MASTER'S THESIS

BLASCHKE'S CONJECTURE AND WIEDERSEHEN MANIFOLDS

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2013

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Preface

The injectivity radius inj$M$ of a complete Riemannian manifold M is always not greater than its diameter diam$M$. A natural question to ask is what are the compact Riemannian manifolds M for which inj$M$ = diam$M$? In other words, what are the compact Riemannian manifolds M on which every unit speed geodesic from any given point of M hits its cut locus at distance diam$M$? It is obvious that equality of the injectivity radius and the diameter is a very strong condition. Oddly, however, the above question has not been answered yet, except in special cases. The only known examples are the (unit) n-sphere S^n, $n \geq 2$, the real projective n-space $\mathbb{R}P^n$, $n \geq 2$, the complex projective n-space $\mathbb{C}P^n$, $n \geq 1$, the quaternionic projective n-space $\mathbb{H}P^n$, $n \geq 1$, and the Cayley projective plane $\mathbb{C}aP^2$, with their standard metrics. These are the compact rank one symmetric spaces (known as CROSS for brevity). The condition that there exists a point $x \in M$ such that all geodesics emanating from x hit its cut locus at constant distance is not strong and one can expect that it has only topological implications. For example, every Riemannian metric of revolution on the 2-sphere S^2 has a point (the "north pole") whose cut locus is a singleton (the "south pole") and has this property. It has been proved by R. Bott [4] and H. Samelson [16] that such a Riemannian manifold has the integral cohomology ring of a CROSS.

In particular, the Riemannian manifolds on which there exists a point with cut locus a singleton are called stigmatic and are important in geometrical optics. We require that our metric is stigmatic at every point. A complete Riemannian n-manifold M is called Wiedersehen if there exists $L > 0$ such that the cut locus of every point of M is a singleton at distance L. Normalizing, we can always take $L = \pi$. This term is originally due to W. Blaschke who in the 1921 edition of his book "Vorlesungen über Differentialgeometrie", Springer-Verlag, conjectured that a "Wiedersehenfläche" is a 2-sphere with the usual Euclidean metric (up to a constant factor). A wrong "proof" of the conjecture due to K. Reidemeister appeared in the second 1924 edition of the book. The problem remained open until 1963 when L. Green proved that the standard round 2-sphere is indeed the only "Wiedersehenfläche".

A connected compact Riemannian n-manifold M, $n \geq 2$, such that inj$M$ = diam$M$ is called a Blaschke manifold. If L = diam$M$, it follows from results of H. Nakagawa and K. Shiohama (see [12] and [13]) that at each point $x \in M$ the exponential map $\exp_x$ is a smooth embedding when restricted on the open ball $S(0, L)$ of radius L centered at 0 in the tangent space T_xM and $\exp_x|_{\partial S(0,L)}$ is a smooth r-sphere fiber bundle for some $r \geq 0$. In the case of the standard n-sphere we have $L = \pi$ and $r = n - 1$, and in the other cases of CROSS $L = \dfrac{\pi}{2}$ and $r = 0, 1, 3, 7$, respectively. Another nice and important feature of a Blaschke manifold is that it has periodic geodesic flow and all of its geodesics are simple, closed and of the same length $2L$.

The Blaschke conjecture is simply stated as follows.

Blaschke conjecture. A Blaschke manifold is isometric (up to a constant factor) to a CROSS.

The Bott-Samelson Theorem reduces the effort to answer the conjecture to a case by case verification and the CROSSes (with their canonical metrics) are the models for Blaschke manifolds. Thus the conjecture can be restated as follows: The only Riemannian metric on a manifold having the cohomology ring of a CROSS is a scalar multiple of the standard metric on the corresponding CROSS. The first natural question to ask is whether we can at least exclude fake CROSSes, namely the conjecture is true topologically.

Topological Blaschke conjecture. A Blaschke Riemannian manifold is homeomorphic (or diffeomorphic) to a CROSS.

This is elementary for spheres and projective spaces, but has not yet been fully established in the other cases. Every Blaschke manifold gives rise to a fibration of a sphere in T_pM by great subspheres as we show in chapter 2. This fibration encodes the diffeomorphism type of the manifold and all the work done on the topological Blaschke conjecture consists of proving that such a fibration is equivalent up to homeomorphism or diffeomorphism to the corresponding Hopf fibration (see [9]). In a series of papers this is done for a fibration by great circles proving a diffeomorphism and deriving that a Blaschke manifold modelled on $\mathbb{C}P^n$ is diffeomorphic to $\mathbb{C}P^n$ (see [17], [23] and [11]). The case of $\mathbb{H}P^n$ was handled by H. Sato and T. Mizutani. H. Sato proves in [19] that such a Blaschke manifold is homotopy equivalent to $\mathbb{H}P^n$, whereas for $n = 2$ he and T. Mizutani obtain in [18] a $PL-$homeomorphism. Furthermore, he states without proof that the $K-$theory of $\mathbb{H}P^n$ and the homotopy equivalence ensure the existence of a homeomorphism. The $\mathbb{C}aP^2$ case was settled by H.Gluck, F.Warner and C.T. Yang in [9] who show homeomoprhism to the model.

The next step towards the resolution of the Blaschke conjecture, which is crucial for the proof of the spherical case, was initiated by the work of Weinstein [21]. Using the fact that a Blaschke manifold M has closed geodesics of length 2diam$M$, Weinstein constructed the manifold of geodesics CM (see chapter 3) and computed the volume of M from the integral cohomology of CM, finally proving that it is an integral multiple of the volume of the standard model. For even dimensional Blaschke manifolds modelled on S^n, A. Weinstein showed that $\mathrm{vol}(CM) = \mathrm{vol}(S^n)$ and shortly after, C.T. Yang proved in [22] the odd dimensional case. This answered the spherical case of the so called Weak Blasche Conjecture, which can be stated as follows.

Weak Blaschke Conjecture. The volume of a Blaschke manifold equals the volume of its model CROSS of the same diameter.

A. Reznikov proved this conjecture for $\mathbb{H}P^n$, $\mathbb{C}aP^2$ and finally $\mathbb{C}P^n$ (see [14]). However, contrary to the spherical case, it is assumed that the Blaschke manifold is homeomorphic to its model. Therefore it is a proof of the above conjecture for $\mathbb{H}P^n$, $n > 2$, modulo the Topological Blaschke conjecture.

The Blaschke Conjecture was finally established for spheres by M. Berger using an analytical inequality of J. Kazdan.

Berger's isoembolic inequality. The volume of a Riemannian manifold is greater or equal to the volume of the standard sphere of the same dimension and diameter. Equality occurs only in the case of isometry to the standard sphere.

Clearly the above result, proved in [1], combined with the work of A. Weinstein and C.T. Yang for the Weak Blascke Conjecture settles the Blaschke conjecture for spheres. There are no known curvature restrictions for Blaschke manifolds, however assuming certain curvature bounds it is possible to show the Blaschke Conjecture, as it is done in [20]. Specifically it is derived that there exist many totally geodesic Blaschke submanifolds and the conjecture is proven by a generalization of the argument of Berger. In [2] and [10] the authors proved the Blaschke Conjecture under additional assumptions concerning the behavior of geodesics.

The main purpose of this work is to give a detailed account of the proof of the Blaschke conjecture in the spherical case, namely for Wiedersehen manifolds.

This work has the following structure. Chapter 1 is mostly introductory and is devoted to the presentation of notions and tools that are basic in the sequel. The Berger-Kazdan isoembolic inequality is proved in detail, since it is an important ingredient of the proof. In Chapter 2 gives an account of the general properties of Blaschke manifolds. Chapter 3 is the core of the proof. The material is a detailed presentation of the papers of A. Weinstein [21] and C.T. Yang [22], which calculate the volume of Riemannian manifolds with closed geodesics and show that Wiedersehen manifolds, i.e. Blaschke manifolds diffeomorphic to spheres, have the right volume. Combination of this with the Berger-Kazdan inequality yields the theorem.

Contents

Chapter 1

Introduction

1.1 The cut locus and the injectivity radius

In this introductory section and the next two we shall recall some basic notions and facts which are necessary in the sequel. Proofs can be found in standard textbooks on Riemannian Geometry such as [7], [8] and [15], which we refer to and are included in the bibliography.

Let (M, g) be a connected Riemannian manifold. For p, $q \in M$ we denote by $d(p, q)$ the distance of p and q induced by the Riemannian metric g, which is by definition the infimum of lengths of piecewise smooth curves from p to q. The function d is a distance on M whose induced topology coincides with the original manifold topology. We will denote by $B(p, r)$ or $B_r(p)$ the open ball on M with respect to d of radius $r > 0$ centered at $p \in M$.

The injectivity radius $Inj(p)$ of M at the point $p \in M$ is the supremum of all $r > 0$ such that the exponential map $\exp_p$ at p maps the open ball $B(0, r)$ in T_pM of radius r centered at $0 \in T_pM$ diffeomorphically onto its image. The fact that $0 < Inj(p) \leq +\infty$ is a consequence of the Inverse Function Theorem. The injectivity radius of M is by definition $Inj(M) = \inf\{Inj(p) : p \in M\}$. If M is compact this is positive and finite. In case M is not compact, it may happen that $Inj(M) = 0$ or $Inj(M) = +\infty$.

A unit speed curve $\gamma : [a, b] \to M$ is a segment if its length equals $d(\gamma(a), \gamma(b))$. Segments are necessarily geodesics of the Riemannian metric, since geodesics locally minimize length and do this uniquely. We shall denote by $seg(p, q)$ the set of all segments from p to q. For example, if $0 < s < Inj(p)$ and $v \in T_pM$ is a unit vector, the geodesic $\gamma_v(t) = \exp_p(tv)$, $0 < t \leq s$ is the unique element of $seg(p, \gamma_v(s))$.

If M is complete, then given any two points p, $q \in M$ the set $seg(p, q)$ is not empty, by the Hopf-Rinow Theorem. However, it may not be a singleton. A standard example is the n-sphere (of any radius) with the usual Euclidean metric. The uniqueness of segments, i.e. minimal geodesics, can be investigated through the notions of cut value and cut locus.

Let $p \in M$, $v \in T_pM$ with $\|v\| = 1$ and γ_v be the unique geodesic with initial conditions $\gamma(0) = p$ and $\dot{\gamma_v}(0) = v$, that is $\gamma_v(t) = \exp_p(tv)$. Put

$$c(v) = \sup\{t > 0 : \gamma_v(t) \text{ is defined and } d(p, \gamma_v(t)) = t\}.$$

Then $0 < c(v) \leq +\infty$ and it is called the cut value, which is the distance of p from the cut point $\gamma_v(c(v))$ along γ_v. For any $0 < s < c(v)$ the restriction of γ_v to $[0, s]$ is the unique segment from p to $\gamma_v(s)$. If M is complete and $c(v)$ is finite, then either $\gamma_v(c(v))$ is the first conjugate point to p along γ_v or $seg(p, \gamma_v(c(v)))$ is not a singleton.

