = TC_X(A)$. Let $h_t : B \to X \times X$ be a homotopy such that h_0 is the inclusion and $h_1(B) \subset A$. There is a homotopy $s_i : U_i \times I \to X$ such that $s_i(\cdot, 0)$, $s_i(\cdot, 1)$ are the projections of U_i of the first and the second factors, respectively. Setting $W_i = h_1^{-1}(U_i)$ we get an open covering of B. On each W_i there is a continuous section $W_i \to PX$ of $\pi : PX \to X \times X$ defined by $(x, y) \mapsto pr_1 h_t(x, y) * s_i(h_1(x, y), \cdot) * pr_2 h_t(x, y)^{-1}$. This shows that $TC_X(A) \ge TC_X(B)$. ∎

Lemma 1.3.4 *If X is an ENR and the subset $A \subset X \times X$ is locally compact then there is an open set $A \subset U \subset X \times X$ such that $TC_X(A) = TC_X(U)$.*

Proof Let $k = TC_X(A)$ and let $A = U_1 \cup \ldots \cup U_k$, where each U_i is open in A and the projections $U_i \to X$ over each U_i on the first and the second factors are homotopic. Since A is locally compact and U_i is open in A, we have $U_i = O_i \cap \overline{A}$, where $O_i \subset X \times X$ is open (See [6], Theorem 6.5, p. 239). ¿From Lemma 1.1.11, there is an open set $U_i \subset \tilde{O}_i \subset O_i$ such that the projections $\tilde{O}_i \to X$ are homotopic. Therefore $TC_X(A) = TC_X(U)$, where $U = \tilde{O}_1 \cup \ldots \cup \tilde{O}_k$. ∎

Proposition 1.3.5 *If X is an ENR and $X \times X = A_1 \cup \ldots \cup A_k$, where the sets $A_1, \ldots, A_k$ are locally compact then $TC(X) \leq TC_X(A_1) + \ldots + TC_X(A_k)$.*

Proof ¿From Lemma 1.3.4, there are open sets $U_1 \supset A_1, \ldots, U_k \supset A_k$ such that $TC_X(A_i) = TC_X(U_i)$ for $i = 1, \ldots, k$. Since the sets U_i form an open covering of $X \times X$, we have

$$\begin{aligned} TC(X) &\leq TC_X(U_1) + \ldots + TC_X(U_k) \\ &= TC_X(A_1) + \ldots + TC_X(A_k). \quad \blacksquare \end{aligned}$$

Chapter 2

Cohomology and topological complexity

2.1 Sectional category weight

Throughout this section we assume that a fixed fibration $p : E \to B$ and a fixed commutative ring R with a unity are given and we are interested in cohomology classes in the cohomology ring $H^*(B; R)$. We denote by $f^*p : f^*(E) \to X$ the fibration induced from p by a continuous map $f : X \to B$ (see [19], Chapter 2, Section 8, p.98) and by *genus*(f^*p) the Schwarz genus of f^*p. Also, we write $H^*(B)$ instead of $H^*(B; R)$. The following notion was introduced in [11].

Definition 2.1.1 *The sectional category weight of a cohomology class $\xi \in H^*(B)$ with respect to p, denoted by $wgt_p(\xi)$, is defined to be the largest integer $k \geq 0$ such that $f^*(\xi) = 0$ for all continuous maps $f : X \to B$ with genus$(f^*p) \leq k$. The sectional category weight of the zero class is defined to be ∞.*

Note that the inequality *genus*$(f^*p) \leq k$ means that there are k open sets $U_1, \ldots, U_k$ that cover X and k continuous maps $\phi_i : U_i \to E$, $i = 1, \ldots, k$, such that $p \circ \phi_i = f|_{U_i}$ for $i = 1, \ldots, k$.

Proposition 2.1.2 *If $\xi \in H^*(B)$ then $wgt_p(\xi) \geq 1$ if and only if $p^*(\xi) = 0$.*

Proof Suppose firstly that $p^*(\xi) = 0$. Let $f : X \to B$ be a continuous map with *genus*$(f^*p) \leq 1$. Then there is a section $g : X \to f^*(E)$ of f^*p. The commutativity of the diagram

$$\begin{tikzcd}
H^*(X) \arrow[dr, equal] & H^*(f^*(E)) \arrow[l, "g^*"'] & H^*(E) \arrow[l, "A"'] \\
 & H^*(X) \arrow[u, "(f^*p)^*"] & H^*(B) \arrow[l, "f^*"] \arrow[u, "p^*"']
\end{tikzcd}$$

shows that $f^*(\xi) = g^*(f^*p)^*f^*(\xi) = g^*Ap^*(\xi) = 0$. Thus $\mathrm{wgt}_p(\xi) \geq 1$.

Conversely, if $\mathrm{wgt}_p(\xi) \geq 1$ then *genus*$(p^*p) = 1$, because the diagonal map $\Delta : E \to E \times E = p^*(E)$ is a section of p^*p, so $p^*(\xi) = 0$. ∎

Definition 2.1.3 *Let X be a space and let R be a commutative ring with a unity. The TC-weight of a cohomology class $u \in H^*(X \times X)$ is defined to be the sectional category weight $wgt_\pi(u)$ of u with respect to the endpoints fibration $\pi : PX \to X \times X$.*

Corollary 2.1.4 *$wgt_\pi(u) \geq 1$ if and only if $\Delta^*(u) = 0$, where $\Delta : X \to X \times X$ is the diagonal map.*

Proof The map $i : X \to PX$ that sends each $x \in X$ to the constant path at the point x is a homotopy equivalence. Thus the conclusion follows from Proposition 2.1.2 and the commutativity of the diagram

$$\begin{tikzcd} & PX \arrow[d, "\pi"] \\ X \arrow[ur, "i"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd} \qquad \blacksquare$$

Proposition 2.1.5 *If $genus(p) < \infty$ then $wgt_p(\xi) < genus(p)$ for all non-zero cohomology classes $\xi \in H^*(B)$.*

Proof Let $\xi \in H^*(B)$ with $\mathrm{wgt}_p(\xi) \geq genus(p)$. Then since $genus(1^*p) = genus(p) \leq \mathrm{wgt}_p(\xi)$, where $1 : B \to B$ is the identity map, we have $\xi = 1^*(\xi) = 0$. $\blacksquare$

The above observation means that in order to find lower bounds of the Schwarz genus of a fibration p it suffices to find non-zero cohomology classes of the highest possible sectional weight with respect to p.

Proposition 2.1.6 *If $genus(p) < \infty$ then*

$$wgt_p(\xi_1 \smile \cdots \smile \xi_l) \geq \sum_{i=1}^{l} wgt_p(\xi_i)$$

for $\xi_1, \ldots, \xi_l \in H^*(B)$.

Proof Suppose that $\xi = \xi_1 \smile \cdots \smile \xi_l \neq 0$ (In the case $\xi_1 \smile \cdots \smile \xi_l = 0$ we have $\mathrm{wgt}_p(\xi_1 \smile \cdots \smile \xi_l) = \infty \geq \sum_{i=1}^{l} \mathrm{wgt}_p(\xi_i)$). We put $k_i = \mathrm{wgt}_p(\xi_i)$ and $k = k_1 + \cdots + k_l$. Let $f : X \to B$ be a map with $genus(f^*p) \leq k$. There is an open covering $U_1 \cup \ldots \cup U_k = X$ of X such that there are k continuous maps $\phi_i : U_i \to E$ with $p \circ \phi_i = f|_{U_i}$. We define the families of sets $\Omega_1, \ldots, \Omega_l$ by

$$\Omega_1 = \{U_1, \ldots, U_{k_1}\}, \Omega_2 = \{U_{k_1+1}, \ldots, U_{k_1+k_2}\}, \ldots, \Omega_l = \{U_{\sum_{i=1}^{l-1} k_i+1}, \ldots, U_l\}.$$

We also set A_i to be the union of the family of sets Ω_i for $i = 1, \ldots, l$.