Let $UM = \{v \in TM : \|v\| = 1\}$ be the unit tangent bundle of M. We have a well defined cut function $c : UM \to (0, +\infty]$ which is upper semicontinuous. However, if M is complete, then c is continuous. We have $Inj(p) = \inf\{c(v) : v \in U_pM\}$ for every $p \in M$.

Assuming that M is complete, the cut locus of p in M is the set

$$cut(p) = \{\exp_p(c(v)v) : v \in U_pM, \quad c(v) < +\infty\}.$$

This is a set of measure zero in M. Note also that for $p, q \in M$ we have that $p \in cut(q)$ if and only if $q \in cut(p)$.

For example the cut locus of a point in a Euclidean n-sphere is a singleton consisting of its antipodal point. On a cylinder in $\mathbb{R}^3$ the cut locus of any point is the opposite vertical line. In general the cut locus can be topologically complicated.

In Chapter 2 we will use repeatedly a useful property of segments called the Acute Angle Property. In order to prove it we need some preparation.

Let $p \in M$ and $f : M \to \mathbb{R}$ be the distance function from p, that is $f(q) = d(p, q)$. This is a continuous function on M, but it is not smooth. For instance, it is never differentiable at p. If $V_p = \{tv \in T_pM : v \in U_pM, \quad 0 \leq t < c(v)\}$, then $\exp_p$ maps V_p diffeomorphically onto $D_p = \exp_p(V_p)$, which is a geodesically star-shaped open neighborhood of p. For every $q \in D_p \setminus \{p\}$ there exists a unique $v_q \in U_pM$ such that $\gamma_{v_q}(t) = \exp_p(tv_q)$, $0 \leq t < f(q)$, is the unique segment from p to q. Obviously, $V_p \setminus \{0\} = \{f(q)v_q : q \in D_p \setminus \{p\}\}$. Since $f(q) = \|\exp_p^{-1}(q)\|$ for every $q \in D_p \setminus \{p\}$, it follows that f is smooth on $D_p \setminus \{p\}$.

We will calculate the gradient of f with respect to the Riemannian metric. Let $X \in T_qM$ and $\delta : (-\epsilon, \epsilon) \to D_p \setminus \{p\}$ be a smooth curve for some $\epsilon > 0$ with $\delta(0) = q$ and $\dot{\delta}(0) = X$. Let $\Gamma(s, t)$ be a small variation of γ_{v_q} so that $\Gamma(s, .)$ is the unique segment from p to $\delta(s)$, that is $\Gamma(s, t) = \gamma_{v_{\delta(s)}}(t)$. Since the variation field at q is X, from the first variation formula for the length functional $L(s) = L(\Gamma(s, .))$ we get

$$(Xf)(q) = \frac{d}{ds}\Big|_{s=0} f(\delta(s)) = L'(0) = g(X, \dot{\gamma}_{v_q}(f(q))).$$

Therefore $\nabla f(q) = \dot{\gamma}_{v_q}(f(q))$.

Proposition 1.1.1 (Acute Angle Property) *Let p, $q \in M$ with $d(p, q) = l > 0$ and let $\gamma : [0, l] \to M$ to be a segment from p to q. Let δ be a smooth curve with $\delta(0) = q$ and such that $g(\dot{\delta}(0), \dot{\gamma}(l)) < 0$. Then $d(p, \delta(s)) < d(p, q)$ for small enough $s > 0$.*

Proof We take any point on γ, for instance the middle point $x = \gamma(\frac{l}{2})$ and consider the function $f : M \to \mathbb{R}$ defined by $f(y) = d(x, y)$. Then $q \notin cut(x)$ and the above calculation of $(\nabla f)(q) = \dot{\gamma}(l)$ shows that

$$\frac{d}{ds}\Big|_{s=0} f(\delta(s)) = g(\dot{\delta}(0), \dot{\gamma}(l)).$$

Therefore, our assumption means that $f(\delta(s)) < \frac{l}{2}$ for $s > 0$ small enough. The triangle inequality now gives

$$d(\delta(s), p) \leq d(\delta(s), x) + d(p, x) < \frac{l}{2} + \frac{l}{2} = d(p, q). \qquad \blacksquare$$

1.2 The geodesic flow

In this section we assume that (M, g) is a complete Riemannian n-manifold. We denote by $\eta \in \Omega^1(TM)$ the pullback of the canonical Liouville 1-form on T^*M by the musical isomorphism $TM \to T^*M$. In other words η is defined by

$$\eta(\xi) = g(\tau_* \xi, v)$$

with $\tau : TM \to M$ is the tangent bundle projection, $v \in TM$ and $\xi \in T_vTM$. Since the pullback commutes with the exterior differentiation, we see that $d\eta$ is a symplectic 2-form on TM. In local coordinates $(x^1, ..., x^n, v^1, ..., v^n)$ on TM, we have

$$\begin{aligned} \eta &= \sum_{i,j=1}^{n} g_{ij} v^i dx^j \\ d\eta &= \sum_{i,j=1}^{n} g_{ij} dv^i \wedge dx^j + \sum_{i,j,k=1}^{n} \frac{\partial g_{ij}}{\partial x^k} v^i dx^k \wedge dx^j \end{aligned}$$

The energy function $E : TM \to \mathbb{R}$ is defined by

$$E(v) = \frac{1}{2} g(v, v).$$

The Liouville vector field $Y \in \mathfrak{X}(TM)$ is defined to be the infinitesimal generator of the flow of dilations in TM, that is $\psi^t(x, v) = (x, e^t v)$.

The geodesic vector field $Z \in \mathfrak{X}(TM)$ can be defined as the Hamiltonian vector field of the energy, which is the unique solution of the equation

$$i_Z d\eta = -dE.$$

We denote by $\zeta^t : TM \to TM$, $t \in \mathbb{R}$, the geodesic flow, i.e the flow of Z.

Next we consider the tangent bundle as a differentiable manifold. Its tangent bundle $\mathcal{T} : TTM \to TM$ has a natural subbundle, consisting of "vertical" vectors. We set $VTM := ker \tau_* \leq TTM$, where $\tau : TM \to M$ is the bundle projection. The bundle

$$\mathcal{T} : VTM \to TM$$

is called the vertical subbundle of TTM.

There are many equivalent formulations for the concept of a connection. Next, we are going to define the connector of the Levi-Civita connection of the Riemannian manifold (M, g). The connector of the Levi-Civita connection is the map

$$K : TTM \to TM$$

defined as follows. Let $z : (-\varepsilon, \varepsilon) \to TM$, $\varepsilon > 0$, be a smooth curve such that $z(0) = v$ and $\dot{z}(0) = \xi$, where $v \in TM$, $\xi \in T_vTM$. Let $\gamma = \tau \circ z : (-\varepsilon, \varepsilon) \to M$ be the projected curve. There exists a smooth vector field X along γ, such that $z(t) = (\gamma(t), X(t))$.

We define

$$K_v(\xi) = \frac{DX}{dt}(0)$$

where $\dfrac{D}{dt}$ is the covariant derivative along γ with respect to the Levi-Civita connection.

We consider the exact sequence

$$0 \to VTM \to TTM \overset{\tau_*}{\to} TM \to 0 \tag{1.1}$$

Noting that K is a natural isomorphism when restricted to VTM, we can consider the connector as a projection $K : TTM \to VTM$. Therefore, setting $HTM := kerK$ we have a splitting of the above exact sequence

$$TTM = HTM \oplus VTM \tag{1.2}$$

and there is a fiberwise isomorphism (τ_*, K) which is shown in the following diagram:

$$\begin{tikzcd} TTM \arrow[r, "{(\tau_*, K)}"] \arrow[d, "\mathcal{T}"'] & TM \oplus TM \arrow[d, "\tau \oplus \tau"] \\ TM \arrow[r, "\tau"'] & M \end{tikzcd}$$

Now, if $v, w \in TM$, there is a unique vector $v^H \in H_wTM$ such that $\tau_* v^H = v$. This is called the horizontal lift of v at $w \in TM$.

Likewise, we denote by v^V the preimage of v with respect to the natural isomorphism

$$V_wTM \to T_{\tau(w)}M$$

This is the vertical lift of v at $w \in TM$.

We now equip TM with a Riemannian metric compatible with the decomposition of its tangent bundle into horizontal and vertical subbundles. We define the Sasaki metric $\overline{g}$ on TM by

$$\overline{g}(\xi_1, \xi_2) = g(\tau_*\xi_1, \tau_*\xi_2) + g(K\xi_1, K\xi_2)$$

for ξ_1, ξ_2 tangent vectors at $v \in TM$.

We observe that the bundle projection $\tau : TM \to M$ becomes a Riemannian submersion, when we equip TM with the above metric. Moreover, the bundles HTM and VTM are orthogonal with respect to this metric. Specifically, let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of T_pM, where $p = \tau(w)$. Then, the basis $\{e_1^H, e_1^V, \ldots, e_n^H, e_n^V\}$ of T_wTM is orthogonal.

Another useful observation is that the geodesic vector field $Z : TM \to TTM$ has a very simple expression. If γ_v denotes the unique geodesic through $\gamma_v(0)$ with initial velocity $\dot{\gamma}_v(0) = v \in TM$, then

$$Z(v) = \left.\frac{d}{dt}\right|_{t=0} (\gamma_v(t), \dot{\gamma}_v(t))$$

Recall that $\dot{\gamma}_v(t)$ is the parallel translation of $\dot{\gamma}_v(0)$ along γ_v. Therefore, by definition

$$KZ(v) = \left.\frac{D}{dt}\right|_{t=0} \dot{\gamma}_v(t) = 0$$

and Z is thus horizontal. Moreover

$$\tau_* Z(0) = \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = \dot{\gamma}_v(0) = v$$

and therefore $Z(v) = v^H$

On the other hand, we observe that $KY(v) = v$ and $\tau_* Y(v) = 0$. Therefore, the Liouville vector field is $Y(v) = v^V$.

The fiberwise isomorphism $(\tau_*, K) : TTM \overset{\simeq}{\to} TM \oplus TM$ allows the introduction of an almost complex structure J on TM, namely the bundle isomorphism

$$TM \oplus TM \to TM \oplus TM \quad (v, w) \longmapsto (-w, v).$$

Explicitly, the almost complex structure J is given by:

$$J(v^H + w^V) = -w^H + v^V$$

Obviously $J^2 = -id$ and J is an isometry of the tangent bundle with respect to $\overline{g}$, because

$$\overline{g}(J\xi_1 J\xi_2) = g(\tau_* J\xi_1, \tau_* J\xi_2) + g(KJ\xi_1, KJ\xi_2) = g(-K\xi_1, -K\xi_2) + g(\tau_*\xi_1, \tau_*\xi_2) = \overline{g}(\xi_1, \xi_2).$$

One can compute that the symplectic form $d\eta \in \Omega^2(TM)$, the Sasaki metric $\overline{g}$ and the almost complex structure J are compatible, that is

$$-d\eta(\xi_1, \xi_2) = \overline{g}(J\xi_1, \xi_2)$$

which is

$$d\eta(\xi_1, \xi_2) = \overline{g}(K\xi_1, \tau_*\xi_2) - \overline{g}(K\xi_2, \tau_*\xi_1).$$

Let us now consider the case of the unit tangent bundle

$$UM = \{v \in TM \mid \|v\| = 1\} = E^{-1}(1/2)$$

for $\xi \in TUM$ we have

$$\overline{g}(Y, \xi) = \overline{g}(JZ, \xi) = -d\eta(Z, \xi) = dE(\xi) = 0$$

and therefore the Liouville vector field Y is perpendicular to UM.

Next we write $\eta|_{UM}$ or just η if no confusion can arise, to denote the restriction of the form η to UM, i.e the pullback of η by the injection $UM \hookrightarrow TM$. Our goal is to describe the canonical volume form and induced Riemannian measure on the manifold UM.

Since $d\eta \in \Omega^2(TM)$ is symplectic, $d\eta^n \in \Omega^{2n}(TM)$. We have just proved that Y is everywhere transverse to UM. We conclude that $i_Y(d\eta)^n$ is a volume form on UM. But

$$i_Y(d\eta)^n = n(i_Y d\eta) \wedge d\eta^{n-1} = n\eta \wedge d\eta^{n-1}$$

and therefore $\eta \wedge d\eta^{n-1}$ is a volume form. We have used the fact that $i_Y d\eta = \eta$. Indeed let $\xi \in T_v TM$. Then

$$(i_Y d\eta)(\xi) = -d\eta(JZ, \xi) = \overline{g}(Z, \xi) = g(v, \tau_*\xi) = \eta(\xi).$$

Hence η is a contact 1-form on UM.

Next we find a Riemannian volume form of the Sasaki metric on UM, which will of course be proportional to $\eta \wedge d\eta^{n-1}$. We select an orthonormal basis $\{v = e_1, \ldots e_n\}$

of $T_{\tau(v)}M$. Since $Y(v) = v^V$ is the unit normal to UM, we have that the basis $\{v^H = Z(v), e_2^H, e_2^V, \ldots, e_n^H, e_n^V\}$ is an orthonormal basis of T_vUM. We compute

$$\eta \wedge d\eta^{n-1}(Z, e_2^H, e_2^V, \ldots) = \eta(Z)d\eta^{n-1}(e_2^H, e_2^V, \ldots) = (n-1)! \prod_{i=2}^{n} d\eta(e_i^H, e_i^V) = (n-1)!$$

Therefore the form

$$dUM_{\overline{g}} := \frac{\eta \wedge d\eta^{n-1}}{(n-1)!}$$

is the Riemannian volume form on UM.

We denote by $d\nu_{\overline{g}}$ the corresponding measure, which is called the Liouville measure on UM. It is easy to show (see [6]) that locally

$$d\nu_{\overline{g}} = d\nu_g \otimes dS^{n-1}$$

where $d\nu_g$ denotes the Riemannian measure on (M, g) and dS^{n-1} denotes the canonical spherical Lebesgue measure on the unit sphere of T_pM for $p \in M$.

The geodesic vector field Z is tangent to UM, since $dE(Z) = -d\eta(Z, Z) = 0$. So, UM is invariant by the geodesic flow.

Proposition 1.2.1 *The contact 1-form $\eta|_{UM}$ on UM and the Liouville measure $d\nu_{\overline{g}}$ are invariant under the geodesic flow, so that*

$$(\zeta^t)^* d\nu_{\overline{g}} = d\nu_{\overline{g}}$$

Proof We have

$$\mathcal{L}_Z \eta|_{UM} = i_Z d\eta|_{UM} + d i_Z \eta|_{UM} = -dE|_{UM} + d(\eta(Z))|_{UM}$$

Now, on UM we have $\eta(Z) = 1$ and E takes the constant value $1/2$. Therefore

$$\mathcal{L}_Z \eta|_{UM} = 0.$$

We conclude now that

$$\mathcal{L}_Z dUM_{\overline{g}} = 0$$

In turn, this implies invariance of the induced measure, which is the last assertion. ∎

Let now $v \in TM$ and $\gamma_v(t) = \tau(\zeta^t v)$ be the geodesic with initial velocity $\dot{\gamma}_v(0) = v$. We are going to identify the space $\mathcal{J}_v$ of Jacobi fields along γ_v with the tangent space T_vTM. This will be useful in subsequent chapters. Let $\xi \in T_vTM$ and $z : (-\varepsilon, \varepsilon) \to TM$ be a smooth curve with $z(0) = v$ and $\dot{z}(0) = \xi$. Then

$$F(s, t) = \tau \circ \zeta^t(z(s))$$

is a variation of γ_v by geodesics and therefore, the variation vector field

$$J_\xi(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} F(s, t) = \frac{\partial F}{\partial s}(0, t)$$

is a Jacobi field along γ_v with initial conditions:

$$J_\xi(0) = \frac{\partial F}{\partial s}(0,0) = \tau_* \xi$$
$$J_\xi^{'}(0) = \frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial F}{\partial t}(0,0) = \left.\frac{D}{ds}\right|_{s=0} z(s) = K\xi$$

If we denote by $\mathcal{J}_v$ the vector space of Jacobi fields along γ_v, the map

$$j_v : T_v TM \to \mathcal{J}_v$$

defined by $j_v(\xi) = J_\xi$ is a linear isomorphism since $ker\tau_* \cap ker K = \{0\}$ and the dimension of both vector spaces is equal to $2n$.

We now restrict to UM and identify a suitable subspace of $T_v UM$ to the space of normal Jacobi fields along γ_v, a fact needed in Chapter 3.

Proposition 1.2.2 *The restriction of the map*

$$j_v : T_v UM \cap ker\eta_v \longrightarrow \mathcal{J}_v$$

with the above notation, is an isomorphism onto the space $\mathcal{J}_v^\perp$, of normal Jacobi fields along γ_v

Proof It is immediate that we can describe the tangent space to UM as

$$T_v UM = \{\xi \in T_v TM |\ \ g(K\xi, v) = 0\}$$

Moreover, from the definition of η we have

$$ker\eta_v = \{\xi \in T_v TM |\ \ g(\tau_*\xi, v) = 0\}.$$

Therefore, for $\xi \in T_v UM \cap ker\eta_v$, the initial conditions $J_\xi(0) = \tau_*\xi$ and $J_\xi^{'}(0) = K\xi$ give a Jacobi vector field normal to $v = \dot\gamma(0)$. By the standard theory of Jacobi fields, J_ξ is normal along γ_v and thus

$$j_v(T_v UM \cap ker\eta_v) \subset \mathcal{J}_v^\perp$$

The conclusion is now obvious since the dimension of both spaces is equal to $2n-2$. ∎

1.3 Integration via geodesic spherical coordinates

Let (M, g) be a complete Riemannian n-manifold. Let $p \in M$ and $r > 0$ be such that the exponential map is a diffeomorphism on $B_r(0) \subset T_pM$, the open ball of radius r centered at 0. ¿From the Theorem of Fubini, the Riemannian volume of the ball $B_r(p)$ of radius r on M centered at p is

$$\begin{aligned} volB_r(p) &= \int_{x\in B_r(0)} \sqrt{\det g_{ij}(x)} dx \\ &= \int_{v\in S^{n-1}} dS^{n-1} \int_0^r t^{n-1}\sqrt{\det g_{ij}(tv)} dt \\ &= \int_{v\in S^{n-1}} dS^{n-1} \int_0^r \theta(t,v) dt \end{aligned}$$

where we have set $\theta(t,v) = t^{n-1}\sqrt{\det g_{ij}(tv)}$ and dS^{n-1} is the canonical Lebesgue measure on the unit sphere of T_pM.

The normal volume element $\theta(t,v)$ can be described in terms of Jacobi fields as follows. Let $\{e_1,\dots,e_n = v\}$ be an orthonormal basis of T_pM. Take Jacobi fields $J_i(t)$, $i = 1,\dots,n-1$, normal along the geodesic $\gamma_v(t) = \exp_p(tv)$ with $J_i(0) = 0$ and $J_i'(0) = e_i$. Then

$$\begin{aligned}\theta(t,v) &= \|J_1(t) \wedge \cdots \wedge J_{n-1}(t)\| \\ &= \sqrt{\det(< J_i(t), J_j(t) >_{i\le i,j\le n-1})}\end{aligned}$$

The proof of this formula can be found on page 65 of [15].

As usual we denote by $(\zeta^t)_{t\in\mathbb{R}}$ the geodesic flow on TM. We shall see how $\theta(t,\zeta^s v)$ is expressed in terms of Jacobi fields.

Let $e_i(t)$, for $i = 1,\cdots,n-1$, be the parallel translation of e_i along the (unit speed) geodesic γ_v, and $J_i(t;s)$, $i = 1,\dots,n-1$ normal Jacobi fields along γ_v satisfying the initial conditions $J_i(s;s) = 0$ and $J_i'(s;s) = e_i(s)$. We may write

$$J_i(t;s) = \sum_{j=1}^{n-1} a_{ji}(t;s)e_j(t)$$

and consider the $(n-1)\times(n-1)$ matrix $A(t;s) = [a_{ji}(t;s)]$. Then we have

$$\begin{cases} A''(t;s) + R(t)A(t;s) = 0, \\ A(s;s) = 0, A'(s;s) = I_{n-1} \end{cases}$$

where the prime stands for the differentiation with respect to t, I_{n-1} is the identity matrix and $R(t) = [R_{ji}(t)]$ denotes the symmetric $(n-1)\times(n-1)$ matrix given by the formula

$$R(e_i(t),\dot\gamma_v(t))\dot\gamma_v(t) = \sum_{j=1}^{n-1} R_{ji}(t)e_j(t)$$

for $i = 1,\dots,n-1$. Then from

$$\theta(t,\zeta^s v) = \|J_1(t+s;s) \wedge \cdots \wedge J_n(t+s;s)\| = |\det A(t+s;s)|$$

we get

$$\theta(t,\zeta^s v) = |\det A(t+s;s)|$$

1.4 The Berger-Kazdan isoembolic inequality

Let (M,g) be a complete Riemannian n-manifold and let $(\zeta^t)_{t\in\mathbb{R}}$ denote the geodesic flow on TM and UM. We let $\alpha_k = Vol(S^k, can)$ and $B_r(p) = \{q \in M | d(p,q) < r\}$, while we refer to the previous sections 2 and 3 for integration using in geodesic polar coordinates in M and also for concepts of integration on UM. We use the normalization $InjM = \pi$. This section is devoted to the detailed proof of an isoembolic inequality originally due to M. Berger.