Since $genus((f|_{A_i})^*p) \leq k_i = \mathrm{wgt}_p(\xi_i)$, we have $f^*(\xi_i)|_{A_i} = (f|_{A_i})^*(\xi_i) = 0$ and so $f^*(\xi_i)$ pulls back to a cohomology class in $H^*(X, A_i)$. Thus we obtain $f^*(\xi) = f^*(\xi_1) \smile \cdots \smile f^*(\xi_l) = 0$. $\blacksquare$

The propositions 2.1.5 and 2.1.6 above give the following.

Proposition 2.1.7 *If $genus(p) < \infty$, $\xi_1, \ldots, \xi_l \in H^*(B)$ and $\xi_1 \smile \cdots \smile \xi_l \neq 0$ then*

$$genus(p) > \sum_{i=1}^{l} wgt_p(\xi_i).$$

Definition 2.1.8 *Any cohomology class $u \in H^*(X \times X; R)$, where R denotes a commutative ring with a unity, is called a zero-divisor if $wgt_\pi(u) \geq 1$.*

Note that from Corollary 2.1.4 the last statement is equivalent to $\Delta^*(u) = 0$.

If $u \in H^q(X; R)$ then the cohomology class $\overline{u} = 1 \times u - u \times 1 \in H^q(X \times X; R)$ is a zero-divisor, since $\Delta^*(\overline{u}) = \Delta^*(1 \times u) - \Delta^*(u \times 1) = 1 \smile u - u \smile 1 = u - u = 0$.

¿From Proposition 2.1.7 follows that

Proposition 2.1.9 *If the cohomology classes $u_1, \ldots, u_k \in H^*(X \times X; R)$ are zero-divisors and $u_1 \smile \cdots \smile u_k \neq 0$, then $TC(X) > k$.*

Example: We have shown in Section 1.1 that $TC(S^n) = 2$ for odd n and $TC(S^n) \leq 3$ for even n. We will show that actually $TC(S^n) = 3$ for n even.

Let $u \in H^n(S^n; \mathbb{Q})$ be a non-zero cohomology class of degree n. Then, setting $\overline{u} = 1 \times u - u \times 1$, we have

$$\begin{aligned}
\overline{u}^2 &= (1 \times u - u \times 1) \smile (1 \times u - u \times 1) \\
&= 1 \times u^2 - (-1)^{n^2} u \times u - u \times u + u^2 \times 1 \\
&= -(-1)^n u \times u - u \times u \\
&= -[1 + (-1)^n] u \times u,
\end{aligned}$$

hence $\overline{u}^2 \neq 0$ if n is even, and from Proposition 2.1.9 we get $TC(S^n) > 2$ for n even.

Example: We will show that $TC(\mathbb{C}P^n) = 2n + 1$. Let $u \in H^2(\mathbb{C}P^n; \mathbb{Q})$ be a generator. Since $(1 \times u) \smile (u \times 1) = u \times u = (u \times 1) \smile (1 \times u)$, we have

$$(1 \times u - u \times 1)^{2n} = (-1)^n \binom{2n}{n} (u^n \times u^n) \neq 0.$$

Therefore, by Proposition 2.1.9, $TC(\mathbb{C}P^n) > 2n$. In addition, we have shown that $cat\mathbb{C}P^n = n + 1$ and Proposition 1.2.11 shows that $TC(\mathbb{C}P^n) \leq 2n + 1$.

Example: We will show that

$$TC(\underbrace{S^m \times \cdots \times S^m}_{n \text{ factors}}) = \begin{cases} n + 1 & \text{if } m \text{ is odd} \\ 2n + 1 & \text{if } m \text{ is even} \end{cases}$$

By Theorem 1.2.14, we have

$$
\begin{aligned}
TC(\underbrace{S^m \times \cdots \times S^m}_{n \text{ factors}}) &\leq TC(\underbrace{S^m \times \cdots \times S^m}_{n-1 \text{ factors}}) + TC(S^m) - 1 \\
&\leq \cdots \\
&\leq TC(\underbrace{S^m \times \cdots \times S^m}_{n-k \text{ factors}}) + kTC(S^m) - k \\
&\leq \cdots \\
&\leq TC(S^m) + (n-1)TC(S^m) - (n-1) \\
&= n(TC(S^m) - 1) + 1,
\end{aligned}
$$

and thus $TC(S^m \times \cdots \times S^m) \leq n+1$ if m is odd and $\leq 2n+1$ if m is even.

Let $a \in H^m(S^m;\mathbb{Q})$ be the fundumental class and let a_i be the image of a under the homomorphism $pr_i^* : H^m(S^m;\mathbb{Q}) \to H^m(X;\mathbb{Q})$ induced by the projection $pr_i : X \to S^m$ of $X = S^m \times \cdots \times S^m$ onto the i-th factor. If $u_i = 1 \times a_i - a_i \times 1 \in H^m(X \times X;\mathbb{Q})$ then

$$u_1 \smile \cdots \smile u_n = \sum_{i_1,\ldots,i_n \in \{0,1\}} (\pm)(a_1^{i_1} \smile \cdots \smile a_n^{i_n}) \times (a_1^{1-i_1} \smile \cdots \smile a_n^{1-i_n}) \neq 0,$$

and hence, by Proposition 2.1.9, $TC(X) > n$. So in the case that m is odd the conclusion follows. In the case that m is even, we have $u_i^2 = 1 \times a_i^2 - (-1)^m a_i \times a_i - a_i \times a_i + a_i^2 \times 1 = -2(a_i \times a_i)$. Thus

$$u_1^2 \smile \cdots \smile u_n^2 = (-2)^n (a_1 \smile \cdots \smile a_n) \times (a_1 \smile \cdots \smile a_n) \neq 0,$$

and therefore, by Proposition 2.1.9, $TC(X) > 2n$ and the conlusion follows.

Example: We will show that $TC(\Sigma_g) = 5$, where Σ_g is a compact orientable 2-dimensional surface of genus $g \geq 2$.

There are cohomology classes $u_1, u_2, \upsilon_1, \upsilon_2 \in H^1(\Sigma_g;\mathbb{Q})$ which form a symplectic system. This means that the following properties are satisfied (see [15]):

1. $u_i^2 = 0$ and $\upsilon_i^2 = 0$.
2. $u_1 \smile \upsilon_1 = u_2 \smile \upsilon_2 = A \neq 0$, where $A \in H^2(\Sigma_g;\mathbb{Q})$ is the fundamental class.
3. $u_i \smile u_j = \upsilon_i \smile \upsilon_j = \upsilon_i \smile u_j = 0$ for $i \neq j$.

Therefore

$$
\begin{aligned}
\prod_{i=1}^{2} (1 \times u_i - u_i \times 1) \smile (1 \times \upsilon_i - \upsilon_i \times 1) &= \prod_{i=1}^{2} (1 \times A + A \times 1 + \upsilon_i \times u_i - u_i \times \upsilon_i) \\
&= 2A \times A \neq 0.
\end{aligned}
$$

¿From Proposition 2.1.9 we have $TC(\Sigma_g) > 4$. Also, the fact that Σ_g is a 2-manifold and Corollary 1.2.12 imply that $TC(\Sigma_g) \leq 2dim(\Sigma_g) + 1 \leq 5$.