Theorem 1.4.1 (Berger-Kazdan) *Let (M,g) be a n-dimensional compact Riemannian manifold with $InjM=\pi$. Then,*

$$Vol(M,g) \geq Vol(S^n, can)$$

and the equality holds if and only if (M,g) is isometric to (S^n, can)

Proof Since $InjM = \pi$, we have $d(\tau u, \tau\zeta^\pi u) = \pi$ for $u \in UM$, which implies that $B_s(\tau u) \cap B_{\pi-s}(\tau\zeta^\pi u) = \emptyset$ for $s \in [0,\pi]$, where $(\zeta^t)_{t\in\mathbb{R}}$ is the geodesic flow. Therefore,

$$VolM \geq VolB_s(\tau u) + VolB_{\pi-s}(\tau\zeta^\pi u).$$

Integrating this inequality with respect to the Liouville measure for $u \in UM$ and using that

$$Vol(UM,\bar{g}) = Vol(M,g)Vol(S^{n-1}, can) = Vol(M,g)\alpha_{n-1}$$

we get:

$$\begin{aligned}
\alpha_{n-1}(VolM)^2 &\geq \int_{UM} VolB_s(\tau u)d\nu_{\overline{g}} + \int_{UM} VolB_{\pi-s}(\tau\zeta^\pi u)d\nu_{\overline{g}} \\
&= \int_{p\in M} d\nu_g \int_{U_pM} VolB_s(p)dS^{n-1} + \int_{u\in UM} VolB_{\pi-s}(\tau\zeta^\pi u)d\nu_{\overline{g}} \\
&= \int_{p\in M} d\nu_g \left(\int_{U_pM} \left(VolB_s(p) + VolB_{\pi-s}(p)\right) dS^{n-1}\right) \\
&= \alpha_{n-1} \int_{p\in M} \left(VolB_s(p) + VolB_{\pi-s}(p)\right) d\nu_g
\end{aligned}$$

where we have used the invariance of the Liouville measure $d\nu_{\overline{g}}$ on $(UM,\overline{g})$ under the geodesic flow.

Now we have

$$\begin{aligned}
(VolM)^2 &\geq \int_{p\in M} \left(VolB_S(p) + VolB_{\pi-s}(p)\right) d\nu_g \\
&= \int_{p\in M} d\nu_g \left(\int_0^s dt \int_{u\in U_pM} \theta(t,u)dS^{n-1} + \int_0^{\pi-s} dt \int_{u\in U_pM} \theta(t,u)dS^{n-1}\right) \\
&= \int_0^s dt \int_{UM} \theta(t,u)d\nu_{\overline{g}} + \int_0^{\pi-s} \int_{UM} \theta(t,u)d\nu_{\overline{g}}
\end{aligned}$$

where we have used the Fubini Theorem again, and the equality

$$VolB_r(p) = \int_0^r dt \int_{U_PM} \theta(t,u)dS^{n-1}$$

for $r \leq InjM$.

Next we employ the Liouville Theorem, to write the last inequality as

$$(VolM)^2 \geq \int_0^s dt \int_{UM} \theta(t,\zeta^{\pi-s}u)d\nu_{\overline{g}} + \int_0^{\pi-s} dt \int_{UM} \theta(t,\zeta^s u)d\nu_{\overline{g}}$$

Averaging this last inequality with respect to $s \in [0, \pi]$ and then changing the variable s to $\pi - s$ in the first integral we obtain:

$$
\begin{aligned}
(VolM)^2 \geq & \frac{1}{\pi} \int_0^\pi ds \left(\int_0^s dt \int_{u \in UM} \theta(t, \zeta^{\pi - s}) d\nu_{\overline{g}} + \int_0^{\pi - s} dt \int_{UM} \theta(t, \zeta^s u) d\nu_{\overline{g}} \right) \\
= & \frac{2}{\pi} \int_0^\pi ds \int_0^{\pi - s} dt \int_{UM} \theta(t, \zeta^s u) d\nu_{\overline{g}} \\
= & \frac{2}{\pi} \int_{UM} d\nu_{\overline{g}} \left(\int_0^\pi ds \int_0^{\pi - s} \det A(t+s; s) dt \right)
\end{aligned}
$$

since $\theta(t, \zeta^s u) = |\det A(t+s; s)|$ and $t \in (0, \pi)$ implies that the determinant is positive. Using the Fubini Theorem and interchanging variables t to s

$$
\int_0^\pi ds \int_0^{\pi - s} \det A(t+s; s) dt = \int_0^\pi ds \int_0^{\pi - s} \det A(t+s; t) dt
$$

Therefore we get the inequality

$$
(Vol(M, g))^2 \geq \frac{2}{\pi} \int_{UM} d\nu_{\overline{g}} \int_0^\pi ds \int_0^{\pi - s} \det A(t+s; t) dt
$$

If we prove the Kazdan inequality

$$
\int_0^\pi ds \int_0^{\pi - s} \det A(t+s; t) dt \geq \pi \int_0^{\frac{\pi}{2}} (\sin s)^{n-1} ds
$$

with equality if and only if $R(t) = I_{n-1}$ and $A(t; s) = \sin(t - s) I_{n-1}$, we get that the above quantity is greater than or equal to

$$
2 \int_{UM} d\nu_{\overline{g}} \int_0^{\frac{\pi}{2}} (\sin s)^{n-1} ds = 2a_{n-1} \int_0^{\frac{\pi}{2}} (\sin s)^{n-1} ds \; Vol(M, g) = Vol(S^n, can) \, Vol(M, g)
$$

as is seen by the specific values of the Jacobi Fields in the sphere. Thus

$$
Vol(M, g) \geq Vol(S^n, can)
$$

as stated in the theorem.

Moreover, equality implies all the above inequalities are actually equalities, and in particular

$$
\int_0^\pi ds \int_0^{\pi - s} \det A(t+s; t) dt = \pi \int_0^{\frac{\pi}{2}} (\sin s)^{n-1} ds, \quad \forall u \in UM
$$

and again using the Kazdan inequality, we get $R(t) = I_{n-1}$, with I_{n-1} the $(n-1) \times (n-1)$ identity matrix. This implies in turn that M has constant sectional curvature, and thus also has (S^n, can) as a Riemannian universal covering space.

Consider the Riemannian covering $(S^n, can) \to (M, g)$ and $p \in M$. Any nontrivial element of $\pi_1(M, p)$ is represented by a geodesic loop γ at p, with minimal length. This loop corresponds to a minimal geodesic $\widetilde{\gamma}$, joining $\widetilde{p_0}, \widetilde{p_1} \in S^n$, which both map to $p \in M$. But then

$$
\pi = InjM \leq \frac{1}{2} L(\gamma) \leq \frac{1}{2} L(\widetilde{\gamma}) \leq \frac{diam(S^n, can)}{2} = \frac{\pi}{2},
$$

which is a contradiction. This proves that (M, g) is simply connected and therefore isometric to (S^n, can), finishing the proof of Theorem 1.4.1. ∎

So, the proof of the Berger-Kazdan inequality is reduced to the proof of the following.

Proposition 1.4.2 (Kazdan inequality) *Set $m = n-1$. Suppose $A(t;s)$ and $R(t)$ are $m \times m$ matrices such that:*
(i) $R(t)$ is self-adjoint,
(ii) $A(t+s;t)$ is non-degenerate for $s \in (0,\pi)$
(iii) $A^{''}(t;s)+R(t)A(t;s) = 0$ with initial conditions $A(s;s) = 0, A^{'}(s;s) = I_m$, where derivatives are taken with respect to t. .
Then

$$\int_0^{\pi} ds \int_0^{\pi-s} \det A(t+s;t)dt \geq \pi \int_0^{\frac{\pi}{2}} (\sin s)^m ds$$

and the equality holds if and only if $R(t) = I_m$ and $A(t;s) = \sin(t-s)I_m$.

The proof consists of a series of intermediate lemmas.

Lemma 1.4.3 *Let S^+ denote the space of $m \times m$ Symmetric Positive Definite matrices, which can be considered as an open convex subset of $\mathbb{R}^{\frac{m(m+1)}{2}}$, if we identify the latter with the vector space S of symmetric matrices. The function $F : S^+ \to \mathbb{R}$ defined by*

$$F(B) = \det B^{-1}$$

is strongly convex, i.e. $D^2F(B)$ is positive definite for all $B \in S^+$.

Proof Let $B \in S^+$ and $A \in S \setminus \{0\}$. If we consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = F(B+tA)$, it suffices to prove $f^{''}(0) > 0$. Set $C(t) = B + tA$. We have:

$$\frac{d}{dt} \det C(t) = \det C(t) tr(AC^{-1}(t)).$$

Differentiating the equality $C(t)C(t)^{-1} = I_m$ at $t = 0$ we get

$$AB^{-1} + B\frac{d}{dt}\Big|_{t=0} C(t)^{-1} = 0,$$

which means that

$$\frac{d}{dt}\Big|_{t=0} C(t)^{-1} = -BAB^{-1}.$$

Now

$$f^{'}(t) = -\det C(t)^{-1} tr(AC(t)^{-1})$$

and

$$f''(t) = \det C(t)^{-1}((tr(AC(t)^{-1}))^2 - \det C(t)^{-1} tr(A\frac{d}{dt}C(t)^{-1})$$

In particular for $t = 0$ we find

$$f''(0) = \det B^{-1}(tr(AB^{-1}))^2 + (\det B)^{-1} tr(AB^{-1}AB^{-1}).$$

By the spectral theorem there is an orthonormal basis $\{e_1, ..., e_m\}$ diagonalising B and A remains symmetric in this basis. If λ_i is the eigenvalue of e_i, we have

$$tr(AB^{-1}AB^{-1}) = \sum_{i,j,k} \delta_{ik}\alpha_{ij}\lambda_j^{-1}\alpha_{jk}\lambda_k^{-1} = \sum_{i,j} (\lambda_j\lambda_i)^{-1}\alpha_{ij}\alpha_{ji} = \sum_{i,j} (\lambda_j\lambda_i)^{-1}{\alpha_{ij}}^2 > 0$$

since $\lambda_i > 0$ and $A = (a_{ij}) \neq 0$. Therefore $f''(0) > 0$ and the lemma is proved. ∎

¿From this we derive a Jensen-type inequality.

Corollary 1.4.4 *Let Ω be a measure space with a positive measure μ. Then, for a function $B : \Omega \to S^+$ in $L^1(\mu)$ we have*

$$\det\Big(\frac{1}{\mu(\Omega)}\int_\Omega B(r)d\mu(r)\Big)^{-1} \leq \frac{1}{\mu(\Omega)}\int_\Omega (\det B(r))^{-1}d\mu(r)$$

with equality if and only if B is constant almost everywhere.

Proof For $B, B_0 \in S^+$ the strong convexity of F implies

$$F(B) \geq F(B_0) + DF(B_0)(B - B_0).$$

¿From Taylor's formula

$$F(B) = F(B_0) + DF(B_0)(B - B_0) + \frac{1}{2}D^2F(B_1)(B - B_0, B - B_0)$$

where B_1 lies in the segment with endpoints B and B_0. We let

$$B_0 = \frac{1}{\mu(\Omega)}\int_\Omega B(r)d\mu(r),$$

$B = B(r)$ and we integrate the above inequality to get

$$\begin{aligned}
\int_\Omega \det B^{-1}(r)d\mu(r) &= \int_\Omega F(B(r))d\mu(r) \\
&\geq \mu(\Omega)F\left(\frac{1}{\mu(\Omega)}\int_\Omega B(r)d\mu(r)\right)^{-1} + DF(B_0)\left(\int_\Omega B(r)d\mu(r) - \mu(\Omega)B_0\right) \\
&= \mu(\Omega)\det\left(\frac{1}{\mu(\Omega)}\int_\Omega B(r)d\mu(r)\right)^{-1}
\end{aligned}$$

which is the desired inequality.

Equality implies that

$$F(B) = F(B_0) + DF(B_0)(B - B_0)$$

μ-almost everywhere in Ω, since μ is a positive measure. By Taylor's theorem this is equivalent to

$$D^2F_(B_1)(B(r) - B_0, B(r) - B_0) = 0$$

or $B(r) = B_0$ μ-almost everywhere since F is strongly convex. ∎

Now we come to the proof of Proposition 1.4.2 and we make the assumptions made there. The proof will be done by successive reductions. This requires several steps.

Lemma 1.4.5 *Let* $\phi(t) = (\det A(t))^{\frac{1}{m}}$*, where* $A(t) = A(t;0)$*. Then*

$$(\det A(t;s))^{\frac{1}{m}} \geq \phi(t)\phi(s) \int_s^t \phi^{-2}(r)dr$$

for $0 < s < t < \pi$ *and equality holds for all* $0 < s < t < \pi$ *if and only if* $A(t) = \phi(t) I_m$*.*

Proof The assumptions of Proposition 1.4.2 imply that $A^*(t)A'(t) = A'^*(t)A(t)$ and

$$A(t;s) = A(t)\Big(\int_s^t (A^*(r)A(r))^{-1}dr\Big)A^*(s).$$

Also, $\phi(0) = 0$ and

$$\phi'(0) = \lim_{t\to 0} \frac{(\det A(t))^{\frac{1}{m}}}{t} = \Big(\lim_{t\to 0} \det \frac{A(t)}{t}\Big)^{\frac{1}{m}} = \Big(\det A'(0)\Big)^{\frac{1}{m}} = 1.$$

Setting $B(r) = (A^*(r)A(r))^{-1}$ we see that $B(r) \in S^+$ for $r \in (0,\pi)$ and

$$\Big(\det A(t;s)\Big)^{\frac{1}{m}} = \phi(r)\phi(s)\Big(\det \int_s^t B(r)dr\Big)^{\frac{1}{m}}.$$

Now we take $\Omega = [s,t]$ with measure $d\mu(r) = \phi^{-2}(r)dr$ and apply the Jensen-type inequality to $\phi^2(r)B(r) \in S^+$. This gives

$$\begin{aligned}
\det\Big(\mu(\Omega)^{-1}\int_s^t \phi^2(r)B(r)d\mu(r)\Big)^{-1} &\leq \int_s^t \det(\phi^2(r)B(r))^{-1}d\mu(r)\Big/\mu(\Omega)\\
&= \int_s^t \det(B(r))^{-1}\phi^{-2m-2}(r)dr\Big/\int_s^t \phi^{-2}(r)dr\\
&= \int_s^t \det(A(r)A^*(r))\phi^{-2m-2}(r)dr\Big/\int_s^t \phi^{-2}(r)dr\\
&= \int_s^t \phi^{-2}(r)dr\Big/\int_s^t \phi^{-2}(r)dr\\
&= 1
\end{aligned}$$

since $\det A^*(r) = \det A(r) = (\phi(r))^m$. This is

$$1 \geq \det\Big(\mu(\Omega)^{-1}\int_s^t \phi^2(r)B(r)\phi^{-2}(r)dr\Big)^{-1} = \mu(\Omega)^m \det\Big(\int_s^t B(r)dr\Big)^{-1}$$

or

$$\Big(\det \int_s^t B(r)dr\Big)^{\frac{1}{m}} \geq \mu(\Omega) = \int_s^t \phi^{-2}(r)dr$$

and substituting

$$\Big(\det(A(t;s)\Big)^{\frac{1}{m}} = \phi(r)\phi(s)\Big(\det \int_s^t B(r)dr\Big)^{\frac{1}{m}} \geq \phi(r)\phi(s)\int_s^t \phi^{-2}(r)dr$$

as claimed.

If we have equality, then $\frac{1}{\phi(r)}A(r)$ is constant on $[s,t]$. If this holds for all $0<s<t<\pi$, then $A(r)=\phi(r)I_m$ for every $r\in(0,\pi)$, because

$$\lim_{r\to 0^+}\frac{1}{\phi(r)}A(r)=\lim_{r\to 0^+}\frac{r}{\phi(r)}\frac{A(r)}{r}=\frac{1}{\phi'(0)}A'(0)=I_m.$$

This concludes the proof of the lemma. ∎

Proof of Proposition 1.4.2 The left hand side of the Kazdan inequality can be written

$$\int_0^\pi ds\int_0^{\pi-s}\det A(t+s;t)dt=\int_0^{\frac{\pi}{2}}ds\int_0^{\pi-s}\det A(t+s;t)dt+\int_{\frac{\pi}{2}}^{\pi}ds\int_0^{\pi-s}\det A(t+s;t)dt$$
$$=\int_0^{\frac{\pi}{2}}ds\Big(\int_0^{\pi-s}\det A(t+s;t)dt+\int_0^{s}\det A(t+\pi-s;t)dt\Big)$$
$$\geq\int_0^{\frac{\pi}{2}}\Big[\int_0^{\pi-s}\phi(t)\phi(t+s)\int_t^{s+t}\phi^{-2}(r)dr)^m dt+\int_0^{s}(\phi(t)\phi(t+\pi-s)\int_t^{t+\pi-s}\phi^{-2}(r)dr)^m dt\Big]$$

using the last lemma. Equality implies

$$\Big(\det A(t+s;t)\Big)^{\frac{1}{m}}=\phi(t+s)\phi(t)\int_s^{t+s}\phi^{-2}(r)dr \text{ for all } 0\leq s\leq t+s<\pi$$

since the equality of integrals gives equality almost everywhere and all the above functions are continuous. This implies in turn that $A(t)=\phi(t)I_m$.

Since we want $\phi(t)=\sin t$ for $t\in[0,\pi]$, we are led to define $u:[0,\pi)\to\mathbb{R}$ by

$$\phi(r)=\sin r e^{u(r)}$$

for $r\in[0,\pi)$ and note that since $\sin\pi=0$, u may become singular at $r=\pi$.

We also define for $t\leq r\leq t+s<\pi$ the functions $v_{t,s},h_{t,s}:[t,t+s]\to\mathbb{R}$ by

$$v_{t,s}(r)=u(t)+u(t+s)-2u(r)$$

and

$$h_{t,s}(r)=\frac{\sin t\sin(t+s)}{\sin^2 r}.$$

Then, substituting we get

$$\int_0^\pi ds\int_0^{\pi-s}\det A(t+s;t)\geq\int_0^{\frac{\pi}{2}}ds\Big[\int_0^{\pi-s}(\int_t^{t+s}h(r)e^{v(r)}dr)^m dt+\int_0^{s}(\int_t^{t+\pi-s}h(r)e^{v(r)}dr)^m dt\Big]$$

suppressing the subscripts t,s. Thus, if we set for $\lambda\in[0,1]$

$$f_\lambda(s):=\int_0^{\pi-s}(\int_t^{t+s}h(r)e^{\lambda v(r)}dr)^m dt+\int_0^{s}(\int_t^{t+\pi-s}h(r)e^{\lambda v(r)}dr)^m dt$$

we have

$$\int_0^\pi ds\int_0^{\pi-s}\det A(t+s;t)\geq\int_0^{\frac{\pi}{2}}f_1(s)ds$$

On the other hand, we compute

$$\int_t^{t+s} h(r)dr = \int_t^{t+s} \frac{\sin t \sin(t+s)}{\sin^2 r} dr = -\sin t \sin(t+s)[\cot t - \cot(t+s)] = \sin s$$

and therefore, since for $\phi = \sin$ we get $u = 0$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} f_0(s)ds &= \int_0^{\frac{\pi}{2}} ds\Big[\int_0^{\pi-s}(\int_t^{t+s} h(r)dr)^m + \int_0^s(\int_t^{t+\pi-s} h(r)dr)^m dt\Big] \\ &= \int_0^{\frac{\pi}{2}} \Big(\int_0^{\pi-s}(\sin s)^m dt + \int_0^s (\sin(\pi-s))^m dt\Big) \\ &= \pi\int_0^{\frac{\pi}{2}} (\sin s)^m ds.\end{aligned}$$