Example: We will show that $TC(\mathbb{R}P^n) \geq 2^r$ provided that $2^{r-1} \leq n < 2^r$. We consider the zero-divisor $1 \times a + a \times 1 \in H^1(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{Z}_2)$, where $a \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the generator. We will show that the $(2^r - 1)$-th power of this zero divisor

$$(1 \times a + a \times 1)^{2^r-1} = \sum_{i=0}^{2^r-1} (\pm) \binom{2^r-1}{i} a^i \times a^{2^r-1-i}$$

is nonzero. In fact, the binomial coefficients $\binom{2^r-1}{i} = \frac{(2^r-1)(2^r-2)\cdots(2^r-i)}{1\cdot 2\cdots i}$ are odd because if $i = 2^m k$, where $0 \leq m < r$ and k is an odd positive integer, then $\frac{2^r-i}{i} = \frac{2^{r-m}-k}{k}$, hence $\binom{2^r-1}{i}$ is a quotient of two odd integers. Also, the term $\binom{2^r-1}{n} a^n \times a^{2^r-1-n}$ is non-zero, since a is the generator and a^n, a^{2^r-1-n} are non-zero. Therefore, by Künneth formula, we get $(1 \times a + a \times 1)^{2^r-1} \neq 0$. (see [15], Theorem 3.16, p.219)

2.2 Cohomology classes of weight greater than 1

In this section we will prove a criterion which provides cohomology classes of weight at least 2 using stable cohomology operations, whose definition we recall first briefly.

Let $p, q \in \mathbb{Z}$ and G, G' be two abelian groups. A cohomology operation of type $(p, q|G, G')$ is a natural transformation

$$\theta : H^p(-; G) \to H^q(-; G')$$

of sets. This means that to every topological pair (X, A) corresponds a function (not necessarily homomorphism)

$$\theta_{(X,A)} : H^p(X, A; G) \to H^q(X, A; G')$$

so that for every continuous map $f : (X, A) \to (Y, B)$ we have a commutative diagramm

$$\begin{tikzcd}
H^p(Y, B; G) \arrow[r, "\theta_{(Y,B)}"] \arrow[d, "f^*"'] & H^q(Y, B; G') \arrow[d, "f^*"] \\
H^p(X, A; G) \arrow[r, "\theta_{(X,A)}"] & H^q(X, A; G')
\end{tikzcd}$$

The cohomology operation θ is called additive if $\theta_{(X,A)}$ is a homomorphism for each topological pair (X, A).

For example, let (C_*, ∂) be a free chain complex and $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ be a short exact sequence of abelian groups. Since C_* is free, we get a short exact sequence of cochain complexes

$$0 \longrightarrow Hom(C, G') \longrightarrow Hom(C, G) \longrightarrow Hom(C, G'') \longrightarrow 0$$

and therefore a natural transformation (the connecting homomorphism) $\beta : H^q(C; G'') \to H^{q+1}(C; G')$ for each $q \in \mathbb{Z}$, called the Bockstein homomorphism. Setting

$C = \Delta_*(X, A)$ for a topological pair (X, A) we get an additive cohomology operation of type $(q, q+1|G'', G')$ for each $q \in \mathbb{Z}$.

Another example of a (non-additive) cohomology operation is provided by taking powers. More precisely, let R be a commutative ring with a unity. For every $p, q \in \mathbb{Z}^+$ let $\theta_p : H^q(X, A; R) \to H^{qp}(X, A; R)$ be defined by $\theta_p(\sigma) = \sigma^p$, where the power is taken with respect to the cup-product. Then, θ_p is a cohomology operation of type $(q, pq|R, R)$. It is in general non-additive. It is however additive if $R = \mathbb{Z}_2$ and $p = 2$.

A stable cohomology operation of degree i is a sequence of additive cohomology operations $\theta : H^q(-; G) \to H^{q+i}(-; G')$ for each $q \in \mathbb{Z}$, that is of type $(q, q+i|G, G')$ respectively, which commute with the (unreduced) suspension isomorphism $H^q(X; G) \cong H^{q+1}(\Sigma X; G)$ for every space X or equivalently, commute with the connecting homomorphisms $\delta^* : H^q(A; G) \to H^{q+1}(X, A; G)$ in the long exact sequence of every topological pair (X, A).

The Steenrod squares is a sequence of stable cohomology operations $(Sq^i)_{i \geq 0}$ each of degree i respectively, where $G = G' = \mathbb{Z}_2$. More precisely, for every topological pair (X, A) and each $i \geq 0$ we have a sequence of homomorphisms

$$Sq^i : H^q(X, A; \mathbb{Z}_2) \to H^{q+i}(X, A; \mathbb{Z}_2)$$

for all $q \in \mathbb{Z}$ and they satisfy the following axioms:

(i) $Sq^0 = id$

(ii) If $\sigma \in H^q(X, A; \mathbb{Z}_2)$, then $Sq^q(\sigma) = \sigma^2 = \sigma \smile \sigma$.

(iii) If $\sigma \in H^q(X, A; \mathbb{Z}_2)$ and $i > q$, then $Sq^i(\sigma) = 0$.

(iv) If $\sigma \in H^*(X, A; \mathbb{Z}_2)$, $\tau \in H^*(Y, B; \mathbb{Z}_2)$ and the pair $\{X \times B, A \times Y\}$ is excisive in $X \times Y$, then

$$Sq^k(\sigma \times \tau) = \sum_{i+j=k} Sq^i(\sigma) \times Sq^j(\tau) \quad \text{(Cartan formula)}$$

It follows from naturality and the definition of the cup-product that

$$Sq^k(\sigma \smile \tau) = \sum_{i+j=k} Sq^i(\sigma) \smile Sq^j(\tau).$$

The stability of Sq^i can be shown to follow from the above four axioms as well the following property:

(v) Sq^1 is the Bockstein homomorphism defined from the coefficient exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

(see [2], [7], [15] and [17] for details)

The Steenrod cyclic reduced power operations are analogues of the Steenrod squares for odd prime $p > 2$.

For a prime $p > 2$, the Steenrod cyclic reduced power operations is a sequence of stable cohomology operations $(P^i)_{i \geq 0}$, each of degree $2i(p-1)$ respectively, where $G = G' = \mathbb{Z}_p$. For every topological pair (X, A) and $i \geq 0$ we have a sequence of homomorphisms

$$P^i : H^q(X, A; \mathbb{Z}_p) \to H^{q+2i(p-1)}(X, A; \mathbb{Z}_p)$$

for all $q \in \mathbb{Z}$, which satisfy the following axioms:

(i) $P^0 = id$
(ii) If $\sigma \in H^{2i}(X, A; \mathbb{Z}_p)$, then $P^i(\sigma) = \sigma^p$.
(iii) If $\sigma \in H^q(X, A; \mathbb{Z}_p)$ and $2i > q$, then $P^i(\sigma) = 0$.
(iv) If $\sigma \in H^*(X, A; \mathbb{Z}_p)$, $\tau \in H^*(Y, B; \mathbb{Z}_p)$ and the pair $\{X \times B, A \times Y\}$ is excisive in $X \times Y$, then

$$P^k(\sigma \times \tau) = \sum_{i+j=k} P^i(\sigma) \times P^j(\tau) \quad \text{(Cartan formula)}$$

As in the case of Steenrod squares, we have a corresponding Cartan formula for cup-products and the stability is implied from these four axioms. (see [2], [15])

Definition 2.2.1 *The excess of a stable cohomology operation θ is the largest integer $e(\theta)$ such that $\theta(u) = 0$ for every $u \in H^q(X; G)$ with $q < e(\theta)$.*

By axioms (i) and (ii), the excess of the Steenrod square Sq^i is $e(Sq^i) = i$ and for odd prime p the excess of the Steenrod cyclic power operation P^i is $e(P^i) = 2i$. If $i_1, \dots, i_n$ are positive integers and $\theta = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_n}$, then axiom (iii) implies that

$$e(\theta) \geq \max\{i_k - i_{k+1} - \cdots - i_n | 1 \leq k \leq n\}.$$

If moreover $i_k \geq 2i_{k+1}$ for all $1 \leq k < n$, then

$$i_k - \sum_{l=k+1}^{n} i_l \leq i_{k-1} - \sum_{l=k}^{n} i_l$$

and therefore

$$\begin{aligned} e(\theta) &\geq \max\{i_k - i_{k+1} - \cdots - i_n | 1 \leq k \leq n\} \\ &= i_1 - i_2 - \cdots - i_n \\ &= \sum_{k=1}^{n-1} (i_k - 2i_{k+1}) + i_n. \end{aligned}$$

We now describe a method of finding cohomology classes of TC-weight greater than 1 given a stable cohomology operation $\theta : H^*(-; R) \to H^{*+i}(-; R')$, where R and R' are two commutative rings with a unity.

If $u \in H^q(X; R)$ then we denote by $\overline{u}$ the cohomology class

$$\overline{u} = 1 \times u - u \times 1 \in H^q(X \times X; R)$$

and recall that $\overline{u}$ is a zero-divisor, i.e. $wgt_\pi(\overline{u}) \geq 1$. Also,

$$\theta(\overline{u}) = \theta(pr_2^*(u) - pr_1^*(u)) = \theta(pr_2^*(u)) - \theta(pr_1^*(u)) = pr_2^*(\theta(u)) - pr_1^*(\theta(u)) = \overline{\theta(u)},$$

since $pr_1^*(u) = u \times 1$ and $pr_2^*(u) = 1 \times u$, where $pr_1, pr_2 : X \times X \to X$ are the projections.

We will need the following.

Lemma 2.2.2 *Let $f = (f_1, f_2) : Y \to X \times X$ be a continuous map and $\pi : PX \to X \times X$ be the endpoints fibration. Then $genus(f^*\pi) \leq k$ if and only if there are k open sets $U_1, \dots, U_k \subset Y$ that cover Y and $f_1|_{U_i} \simeq f_2|_{U_i}$ for all $i = 1, \dots, k$.*

Proof Let $U \subset Y$ be an open set. It suffices to show that there exist a local section of $f^*\pi$ over U if and only if $f_1|_U \simeq f_2|_U$. A local section $s : U \to f^*(PX)$ exists if and only if there exist a continuous map $J : U \to PX$ with $\pi \circ J = f|_U$. In fact, if such a map J exists then we define $s(y) = (y, Jy)$. Also, a continuous map $J : U \to PX$ with $\pi \circ J = f|_U$ is assosiated to a homotopy $F : f_1|_U \simeq f_2|_U$, defined by evaluation, and conversely. ∎

Theorem 2.2.3 *Let $\theta : H^*(-; R) \to H^{*+i}(-; R')$ be a stable cohomology operation of degree i, where R and R' are two commutative rings with a unity, and let $u \in H^q(X; R)$ be a cohomology class with $q \leq e(\theta)$. Then $wgt_\pi(\theta(\overline{u})) = wgt_\pi(\overline{\theta(u)}) \geq 2$.*

Proof Let $f = (f_1, f_2) : Y \to X \times X$ be a continuous map such that $genus(f^*\pi) \leq 2$. It suffices to show that $f^*(\overline{\theta(u)}) = 0$. By Lemma 2.2.2, there are open subsets $A, B \subset Y$, $A \cup B = Y$ with $f_1|_A \simeq f_2|_A$ and $f_1|_B \simeq f_2|_B$. We consider the element in $H^q(Y; R)$

$$f^*(\overline{u}) = f^*(pr_2^*(u)) - f^*(pr_1^*(u)) = (pr_2 f)^*(u) - (pr_1 f)^*(u) = f_1^*(u) - f_2^*(u),$$

where $pr_1, pr_2 : X \times X \to X$ are the projections. We take the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^{q-1}(A \cap B; R) \xrightarrow{\delta} H^q(Y; R) \xrightarrow{j_A^* - j_B^*} H^q(A; R) \bigoplus H^q(B; R) \longrightarrow \cdots$$

where $j_A : A \hookrightarrow Y$, $j_B : B \hookrightarrow Y$ are the inclusion maps. Since $j_A^* f^*(\overline{u}) = 0$ and $j_B^* f^*(\overline{u}) = 0$, there exists $w \in H^{q-1}(A \cap B; R)$ such that $f^*(\overline{u}) = \delta(w)$. Therefore $f^*(\overline{\theta(u)}) = f^*(\theta(\overline{u})) = \theta(f^*(\overline{u})) = \theta(\delta(w)) = \delta(\theta(w)) = 0$, by naturality and since θ is stable. ∎

Example: The short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ induces a long exact sequence

$$\cdots \longrightarrow H^q(X; \mathbb{Z}) \xrightarrow{2} H^q(X; \mathbb{Z}) \longrightarrow H^q(X; \mathbb{Z}_2) \xrightarrow{\beta} H^{q+1}(X; \mathbb{Z}) \longrightarrow \cdots$$

for any space X, where β is the corresponding Bockstein homomorphism.

Recall that for even n

$$H_q(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}_2 & \text{for } 0 < q < n, q \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

and for odd n

$$H_q(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, n \\ \mathbb{Z}_2 & \text{for } 0 < q < n, q \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

(see [2], chapter 4, section 14, p. 218). If $0 < q < n$, it follows from the Universal Coefficient Theorem that $H^q(\mathbb{R}P^n; \mathbb{Z}) \cong Ext(H_{q-1}(\mathbb{R}P^n; \mathbb{Z}); \mathbb{Z})$, since $Hom(H_q(\mathbb{R}P^n; \mathbb{Z}), \mathbb{Z}) = 0$. Therefore for even n

$$H^q(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}_2 & \text{for } q = n \\ \mathbb{Z}_2 & \text{for } 0 < q < n, q \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and for odd n

$$H^q(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, n \\ \mathbb{Z}_2 & \text{for } 0 < q < n, q \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

In the case of the real projective space $\mathbb{R}P^n$, $n \geq 2$, the above long exact sequence implies that the Bockstein homomorphism $\beta : H^1(\mathbb{R}P^n;\mathbb{Z}_2) \to H^2(\mathbb{R}P^n;\mathbb{Z})$ is an isomorphism.

If $y \in H^2(\mathbb{R}P^n;\mathbb{Z})$ is a generator then $y = \beta(x)$ for a generator $x \in H^1(\mathbb{R}P^n;\mathbb{Z}_2)$. Since the excess of the Bockstein stable cohomology operation is 1, it follows from Theorem 2.2.3 that

$$wgt_\pi(1 \times y - y \times 1) \geq 2.$$

2.3 Topological Complexity of lens spaces

In this section we will apply Theorem 2.2.3 in order to compute the topological complexity of lens spaces.