It suffices now to prove that

$$\int_0^{\frac{\pi}{2}} f_1(s)ds \geq \int_0^{\frac{\pi}{2}} f_0(s)ds.$$

We are going to investigate the behaviour of $f_\lambda(s)$ with respect to $\lambda \in [0,1]$. Differentiating twice with respect to λ we get

$$\begin{aligned}\frac{d}{d\lambda}f_\lambda(s) =& m\int_0^{\pi-s}(\int_t^{t+s} h(r)^{\lambda v(r)}dr)^{m-1}(\int_t^{t+s} h(r)v(r)e^{\lambda v(r)}dr)dt \\ &+ m\int_0^s(\int_t^{t+\pi-s} h(r)^{\lambda v(r)}dr)^{m-1}(\int_t^{t+\pi-s} h(r)v(r)e^{\lambda v(r)}dr)dt\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{d\lambda^2}f_\lambda(s) =& m(m-1)\int_0^{\pi-s}(\int_t^{t+s} h(r)e^{\lambda v(r)}dr)^{m-2}(\int_t^{t+s} h(r)v(r)e^{\lambda v(r)}dr)^2 dt \\ &+ m\int_0^{\pi-s}(\int_t^{t+s} h(r)e^{\lambda v(r)}dr)^{m-1}(\int_t^{t+s} h(r)v^2(r)e^{\lambda v(r)}dr)dt \\ &+ m(m-1)\int_0^s(\int_t^{t+\pi-s} h(r)e^{\lambda v(r)}dr)^{m-2}(\int_t^{t+\pi-s} h(r)v(r)e^{\lambda v(r)}dr)^2 dt \\ &+ m\int_0^s(\int_t^{t+\pi-s} h(r)e^{\lambda v(r)}dr)^{m-1}(\int_t^{t+\pi-s} h(r)v(r)e^{\lambda v(r)}dr)^2 dt \geq 0\end{aligned}$$

We observe $\frac{d^2}{d\lambda^2}f_\lambda(s) \geq 0$, since $h(r) = \frac{\sin t \sin(t+s)}{\sin^2 r} > 0$ for $0 < t < r < t+s < \pi$. Hence,

$$f_1(s) \geq f_0(s) + \frac{d}{d\lambda}\Big|_{\lambda=0} f_\lambda(s).$$

¿From Taylor's theorem and we will be done if we show

$$\frac{d}{d\lambda}\Big|_{\lambda=0} f_\lambda(s) = 0$$

Given this, suppose we have $\int_0^{\frac{\pi}{2}} f_1(s)ds = \int_0^{\frac{\pi}{2}} f_0(s)ds$ or $f_1(s) = f_0(s)$ for all $s \in (0, \frac{\pi}{2})$. Then, $\frac{d}{d\lambda}\Big|_{\lambda=0} f_\lambda = 0$ and $\frac{d^2}{d\lambda^2}f_\lambda \geq 0$ imply f_λ is increasing with respect to λ and is thus constant for $\lambda \in [0,1]$. Therefore, $\frac{d^2}{d\lambda^2} f_\lambda = 0$, which in turn, using the previous calculation, implies $v_{t,s} = 0$ for all t, s, since $h_s e^{\lambda v} > 0$. Now, $\frac{du}{dr} = -\frac{1}{2}\frac{dv}{dr} = 0$ and therefore $u(r) = C$, a constant, and $\phi(r) = \sin r e^{u(r)} = C \sin r$ with

$$\frac{\phi(r)}{\sin(r)} = \lim_{r\to 0} \frac{\phi(r)}{\sin r} = \lim_{r \to 0} \frac{\phi(r)}{r} = \phi'(0) = 1$$

which gives $\phi(r) = \sin r$ as claimed.

So the final step will be to prove that $\left.\frac{d}{d\lambda}\right|_{\lambda=0} f_\lambda(s) = 0$.

We compute

$$\begin{aligned}
\left.\frac{d}{d\lambda}\right|_{\lambda=0} f_\lambda(s) =& m \int_0^{\pi-s} \Big(\int_t^{t+s} h(r)dr\Big)^{m-1} \Big(\int_t^{t+s} h(r)v(r)dr\Big)dt \\
&+ m \int_0^{s} (\int_t^{t+\pi-s} h(r)dr)^{m-1} (\int_t^{t+\pi-s} h(r)v(r)dr)dt \\
&= m(\sin s)^{m-1}\Big[\int_0^{\pi-s}(\int_t^{t+s} h(r)v(r)dr)dt + \int_0^s (\int_t^{t+\pi-s} h(r)v(r)dr)dt\Big]
\end{aligned}$$

and setting

$$k(s) = \int_0^{\pi-s} (\int_t^{t+s} h_{t,s}(r) v_{t,s}(r) dr)dt$$

we have

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} f_\lambda(s) = m(\sin s)^{m-1}[k(s) + k(\pi - s)]$$

Thus, we are reduced to showing that $k(s) + k(\pi - s) = 0$. We define

$$\psi(t) = \int_t^{\frac{\pi}{2}} \frac{u(r)}{(\sin r)^2} dr$$

and compute,

$$\begin{aligned}
k(s) =& \int_0^{\pi-s} \Big[\int_t^{t+s} h(r)(u(s+t) + u(t) - 2u(r))dr\Big]dt \\
=& \int_0^{\pi-s} (u(s+t) + u(t))(\int_t^{t+s} h(r)dr)dt - 2\int_0^{\pi-s} \sin(t+s)\sin t(\int_t^{t+s} \frac{u(r)}{\sin^2 r} dr)dt \\
=& \int_0^{\pi-s} \sin s(u(s+t) + u(t))dt - 2\int_0^{\pi-s} \sin t \sin(t+s)(\psi(t) - \psi(t+s))dt \\
=& \sin s(\int_0^{\pi-s} u(t)dt + \int_s^{\pi} u(t)dt) \\
&- 2(\int_0^{\pi-s} \sin(t+s)\sin t\psi(t)dt + \int_s^{\pi} \sin t \sin(s-t)\psi(t)dt)
\end{aligned}$$

and in a similar manner

$$k(\pi-s) = \sin s(\int_0^s u(t)dt + \int_{\pi-s}^{\pi} u(t)dt) - 2(\int_0^s \sin(s-t)\sin t\psi(t)dt + \int_{\pi-s}^{\pi} \sin t \sin(s+t)\psi(t)dt)$$

Therefore

$$\begin{aligned} k(s)+k(\pi-s) =& 2\sin s\int_0^\pi u(t)dt - 2\int_0^\pi (\sin(s-t)\sin t + \sin t\sin(s+t))\psi(t)dt \\ &= 2\sin s(\int_0^\pi u(t)dt - \int_0^\pi \sin 2t\psi(t)dt) \end{aligned}$$

The second integral in this expression is

$$\begin{aligned} \int_0^\pi \sin 2t\psi(t)dt &= \int_0^\pi \sin 2t(\int_t^{\frac{\pi}{2}} \frac{u(r)}{\sin^2 r}dr)dt \\ &= \int_0^{\frac{\pi}{2}} \frac{u(r)}{\sin^2 r}(\int_0^r \sin 2tdt)dr - \int_{\frac{\pi}{2}}^\pi \frac{u(r)}{\sin^2 r}(\int_r^\pi \sin 2tdt) \\ &= \int_0^{\frac{\pi}{2}} u(r)dr + \int_{\frac{\pi}{2}}^\pi u(r)dr = \int_0^\pi u(r)dr \end{aligned}$$

Consequently,

$$k(s)+k(\pi-s) = 2\sin s(\int_0^\pi u(t)dt - \int_0^\pi u(r)dr) = 0$$

and the proof of the Kazdan inequality is complete.

The equality above implies $A(t) = \phi(t)I_m$ and $\phi = \sin$. Thus $A(t) = (\sin t)I_m$ and $A''(t) + R(t)A(t) = 0$ which give $R(t) = I_m$. Solving

$$A(t;s) + R(t)A(t;s) = 0$$

with the given initial conditions, we get $A(t;s) = (\sin(t-s))I_m$, as claimed. ∎

Chapter 2

Blaschke manifolds

A Blaschke manifold is a compact Riemannian manifold (M, g) with the property $Inj(M) = diam(M)$. In this chapter we are going to investigate various properties of Blaschke manifolds and explore the most readily accessible consequences of the Blaschke condition, showing that it is indeed very restrictive.

Further on, the aim of the chapter is to get a feeling of the geometry of Blaschke manifolds as well as a primary and intuitive justification of the Blaschke conjecture, still unproved in most cases, concerning their classification up to isometry. The desired classification, even up to homeomorphism was considerable amount of work by several authors and already out of the scope of this work.

2.1 The cut locus of a pointed Blaschke manifold

In this section we will be concerned with pointed Blaschke manifolds.

Let (M, g) be a compact Riemannian manifold and $p \in M$. We call (M, g) a *pointed Blaschke manifold at* p if $cut(p)$ is spherical, which means that $d(p, q)$ does not depend on $q \in cut(p)$.

If M is a Blaschke manifold and $p \in M$, then

$$Inj(M) \leq Inj(p) \leq d(p, q) \leq diam(M)$$

for every $q \in cut(p)$ and we conclude that M is Blaschke at p. The converse is also true but we postpone its proof until the next section of the present chapter, since it will be rather obvious after subsequent development of the theory of pointed Blaschke manifolds.

Our aim now is to find a suitable description for manifolds that are Blaschke at a point. First we need the following definition.

Let $p \in M$ and $q \in cut(p)$. We define the link from p to q to be

$$\Lambda(p, q) = \{\dot{\gamma}(0) \in U_q M : \gamma \in seg(q, p)\}$$

where $seg(q, p)$ as usual denotes the set of all segments, i.e. minimizing unit speed geodesics, from q to p.

Proposition 2.1.1 *Let (M, g) be a pointed Blaschke manifold at $p \in M$. Then, for all $q \in cut(p)$, $\Lambda(p, q)$ is a great sphere in $U_q M$, i.e. $\Lambda(p, q) = U_q M \cap V$, where V is some vector subspace of $T_q M$.*

The proof uses a sequence of lemmas. We normalize to $d(p, cut(p)) = \frac{\pi}{2}$ and denote by $g = (\quad,\quad)$ the Riemannian metric. Let the point p be fixed and $q \in cut(p)$. We let

- $\Lambda(q) := \Lambda(p, q)$,
- $\hat{v} := \frac{v}{\|v\|}$ for non-zero $v \in T_qM$ and
- $N(q)$ be the set of all $w \in U_qM$ such that there exists a sequence $(q_n)_{n\in\mathbb{N}}$ in $cut(p) \setminus \{q\}$ with $q_n \to q$ and $\widehat{\exp_q^{-1}(q_n)} \to w$.