Let $m > 1$ be an integer. Recall that the lens space L_m^{2n+1} is defined to be the orbit space $S^{2n+1}/\mathbb{Z}_m$ of the action of $\mathbb{Z}_m$, regarding it as the multiplicative group $\{z \in \mathbb{C} | z^m = 1\}$, on the unit sphere $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} || z_0|^2 + |z_1|^2 + \ldots + |z_n|^2 = 1\} \subset \mathbb{C}^{n+1}$ defined by pointwise multiplication (see [15], example 2.43, p. 144). The Hopf fibration $\eta : S^{2n+1} \to \mathbb{C}P^n$ factors through L_m^{2n+1}, so we have a map $\tilde{\eta} : L_m^{2n+1} \to \mathbb{C}P^n$. Over the set $U_i = \{[z_0, \ldots, z_n] \in \mathbb{C}P^n | z_i \neq 0\}$ there is a trivialization $\phi_i : \eta^{-1}(U_i) \to U_i \times S^1$ with $\phi_i(z_0, \ldots, z_n) = ([z_0, \ldots, z_n], z_i/|z_i|)$, where $[z_0, \ldots, z_n]$ are the homogeneous coordinates of a point. This means that ϕ_i is homeomorphism and the following diagram is commutative

$$\begin{tikzcd} \eta^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "\eta"'] & & U_i \times S^1 \arrow[dl, "projection"] \\ & U_i & \end{tikzcd}$$

The inverse of ϕ_i is given by $\phi_i^{-1}([z_0, \ldots, z_n], \lambda) = |z_i| \frac{\lambda}{z_i}(z_0, \ldots, z_n)$. It follows that ϕ_i induces a trivialization $\tilde{\phi}_i : \eta^{-1}(U_i)/\mathbb{Z}_m \to U_i \times (S^1/\mathbb{Z}_m) \approx U_i \times S^1$, i.e., $\tilde{\phi}_i$ is homeomorphism and we have the commutative diagam

$$\begin{tikzcd} \eta^{-1}(U_i)/\mathbb{Z}_m \arrow[rr, "\tilde{\phi}_i"] \arrow[dr, "\tilde{\eta}"'] & & U_i \times (S^1/\mathbb{Z}_m) \arrow[dl, "projection"] \\ & U_i & \end{tikzcd}$$

(Note that $\eta^{-1}(U_i)/\mathbb{Z}_m = \tilde{\eta}^{-1}(U_i)$ and the orbit space topology of the action of $\mathbb{Z}_m$ on $\eta^{-1}(U_i)$ coincides with the subspace topology induced by L_m^{2n+1}).

Proposition 2.3.1 *$TC(L_m^{2n+1}) \leq 4n + 2.$*

Proof The map $\tilde{\eta} : L_m^{2n+1} \to \mathbb{C}P^n$ is a fibration with fiber S^1, since it is a fiber bundle with trivializations the maps $\tilde{\phi}_i$, $i = 0, 1, \ldots, n$ and the base space $\mathbb{C}P^n$ is metrizable. Proposition 1.2.15 implies that $TC(L_m^{2n+1}) \leq TC(S^1) \cdot cat(\mathbb{C}P^n \times \mathbb{C}P^n) = 2cat(\mathbb{C}P^n \times \mathbb{C}P^n)$. By Proposition 1.2.10 and the proof of Proposition 1.2.11 we have $2n+1 = TC(\mathbb{C}P^n) \leq cat(\mathbb{C}P^n \times \mathbb{C}P^n) \leq 2cat(\mathbb{C}P^n) - 1 = 2n+1$. Hence, $cat(\mathbb{C}P^n \times \mathbb{C}P^n) = TC(\mathbb{C}P^n) = 2n+1$ and the inequality follows. ∎

The homology groups of the lens space L_m^{2n+1} are given by

$$H_q(L_m^{2n+1}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, 2n+1 \\ \mathbb{Z}_m & \text{for } 0 < q < 2n+1, q \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

and, by the Universal Coefficient Theorem, it follows that

$$H^q(L_m^{2n+1}; \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & \text{for } 0 \leq q \leq 2n+1 \\ 0 & \text{for } q > 2n+1. \end{cases}$$

We choose a generator $x \in H^1(L_m^{2n+1}; \mathbb{Z}_m)$ and we consider the Bockstein homomorphism $\beta : H^1(L_m^{2n+1}; \mathbb{Z}_m) \to H^2(L_m^{2n+1}; \mathbb{Z}_m)$ assosiated to the short exact sequence of abelian groups $0 \longrightarrow \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \longrightarrow \mathbb{Z}_m \longrightarrow 0$. The element $y = \beta(x) \in H^2(L_m^{2n+1}; \mathbb{Z}_m)$ is a generator since β is an isomorphism (see [15], example 3E.1, p. 303). Also, for all i the elements $y^i \in H^{2i}(L_m^{2n+1}; \mathbb{Z}_m)$ and $x \smile y^i \in H^{2i+1}(L_m^{2n+1}; \mathbb{Z}_m)$ are generators. As a ring,

$$H^*(L_m^{2n+1}; \mathbb{Z}_m) \cong \mathbb{Z}_m[x, y] / < y^{n+1}, x^2 - ky >,$$

where $k = m/2$ if m is even and $k = 0$ if m is odd (see [15], example 3.41, p. 251 and example 3E.2, p. 304).

Proposition 2.3.2 *Let k, l be two integers, $0 \leq k, l \leq n$, $k + l > 0$ and m does not divide $\binom{k+l}{k}$. Then $TC(L_m^{2n+1}) \geq 2(k + l + 1)$.*

Proof By Künneth formulas, the cross product homomorphism

$$\times : H^*(L_m^{2n+1}; \mathbb{Z}_m) \bigotimes H^*(L_m^{2n+1}; \mathbb{Z}_m) \to H^*(L_m^{2n+1} \times L_m^{2n+1}; \mathbb{Z}_m)$$

is a ring isomorphism (see [15], Theorem 3.16, p. 219). Therefore $H^*(L_m^{2n+1} \times L_m^{2n+1}; \mathbb{Z}_m)$ is a free $\mathbb{Z}_m$-module with basis the elements $x^{s_1} y^{r_1} \times x^{s_2} y^{r_2}$, where $s_1, s_2 \in \{0, 1\}$ and $0 \leq r_1, r_2 \leq n$. Also, the excess of the Bockstein stable cohomology operation is 1, hence Theorem 2.2.3 implies that $wgt_\pi(\overline{y}) \geq 2$. We have

$$(\overline{y})^{k+l} = (1 \times y - y \times 1)^{k+l} = 1 \times y^{k+l} + \cdots + (-1)^k \binom{k+l}{k} y^k \times y^l + \cdots + (-1)^{k+l} y^{k+l} \times 1$$

and $\overline{x} \smile (\overline{y})^{k+l} = A - B$, where A and B are given by

$$A = 1 \times (xy^{k+l}) + \cdots + (-1)^k \binom{k+l}{k} y^k \times (xy^l) + \cdots + (-1)^{k+l} y^{k+l} \times x$$

and

$$B = x \times y^{k+l} + \cdots + (-1)^k \binom{k+l}{k} (xy^k) \times y^l + \cdots + (-1)^{k+l}(xy^{k+l}) \times 1.$$

Since $\binom{k+l}{k}$ is not divisible by m, it follows that the terms $(-1)^k\binom{k+l}{k}y^k \times (xy^l)$ and $(-1)^k\binom{k+l}{k}y^k \times (xy^l)$ are not 0, and thus $\overline{x} \smile (\overline{y})^{k+l} \neq 0$. Therefore by Proposition 2.1.7 $TC(L_m^{2n+1}) > 2(k+l)+1$. ∎

Corollary 2.3.3 *If $m \geq 3$ then $TC(L_m^3) = 6$. In the case $m = 2$, $TC(L_2^3) = 4$.*

Proof By Proposition 2.3.1, $TC(L_m^3) \leq 6$, and by Proposition 2.3.2 with $k = l = 1$ we obtain $TC(L_m^3) \geq 6$ for $m > 2$. In the case $m = 2$ we observe that $L_2^3 = \mathbb{R}P^3$. ∎

Let p be a prime. We denote the p-adic representation of a positive integer n by $n = n_0 + n_1 p + \cdots + n_k p^k = n_0 n_1 \ldots n_k$, where $0 \leq n_i < p$, $n_k \neq 0$. Also, we set $n_i = 0$ for $i > k$.

Lemma 2.3.4 *Let p be a prime and let m, n be positive integers with p-adic representations $n = n_0 n_1 \ldots$ and $m = m_0 m_1 \ldots$, respectively. The maximal value of $k \geq 0$ such that p^k divides $\binom{n+m}{n}$ equals to the number of the values of $i \geq 0$ for which either (a) $n_i + m_i \geq p$ or (b) there exists $r \geq 0$ such that $n_i + m_i = n_{i-1} + m_{i-1} = \cdots = n_{i-r} + m_{i-r} = p-1$ and $n_{i-r-1} + m_{i-r-1} \geq p$.*

Proof We first show that for the integer $n = n_0 n_1 \ldots n_k$ the maximal integer $l \geq 0$ such that p^l divides $n!$ is equal to $l = [\frac{n}{p}] + \cdots + [\frac{n}{p^k}]$. We have $[\frac{n}{p}] = n_1 \ldots n_k, [\frac{n}{p^2}] = n_2 \ldots n_k, \; \ldots \; , [\frac{n}{p^k}] = n_k$ and we observe that each factor of $n!$ that the first term at its p-adic representation is not 0, is relative prime to p and each other factor is divisible by p. Therefore

$$n! = p^{[\frac{n}{p}]}(n_1 \ldots n_k)!C = p^{[\frac{n}{p}]+[\frac{n}{p^2}]}(n_2 \ldots n_k)!C = \cdots = p^{[\frac{n}{p}]+\cdots+[\frac{n}{p^k}]} n_k! C = p^l C$$

(Here C denotes an integer relative prime to p).

Let p^s be the maximal power of p that divides $\binom{n+m}{n}$. Then $p^{-s}\binom{n+m}{n} = \frac{(n+m)!}{p^s n! m!}$ is an integer relative prime to p and the maximal power of p that divides the numerator must be equal to the maximal power of p that divides the denominator, that is

$$s + \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right] + \sum_{i=1}^{\infty} \left[\frac{m}{p^i}\right] = \sum_{i=1}^{\infty} \left[\frac{n+m}{p^i}\right],$$

hence

$$s = \sum_{i=1}^{\infty} \left(\left[\frac{n+m}{p^i}\right] - \left[\frac{n}{p^i}\right] - \left[\frac{m}{p^i}\right] \right) = \sum_{i=1}^{\infty} \left(\left\{\frac{n}{p^i}\right\} + \left\{\frac{m}{p^i}\right\} - \left\{\frac{n+m}{p^i}\right\} \right),$$

where $\{x\} = x - [x]$ is the fractional part of x. Each term of this sum is 0 or 1, because it is an integer and belongs to $(-1, 2)$. Also, a term of the sum is 1 if and only if the number $\{\frac{n}{p^i}\} + \{\frac{m}{p^i}\} = \frac{n_0+m_0}{p^i} + \frac{n_1+m_1}{p^{i-1}} + \cdots + \frac{n_{i-1}+m_{i-1}}{p}$ is at least 1. It suffices to show that the second statement is true if and only if at least one of the properties (a) and (b) is true for the integer $i-1$. If (a) or (b) is true for $i-1$ then it is obvious that $\{\frac{n}{p^i}\}+\{\frac{m}{p^i}\}\geq 1$. For the converse, we suppose that this number is ≥ 1 and that the properties (a) and (b) are false and we will arrive at a contradiction. Then $n_{i-1}+m_{i-1}=p-1$, otherwise since $n_i+m_i\leq 2p-2$ we take

$$\frac{n_0+m_0}{p^i}+\frac{n_1+m_1}{p^{i-1}}+\cdots+\frac{n_{i-1}+m_{i-1}}{p}<\frac{p-2}{p}+(2p-2)\Big(\frac{1}{p^2}+\frac{1}{p^3}+\cdots\Big)=1.$$

Similarly, we have $n_{i-1}+m_{i-1}=\cdots=n_1+m_1=n_0+m_0=p-1$, hence

$$\frac{n_0+m_0}{p^i}+\frac{n_1+m_1}{p^{i-1}}+\cdots+\frac{n_{i-1}+m_{i-1}}{p}<(p-1)\Big(\frac{1}{p}+\frac{1}{p^2}+\cdots\Big)=1,$$

which contradicts to the hypothesis that the above sum is ≥ 1. ∎

Let p be an odd prime and let n be a positive integer with p-adic repesentation $n=n_0n_1\ldots$. We define a sequence $r_0(n),r_1(n),\ldots,r_i(n),\ldots$ of nonegative integers as follows: If $2n_i<p$ then $r_i(n)=0$ and if $2n_i\geq p$ then $r_i(n)$ is the maximal value of $k\geq 1$ such that $n_{i+1}=n_{i+2}=\cdots=n_{i+k-1}=(p-1)/2$. Also we set

$$\alpha_p(n)=\sum_{i=0}^{\infty}r_i(n).$$

In the case $p=2$, we define $\alpha_2(n)$ to be the number of ones in the dyadic representation of n.

Proposition 2.3.5 *If p is a prime and $p^{\alpha_p(n)+1}$ divides m then $TC(L_m^{2n+1})=4n+2$.*

Proof Since the number $\alpha_p(n)$ counts the integers $i\geq 0$ for which $2n_i\geq p$ and the integers $i\geq 0$ for which $n_i=n_{i-1}=\cdots=n_{i-r}=(p-1)/2$ and $2n_{i-r-1}\geq p$, it follows by Lemma 2.3.4 that $p^{\alpha_p(n)}$ is the maximal power of p that divides $\binom{2n}{n}$. Hence m does not divide $\binom{2n}{n}$ and by Propositions 2.3.1 and 2.3.2 we have $TC(L_m^{2n+1})=4n+2$. ∎

Corollary 2.3.6 *If p is an odd prime divisor of m and $n_i\leq(p-1)/2$ for all i, where $n=n_0n_1\ldots$ is the p-adic representation of n, then $TC(L_m^{2n+1})=4n+2$.*

Corollary 2.3.7 *If $k\geq 1$ and $\alpha_2(n)\leq k-1$ then $TC(L_{2^k}^{2n+1})=4n+2$.*

Chapter 3

Topological Complexity of real projective spaces

3.1 An upper bound for $TC(\mathbb{R}P^n)$

In this section we will give an upper bound of $TC(\mathbb{R}P^n)$, $n > 1$, connected to the minimal dimension of $\mathbb{R}^k$ in which $\mathbb{R}P^n$ can be immersed. As we know from Chapter I, we have

$$n + 1 \leq TC(\mathbb{R}P^n) \leq 2n + 1.$$

Especially, if n is a power of 2, then the last example of section 2.1 gives $TC(\mathbb{R}P^n) = 2n$ or $2n + 1$. We will prove that the former holds.

We regard a point of $\mathbb{R}P^n$ as a line in $\mathbb{R}^{n+1}$ which passes from the origin, i.e. as a 1-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Theorem 3.1.1 *If an immersion $i : \mathbb{R}P^n \to \mathbb{R}^k$ exists then $TC(\mathbb{R}P^n) \leq k + 1$.*

Proof Projecting orthogonally the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ on $\mathbb{R}^k$, where $x_1, \ldots, x_k$ are the standard coordinates of $\mathbb{R}^k$, we define k smooth vector fields $v_1, \ldots, v_k$ on $\mathbb{R}P^n$, i.e. $v_j(A)$ is the orthogonal projection of $\frac{\partial}{\partial x_j}(i(A))$ onto $T_A\mathbb{R}P^n$ for all $A \in \mathbb{R}P^n$. It is obvious that the tangent vectors $v_1(A), \ldots, v_k(A)$ span the tangent space $T_A\mathbb{R}P^n$ for all $A \in \mathbb{R}P^n$.