Lemma 2.1.2 *For all $v_1 \in \Lambda(q)$ and $v_2 \in N(q)$, we have $(v_1, v_2) \le 0$.*

Proof Let $(v_1, v_2) > 0$ and $\gamma \in seg(q, p)$ be such that $v_1 = \dot{\gamma}(0)$. By the Acute Angle Property (see Proposition 1.1.1) there is a neighbourhood V of v_2 in U_qM and $\delta > 0$ small enough such that $d(p, \exp_q(tw)) < \frac{\pi}{2}$ for all $w \in V$ and $0 < t < \delta$. This implies that v_2 cannot belong to $N(q)$. ∎

Lemma 2.1.3 *For all $v \in U_qM \setminus \Lambda(q)$ there exist $v_1 \in \Lambda(q)$ and $v_2 \in N(q)$ such that $v \in \mathbb{R}_+v_1 + \mathbb{R}_+v_2$ and $(v_1, v_2) = 0$.*

Proof Let $v \notin N(q)$. By assumption, $d(p, x) < \frac{\pi}{2}$ for $x \notin cut(p)$. Since $v \notin N(q)$, if we set $x(\epsilon) := \exp_q(\epsilon v)$, then $d(p, x(\epsilon)) < \frac{\pi}{2}$ for $\epsilon > 0$ small, and there is a unique $\gamma_\epsilon \in seg(p, x(\epsilon))$ which we can extend until it hits $cut(p)$ at a point $q(\epsilon) = \gamma_\epsilon(\pi/2)$. Since for ϵ small enough there is a unique minimal geodesic of small length joining $x(\epsilon)$ and q, we have $q(\epsilon) \neq q$, for if $q(\epsilon) = q$ then the curve $x(t)$ is actually γ_ϵ traversed backwards and thus $v \in \Lambda(q)$, contrary to the hypothesis.

Now $\exp_q^{-1} q(\epsilon) \neq 0$ for small ϵ and thus $\widehat{\exp_q^{-1} q(\epsilon)}$ is a well defined vector in U_qM. Moreover, it is obvious that $\lim_{\epsilon\to 0} x(\epsilon) = \lim_{\epsilon\to 0} q(\epsilon) = q$.

By compactness of $UM \times UM$, there a sequence $\epsilon_n \to 0$, and write from now on n for the subscript case of ϵ_n, such that $-\dot{\gamma_n}(\frac{\pi}{2})$ converges to some v_1 and $\widehat{\exp_q^{-1}(q_n)}$ converge to some v_2. Now $v_2 \in N(q)$ immediately from the construction, and we also see that $\exp_q(\frac{\pi}{2}v_1) = p$ from the continuity of the exponential map on TM. Thus we also have $v_1 \in \Lambda(q)$.

It will suffice to show $v \in \mathbb{R}_+v_1 + \mathbb{R}_+v_2$ and $(v_1, v_2) = 0$. To show $v \in \mathbb{R}_+v_1 + \mathbb{R}_+v_2$, we work in exponential local coordinates at q. We will write $\tilde{x}_n$ for $\exp_q^{-1} x_n$ or $\tilde{x} - \tilde{y}$ to denote a vector with application point at $\tilde{y} \in T_qM$. We have

$$\widehat{\tilde{x}_n - \tilde{q}_n} = \frac{\tilde{x}_n - \tilde{q}_n}{\|\tilde{x}_n - \tilde{q}_n\|} = \frac{\|\tilde{x}_n\| v}{\|\tilde{x}_n - \tilde{q}_n\|} - \frac{\|\tilde{q}_n\| \hat{\tilde{q}}_n}{\|\tilde{x}_n - \tilde{q}_n\|}.$$

Therefore, since $\hat{\tilde{q}}_n \to v_2$, if there's a limit of $\widehat{\tilde{x}_n - \tilde{q}_n}$, it belongs to the limit plane of the sequence of planes spanned by $\{v, \hat{\tilde{q}}_n\}$, i.e. the plane spanned by $\{v, v_2\}$. Moreover if there is such a limit, $\lim_{n\to+\infty} \widehat{\tilde{x}_n - \tilde{q}_n} = av - bv_2$ for a, $b \ge 0$.

We are going to show that this limit indeed exists and equals v_1, thus $v_1 = av - bv_2$ and since $v \notin \Lambda(q)$ we have that $v = \frac{1}{a}v_1 + \frac{b}{a}v_2$ as desired.

We now prove that $\lim_{n\to+\infty} \tilde{x}_n - \tilde{q}_n = v_1 = \lim_{n\to+\infty}(-\dot{\gamma}_n(\frac{\pi}{2}))$.

We choose $r > 0$ so small that the open ball $B(q,r)$ is geodesically convex, and $n \in \mathbb{N}$ so that $x_n, q_n \in B(q,r)$. We may choose the exponential local coordinates so that $\dot{\tilde{\bar{\gamma}}}_n(0)$ is proportional to $\frac{\partial}{\partial x^1}$ in T_qM, where $\bar{\gamma}_n(t) = \gamma_n(\frac{\pi}{2} - t)$. We just rotate the coordinates so that $\dot{\tilde{\bar{\gamma}}}_n(0)$ becomes $g^{-1/2}(q_n)\frac{\partial}{\partial x^1}$

We also set $\Gamma_r := \sup\{|\Gamma^i_{jk}(x)| : 1 \le i,j,k \le \dim M \text{ and } x \in B(q,r)\}$. We know that $\Gamma_r \to 0$ for $r \to 0$.

Let $0 < d < 2r$ be the distance of x_n and q_n in M. We also write δ for $\tilde{\bar{\gamma}}_n|_{[0,d]}$ and ϕ_n be the Euclidean angle of $\dot{\delta}(0)$ and $\tilde{x}_n - \tilde{q}_n$. We now have

$$\tan\phi_n = \Big(\sum_{i>1}(\tilde{x}^i_n - \tilde{q}^i_n)^2\Big)^{1/2} \Big/ |\tilde{x}^1_n - \tilde{q}^1_n|$$

¿From the equations of geodesics we deduce that $|\ddot{\delta}^i| \le C\Gamma_r$, where $C > 0$ is a constant, since $(\dot{\delta},\dot{\delta}) = 1$, and $g_{ij} \to \delta_{ij}$ for $r \to 0$. From Taylor's theorem

$$\tilde{x}^i_n - \tilde{q}^i_n = \delta^i(d) - \delta^i(0) = \dot{\delta}^i(0)d + \frac{1}{2}\ddot{\delta}^i(t_i)d^2$$

for some $0 < t_i \le d$ and for $i > 1$. Since $\dot{\delta}(0) = g_{11}^{-1/2}\frac{\partial}{\partial x^1}$ and $|\ddot{\delta}^i_{(t_i)}| \le C\Gamma_r$, we have

$$\Big(\sum_{i>1}(\tilde{x}^i_n - \tilde{q}^i_n)^2\Big)^{1/2} \le C\Gamma_r d^2.$$

On the other hand, $|\tilde{x}^1_n - \tilde{q}^1_n| \ge g_{11}^{-1/2}d - C\Gamma_r d^2$ for the same reasons. Now

$$\tan\phi_n \le \frac{C\Gamma_r d}{g_{11}^{-1/2} - C\Gamma_r d} \to 0$$

for $r \to 0$. This establishes $\phi_n \to 0$ and thus we end up with $v \in \mathbb{R}_+v_1 + \mathbb{R}_+v_2$ as desired.

It remains to prove that $(v_1, v_2) = 0$. This will be a consequence of the Toponogov Comparison Theorem (see Theorem 2.2 in [8]) and Lemma 2.1.2. Since M is compact, the sectional curvature has a lower bound, say $-k$.

Let H_k denote the hyperbolic space of constant curvature equal to $-k$ of the same dimension as M. Let $\tilde{p}, \tilde{q} \in H_k$ with $d(\tilde{p},\tilde{q}) = \frac{\pi}{2}$ and $\{\tilde{\gamma}\} = seq(\tilde{p},\tilde{q})$. Given $\epsilon > 0$, the Acute Angle Property implies the existence of some $\delta > 0$ such that for all $0 < t < \delta$ and $\tilde{u} \in U_{\tilde{q}}H_k$ with $(\tilde{u}, \dot{\tilde{\gamma}}(\frac{\pi}{2})) \le -\epsilon$ we have

$$d(\tilde{p}, \exp_{\tilde{q}}(t\tilde{u})) < \frac{\pi}{2}.$$

Now let $\gamma \in seg(p,q)$ and $u \in U_qM$ be such that $(u, \dot{\gamma}(\frac{\pi}{2})) \le -\epsilon$ and $u \in U_qM$ with $(u, \dot{\gamma}(\frac{\pi}{2})) \le -\epsilon$. We find $\tilde{u} \in U_{\tilde{q}}H_k$ with

$$(\tilde{u}, \dot{\tilde{\gamma}}(\frac{\pi}{2})) = (u, \dot{\gamma}(\frac{\pi}{2})) \le -\epsilon$$

and compare the geodesic hinges at $q \in M$ and $\tilde{q} \in H_k$, spanned by the above vectors. The Toponogov Comparison Theorem gives

$$d(p, \exp_q(tu)) \leq d(\tilde{p}, \exp_{\tilde{q}}(t\tilde{u})) < \frac{\pi}{2}$$

for $0 < t < \delta$.

Now we return to the proof of $(v_1, v_2) = 0$. Let $q_n \to q$ and $\gamma_n \to \gamma$ be as above, with $v_1 = -\dot{\gamma}(\frac{\pi}{2})$. We suppose $(v_1, v_2) < 0$ and derive a contradiction. We set

$$w_n = \frac{\exp_{q_n}^{-1}(q)}{\| \exp_{q_n}^{-1}(q) \|} \in U_{q_n}M$$

and observe that $w_n \to -u_2$. The existence of such a sequence leads to a contradiction because

$$\lim_{n \to +\infty} (w_n, \dot{\gamma}_n(\frac{\pi}{2})) = (-v_2, -v_1) = (v_1, v_2) < 0$$

and setting $t_n = d(q, q_n)$ we have $t_n \to 0$ and $d(p, \exp_{q_n}(t_n w_n)) = d(p, q) = \frac{\pi}{2}$.