Recall that the tangent space $T_A\mathbb{R}P^n$ is naturally identified with the orthogonal complement of the line A in $\mathbb{R}^{n+1}$.Therefore, each nonzero tangent vector $v \in T_A\mathbb{R}P^n$, where $A \in \mathbb{R}P^n$, is assosiated with a line $\hat{v}$ in $\mathbb{R}^{n+1}$ which passes through the origin and it is orthogonal to A. Also, the vector v induces an orientation of the 2-dimensional linear subspace of $\mathbb{R}^{n+1}$ that contains the lines A and $\hat{v}$.

We define the subsets $U_0, U_1, \ldots, U_k \subset \mathbb{R}P^n \times \mathbb{R}P^n$ by setting $(A, B) \in U_0$ if and only if the lines A and B make an acute angle and, for $j = 1, \ldots, k$, $(A, B) \in U_j$ if and only if $v_j(A) \neq 0$ and the lines B and $\widehat{v_j(A)}$ make an acute angle. The map $S^n \times S^n \to \mathbb{R}, (x, y) \mapsto |< x, y >|$, where $<,>$ is the usual inner product, factors through a map $\phi : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}$ under the quotient map $p \times p : S^n \times S^n \to \mathbb{R}P^n \times \mathbb{R}P^n$ and $U_0 = \phi^{-1}(\mathbb{R} - \{0\})$, hence the set U_0 is open in $\mathbb{R}P^n \times \mathbb{R}P^n$. Each set U_j, $j = 1, \ldots, k$, is also open, since it is the inverse image of the open set U_0 under the map $q_j \times 1 : O_j \times \mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}P^n$, where $O_j = \{A \in \mathbb{R}P^n | v_j(A) \neq 0\}$ and

$q_j : O_j \to \mathbb{R}P^n$ is the map $q_j(A) = \widehat{v_j(A)}$. In addition, we will show that the sets U_j, $0 \le j \le k$, cover the space $\mathbb{R}P^n \times \mathbb{R}P^n$. Let $(A, B) \in \mathbb{R}P^n \times \mathbb{R}P^n$. Since the vectors $v_j(A)$, $1 \le j \le k$, span the tangent space $T_A\mathbb{R}P^n$, the lines A and $\widehat{v_j(A)}$ for all $j = 1, \ldots, k$ with $v_j(A) \neq 0$ span the space $\mathbb{R}^{n+1}$. We choose nonzero vectors $\overline{A}, \overline{B}$ and $\overline{v_j(A)}$ in the lines A, B and $\widehat{v_j(A)}$, respectively. There are scalars λ and λ_j such that

$$\overline{B} = \lambda\overline{A} + \sum_j \lambda_j \overline{v_j(A)},$$

thus

$$0 < |\overline{B}|^2 =< \overline{B}, \overline{B} >= \lambda < \overline{A}, \overline{B} > + \sum_j \lambda_j < \overline{v_j(A)}, \overline{B} > .$$

Hence either $< \overline{A}, \overline{B} > \neq 0$ or $< \overline{v_j(A)}, \overline{B} > \neq 0$ for some j. This means that at least one of the sets U_j contains the pair (A, B).

Now, we will construct a continuous section $s_j : U_j \to P\mathbb{R}P^n$ of the endpoints fibration $\pi : P\mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}P^n$ over each set U_j. If $(A, B) \in U_0$ then we define $s_0(A, B)$ to be the path in $\mathbb{R}P^n$ which follows from rotation of the line A towards the line B with constant velocity in the 2-plane containing the lines A and B in the direction of the acute angle. For $j = 1, \ldots, k$ and $A \in \mathbb{R}P^n$ with $v_j(A) \neq 0$, we define $R_j(A)$ to be the path in $\mathbb{R}P^n$ which follows from rotation of the line A towards to the line $\widehat{v_j(A)}$ with constant velocity in the 2-plane containing the lines A and $\widehat{v_j(A)}$ in the direction of the orientation determined by the tangent vector $v_j(A)$. We define s_j by setting $s_j(A, B) = R_j(A) * s_0(\widehat{v_j(A)}, B)$ for all pairs $(A, B) \in U_j$. Therefore, $TC(\mathbb{R}P^n) \le k + 1$. ∎

The following corollary is an application of the Whitney theorem, which says that every C^∞-manifold of dimension $n > 1$ can be immersed into $\mathbb{R}^{2n-1}$ (see [1], Theorem 3.8, p. 86).

Corollary 3.1.2 *$TC(\mathbb{R}P^n) \le 2n$ for all n.*

In section 2.1 we have shown that $TC(\mathbb{R}P^n) \ge 2^r$ whenever $n \ge 2^{r-1}$. So we obtain the following.

Corollary 3.1.3 *If n is a power of 2 then $TC(\mathbb{R}P^n) = 2n$.*

3.2 Nonsingular maps

Another upper bound for $TC(\mathbb{R}P^n)$ can be obtained from the existece of a certain kind of maps.

Definition 3.2.1 *A continuous map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ such that*
(i) $f(\alpha x, \beta y) = \alpha\beta f(x, y)$ for all $\alpha, \beta \in \mathbb{R}$ and all $x, y \in \mathbb{R}^n$ and
(ii) $f(x, y) \neq 0$ for all $x, y \in \mathbb{R}^n - \{0\}$
is called a nonsingular map.

Example: We will construct a nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n-1}$. We recall that there is an infinite number of vectors in a n-dimensional vector space such that any n of them are linearly independent. Indeed, given vectors $v_1, \dots, v_k$, $k \geq n$, in the Euclidean space $\mathbb{R}^n$ such that any n of them are linearly independent, we may find a vector w which does not belong to the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors $v_{i_1}, \dots, v_{i_{n-1}}$ for all $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq k$. This is true, because a $(n-1)$-dimensional suspace V of $\mathbb{R}^n$ has measure 0 in $\mathbb{R}^n$ in the sence that there is a sequence of rectangles whose union covers V and of total length less than a given positive number, hence, the space $\mathbb{R}^n$ cannot be covered by a finite number of $(n-1)$-dimensional subspaces. Now, we consider $2n-1$ linear transformations $\alpha_1, \alpha_2, \dots, \alpha_{2n-1} : \mathbb{R}^n \to \mathbb{R}$ such that any n of them are linearly independent in the dual space of $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$ we define $f(x, y)$ to be the point of $\mathbb{R}^{2n-1}$ whose j-th coordinate is $\alpha_j(x)\alpha_j(y)$. The property (ii) of the definition of the nonsingular map follows from the fact that for $x \in \mathbb{R}^n - \{0\}$ the number of the real numbers $\alpha_1(x), \dots, \alpha_{2n-1}(x)$ which are nonzero is at least n. In fact, if this is not true, then n of the numbers $\alpha_1(x), \dots, \alpha_{2n-1}(x)$ are zero, say $\alpha_1(x) = 0, \dots, \alpha_n(x) = 0$. Since the linear functionals $\alpha_1, \dots, \alpha_n$ span the dual space of $\mathbb{R}^n$, we have that $\alpha(x) = 0$ for every linear functional α, which is a contradiction.

Example: Nonsingular maps do not always exist. If $k < n$ then there is no nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$. Indeed, if such a map f exists then applying the Borsuk-Ulam theorem to the map $S^{n-1} \to \mathbb{R}^k \subset \mathbb{R}^{n-1}, x \mapsto f(x, y)$, where $y \in \mathbb{R}^n - \{0\}$ is fixed, we have $f(x, y) = f(-x, y)$ for some $x \in S^{n-1}$, therefore, $f(x, y) = 0$, which contradicts property (ii).

Proposition 3.2.2 *If there exists a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ such that the first coordinate of $f(x, x)$ is positive for all $x \neq 0$, then $TC(\mathbb{R}P^n) \leq k$.*

Proof Suppose that $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function such that
(i) $\phi(\lambda x, \mu y) = \lambda\mu\phi(x, y)$ for every x, $y \in \mathbb{R}^{n+1}$ and λ, $\mu \in \mathbb{R}$, and
(ii) $\phi(x, x) > 0$ for $x \neq 0$.

The set
$$V_\phi = \{(u, v) \in S^n \times S^n : \phi(u, v) > 0\}$$
is an open neighborhood of the diagonal in $S^n \times S^n$, by property (ii). On V_ϕ we can define a continuous section τ of the endpoints fibration of S^n such that $\tau(u, v)$ is the path obtained by rotating u toward v, if $u \neq v$, and is the constant path with value u, if $u = v$. More precisely, the oriented angle $0 \leq \theta \leq \pi$ from u to v is determined by $\cos\theta = \langle u, v\rangle$. Let
$$J(u, v) = \frac{v - \langle u, v\rangle u}{(1 - \langle u, v\rangle^2)^{1/2}}$$
be the unique unit vector in the plane spanned by u and v such that $(u, J(u, v))$ is an orthonormal basis which defines the same orientation with (u, v). For $0 \leq t \leq 1$, we define
$$\tau(u, v)(t) = \begin{cases} (\cos\theta t)u + (\sin\theta t)J(u, v), \text{if } u \neq v \\ u, \text{if } u = v. \end{cases}$$
Let $p : S^n \to \mathbb{R}P^n$ be the quotient map. If (u, v), $(u', v') \in V_\phi$ are such that $(p \times p)(u, v) = (p \times p)(u', v')$, then $(-u, -v) = (u', v')$, and therefore $\tau(u', v')(t) = -\tau(u, v)(t)$. This

shows that τ induces a continuous section s_ϕ of the endpoints fibration of $\mathbb{R}P^n$ on the open neighborhood $U_\phi = (p \times p)(V_\phi)$ of the diagonal in $\mathbb{R}P^n \times \mathbb{R}P^n$.

If ϕ has only property (i), then we put

$$V_\phi = \{(u,v) \in S^n \times S^n : u \neq \pm v \text{ and } \phi(u,v) > 0\}$$

and we define a continuous section τ on V_ϕ of the endpoints fibration of S^n by the same formula

$$\tau(u,v)(t) = (\cos\theta t)u + (\sin\theta t)J(u,v).$$

Again τ induces a continuous section $s_\phi : U_\phi \to P\mathbb{R}P^n$ of the endpoints fibration of $\mathbb{R}P^n$ on the open set $U_\phi = (p \times p)(V_\phi)$.

Now if we have a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ and $f = (f_1, f_2, ..., f_k)$ such that $f_1(x,x) > 0$ for all $x \neq 0$, then $\{U_{f_1}, U_{f_2}, ..., U_{f_k}\}$ is an open covering of $\mathbb{R}P^n \times \mathbb{R}P^n$ on each member of which there is a continuous section of the endpoints fibration of $\mathbb{R}P^n$. This completes the proof. ∎

We will construct a nonsingular map $f : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$ such that the first coordinate of $f(x,x)$ is positive for $x \neq 0$. For this purpose, we identify the set $\mathbb{R}^8$ with the set $\mathbb{O}$ of octonions, that is, we write an element $(t,x,y,z,s,u,v,w) \in \mathbb{R}^8$ in the form $t + ix + jy + kz + ls + mu + nv + ow$, where i,j,k,l,m,n,o are generalized square roots of -1. The multiplication of i,j,k,l,m,n,o is defined by the following table.

$$\begin{array}{c c}
 & \begin{array}{ccccccc} i & j & k & l & m & n & o \end{array} \\
\begin{array}{c} i \\ j \\ k \\ l \\ m \\ n \\ o \end{array} & \begin{pmatrix}
-1 & k & -j & m & -l & -o & n \\
-k & -1 & i & n & o & -l & -m \\
j & -i & -1 & o & -n & m & -l \\
-m & -n & -o & -1 & i & j & k \\
l & -o & n & -i & -1 & -k & j \\
o & l & -m & -j & k & -1 & -i \\
-n & m & l & -k & -j & i & -1
\end{pmatrix}
\end{array}$$

Also, we identify the set $\mathbb{R}^4$ of quadraples (t,x,y,z) of real numbers with the set $\mathbb{H}$ of quaternions $t + ix + jy + kz$, where i,j,k are generalized square roots of -1 with $ij = k = -ji, jk = i = -kj, ki = j = -ik$. Octonions are written in the form $t+ix+jy+kz+ls+mu+nv+ow = Q+Rl$, where Q and R are the quaternions $Q = t+ix+jy+kz$ and $R = s + iu + jv + kw$. In addition, we define the conjugate of a quaternion and an octonion by

$$\overline{t + ix + jy + kz} = t - ix - jy - kz$$

and

$$\overline{t + ix + jy + kz + ls + mu + nv + ow} = t - ix - jy - kz - ls - mu - nv - ow.$$

We define now a nonsingular map $f : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$ by setting $f(A,B) = A\overline{B}$. Hence, Proposition 3.2.2 implies that $TC(\mathbb{R}P^7) \leq 8$. Also, we obtain by Proposition 1.2.11 and the fact that $cat\mathbb{R}P^n = n+1$ that $TC(\mathbb{R}P^n) \geq n+1$, so we have the following corollary.

Corollary 3.2.3 $TC(\mathbb{R}P^7) = 8$

Remark: In analogy with octonions we can use complex numbers and quaternions to define nonsingular maps $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$. In the same way, we define $g(z,w) = z\overline{w}$ for complex numbers z,w and $h(A,B) = A\overline{B}$ for quaternions A,B.

3.3 Topological Complexity of $\mathbb{R}P^n$ and the immersion problem

The calculation of $TC(\mathbb{R}P^n)$ for all $n > 1$, by finding a general formula as in the case of $TC(\mathbb{C}P^n)$, turns out to be a very difficult problem. Actually, the main results of the last two sections can be reversed. Firstly, the following is proved in [13].

Theorem 3.3.1 *The topological complexity of $\mathbb{R}P^n$ is equal to the smallest positive integer k such that there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$.*

Secondly, a converse of Theorem 3.1.1 holds in the following form.

Theorem 3.3.2 *For $n \neq 1,\ 3,\ 7$, the topological complexity of $\mathbb{R}P^n$ is equal to the smallest positive integer k such that $\mathbb{R}P^n$ can be immersed into $\mathbb{R}^{k-1}$.*

Thus, the problem of computing $TC(\mathbb{R}P^n)$ is equivalent to the immersion problem for real projective spaces. This is a classical problem in Topology, on which a lot of work has been done starting with results of H. Hopf and H. Whitney around 1940, but nevertheless remains unsolved in general. By now many important immersion and nonimmersion results for $\mathbb{R}P^n$ have been proved. The proof of Theorem 3.3.2 is based on some of them. We refer to [4] for a historical survey.

¿From the above follows that $TC(\mathbb{R}P^n)$ is a nondecreasing function of n, i.e. $n \leq m$ implies that $TC(\mathbb{R}P^n) \leq TC(\mathbb{R}P^m)$. Another proof of this can be found in [14]. It would be desirable to have a simple direct proof of this.

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