The above argument also shows that for given $u \in N(q)$, after repeating the procedure of establishing the existence of $\gamma_n \to \gamma$ and $v_1 = -\dot{\gamma}(\frac{\pi}{2}) \in \Lambda(q)$ we have $(v_1, u) = 0$ which is the last assertion of the lemma. ∎

Lemma 2.1.4 *The subset $\Lambda(q)$ of U_qM is convex, meaning that for $u, w \in \Lambda(q)$, such that $u \neq -w$, we have $(\mathbb{R}_+u + \mathbb{R}_+w) \cap U_qM \subset \Lambda(q)$. Moreover, there is some $u \in \Lambda(q)$ such that $-u \in \Lambda(q)$.*

Proof Let $v \in (\mathbb{R}_+u + \mathbb{R}_+w) \cap U_qM$, for $u, w \in \Lambda(q)$, $u \neq \pm w$. So $v = \lambda u + \mu w$ for some $\lambda, \mu \geq 0$. Certainly $v \notin N(q)$, for otherwise

$$1 = (v, v) = \lambda(u, v) + \mu(w, v) \leq 0$$

by Lemma 2.1.3.

Suppose that $v \notin \Lambda(q)$. Then, Lemma 2.1.3 gives $v = av_1 + bv_2$ for some a, $b > 0$ and $v_1 \in \Lambda(q)$, $v_2 \in N(q)$, $(v_1, v_2) = 0$. We have now

$$0 < b|v_2|^2 = \lambda(u, v_2) + \mu(w, v_2) \leq 0.$$

Note that $\Lambda(q)$ is closed from the continuity of the exponential map and thus it is a compact subset of U_qM. Since it is also convex in the above sense, we will deduce that it contains a pair of antipodal points by contradiction.

Suppose it does not. We claim that $\Lambda(q)$ is contained in an open hemisphere of U_qM. Indeed, there is a point e_0 in the complement such that $d(e_0, \Lambda(q))$ is maximum, where d denotes the geodesic distance on a euclidean sphere of unit radius. If $d(e_0, \Lambda(q)) < \frac{\pi}{2}$ and is attained at $e_1 \in \Lambda(q)$, then there is no other point $e_2 \in \Lambda(q)$ with $d(e_2, \Lambda(q)) = d(e_1, \Lambda(q))$, because otherwise, e_1, e_2 cannot be antipodal since $(e_1, e_0) = (e_2, e_0) > 0$ and the segment joining e_1 to e_2 is contained in $\Lambda(q)$. However, it contains points v with $d(v, e_0) < d(e_1, e_0) = d(e_0, \Lambda(q))$, a contradiction. Now, since there is no such e_2, it is easy to see that moving e_0 away from e_1 along the plane spanned by them, locally increases $d(e_0, \Lambda(q))$ which is contrary to the choice of e_0.

Now let us suppose $d(e_0, \Lambda(q)) = \dfrac{\pi}{2}$ and set $\Lambda'(q) := \Lambda(q) \cap e_0^\perp$ where

$$e_0^\perp := \{v \in U_qM : d(v, e_0) = \frac{\pi}{2}\}.$$

Then $\Lambda'(q)$ is also convex and compact, and we can assume inductively that it is contained in the open southern hemisphere of $e_0^\perp$ whose north pole we can assume to be a vector e_1. Then, moving e_0 towards e_1 along the segment in the plane spanned by them, also increases the distance from $\Lambda(q)$, again contradicting the definition of e_0.

Thus $\Lambda(q)$ is contained in the open hemisphere of U_qM with opposite pole e_0. Now using Lemma 2.1.3 we may write $e_0 = \lambda v_1 + \mu v_2$, for some $v_1 \in \Lambda(q), v_2 \in N(q)$ and $\lambda, \mu \geq 0$. Then $(v_1, e_0) = \lambda \|v_1\|^2 \geq 0$. This contradiction completes the proof of the lemma. ∎

Lemma 2.1.5 *For every $u \in \Lambda(q)$, there exists $v \in \Lambda(q)$ such that $(u, v) < 0$.*

Proof By Lemma 2.1.4 we can pick a point u_0 in $\Lambda(q)$ such that $-u_0$ is also contained in $\Lambda(q)$ and assume $(u, u_0) = 0$, for if not, then $(u, u_0) = -(u, -u_0)$ and we are done.

We assert that $d(p, \exp_q(\epsilon v)) < \dfrac{\pi}{2}$ for every v close to $-u$ and $\epsilon > 0$ small.

By Lemma 2.1.4 there is a smooth variation of geodesics in $seg(p, q)$ whose velocity vectors at q run through the great half circle defined by $u_0, -u_0$, and $-u$. In particular, along the geodesic γ in $seg(p, q)$ with $\dot\gamma(\frac{\pi}{2}) = -u_0$ there is a Jacobi field J with $J(0) = 0$, $J(\frac{\pi}{2}) = 0$ and $J'(\frac{\pi}{2}) = -u$. Now choose any vector field Z along γ with $Z(0) = 0$ and $Z(\frac{\pi}{2}) = -u$ and define for $\delta > 0$, small the one parameter family of curves

$$\gamma_{\delta,s}(t) = \exp_{\gamma(t)}(s(\delta Z - J)_{\gamma(t)}).$$

¿From the first variation of energy we have

$$\left.\frac{\partial}{\partial s}\right|_{s=0} E_\delta(s) = (\dot\gamma, \delta Z - J)|_0^{\frac{\pi}{2}} = 0$$

and

$$\left.\frac{\partial^2}{\partial s^2}\right|_{s=0} E_\delta(s) = I(\delta Z - J, \delta Z - J)$$

with I the index form along γ. Since J is a Jacobi field vanishing at 0 and $\dfrac{\pi}{2}$ we have $I(J, J) = 0$. We also have

$$I(J, Z) = (J'(\frac{\pi}{2}), Z(\frac{\pi}{2})) - (J'(0), Z(0)) = (-u, u) = -1$$

hence for δ small we have

$$I(\delta Z - J, \delta Z - J) = -2\delta + \delta^2 I(Z, Z) < 0$$

and thus $E_\delta(s) < E_\delta(0)$. This shows that for small $\delta > 0$ we have,

$$d(\exp_q(\delta(-u)), p) \leq L(\gamma_s) < \frac{\pi}{2}$$

and continuity of the index form (see Proposition 1.98 in [3]) shows $d(\exp_q(\delta v), p) < \frac{\pi}{2}$ for all $v \in U_qM$ close enough to $-u$, which finishes the proof of the assertion.

Now, this implies in turn that $-u$ is not in $N(q)$. Applying Lemma 2.1.3 to $-u$ we get $-u = av_1 + bv_2$ for some $a > 0$, $b \geq 0$, and $v_1 \in \Lambda(q)$, $v_2 \in \Lambda(q)$ with $(v_1, v_2) = 0$. Then $(-u, v_1) = a > 0$ and thus $(u, v_1) < 0$ which completes the proof. ∎

Proof of Proposition 2.1.1 We shall prove that $\Lambda(q)$ is a great sphere.

By Lemma 2.1.3, it is enough to show that for all u in $\Lambda(q)$, $-u$ is also in $\Lambda(q)$. If we have that, along with convexity of $\Lambda(q)$ as established above, we can assume a maximal set of linearly independent vectors $\{v_1, \ldots, v_k\}$ in $\Lambda(q)$ spanning a vector subspace V of T_qM, and it is immediate that $\Lambda(q) = V \cap U_qM$. We assume $S = V \cap U_qM$ where S is a maximal great subsphere contained in $\Lambda(q)$.

By Lemma 2.1.4, S is not empty, since there is one element of $\Lambda(q)$ such that its antipodal element is also in $\Lambda(q)$. We set $S^\perp := V^\perp \cap U_qM$. If $\Lambda(q) = S$, then we are done. If not, convexity implies $\Lambda(q) \cap S^\perp \neq \emptyset$.

Now by assumption, no pair of antipodal points is contained in $\Lambda^\perp(q) := S^\perp \cap \Lambda(q)$ and thus $\Lambda^\perp(q)$ is forced to be contained in an open hemisphere of $S^\perp$, as shown in the proof of Lemma 2.1.4. Let e be a north pole to that hemisphere, all points $u \in \Lambda^\perp(q)$ satisfying $(u, e) > 0$.

We note $(w, e) > 0$ for all points $w \in \Lambda(q) \setminus S$, for otherwise the great half sphere of dimension one greater than that of S and spanned by S and w would have a non trivial intersection with $S^\perp$, forced to be contained in $\Lambda^\perp(q)$ by convexity. However $(u, e) > 0$ for all $u \in \Lambda^\perp(q)$ as noted earlier, and this would be contradicting such an intersection.

In particular $(u, e) \geq 0$ for all $u \in \Lambda(q)$, and thus e cannot be in $\Lambda(q)$ by the last lemma. We choose $v_1 \in \Lambda^\perp(q)$ such that (v_1, e) is maximum, implied by compactness of $\Lambda^\perp(q)$.

We use the last lemma again to find $v_2 \in \Lambda(q)$ with $(v_1, v_2) < 0$ and since $v_2 \notin S$ we argue as above and use the half sphere spanned by v_2 and S to replace v_2 by $v_3 \in \Lambda^\perp(q)$ such that $(v_3, e) > 0$ and $(v_3, v_1) < 0$. We now let $\{v_1, v_0\}$ be obtained from $\{v_1, v_3\}$ by Gram-Schmidt and note $(v_0, e) > 0$ as well.

Next we consider the convex arc $v(t)$ with $v(0) = v_0$, $v(\frac{\pi}{2}) = v_1$ given by the formula

$$v(t) = \cos t v_0 + \sin t v_1, \qquad t \in [0, \frac{\pi}{2}]$$

contained in $\Lambda^\perp(q)$, since $\Lambda^\perp(q) = \Lambda(q) \cap S^\perp$ is convex, and it is a sub-arc of the convex arc spanned by $v_1, v_3 \in \Lambda^\perp(q)$. However the function $d(t) = (v(t), e)$ takes a maximum at some $t \in (0, \frac{\pi}{2})$, which contradicts maximality of (v_1, e) and finishes the proof. ∎

We note without proof that sphericity of $\Lambda(p, q)$ for all $q \in cut(p)$ is indeed equivalent to the pointed Blaschke condition, a fact proved in [3]. In fact, slight modification of the proof of the next proposition would allow not to use sphericity of $cut(p)$ and finally derive it as a corollary. This aspect is however not of great importance to us and we will proceed to derive further properties of pointed Blaschke manifolds. We begin with a definition.

Definition 2.1.6 *An SL^p_{2l} manifold is a Riemannian manifold such that all unit speed geodesics emanating from the point p return back at p in time $2l$ and are simple geodesic loops on the interval $[0, 2l]$. An SL_{2l} manifold is one that is SL^p_{2l} for all its points $p \in M$.*

Proposition 2.1.7 *If (M, g) is a Blaschke manifold at $p \in M$, then (M, g) is SL^p_{2l} for $l = d(p, cut(p))$. Moreover, all geodesic loops with length $2l$ emanating from p, have the same index, say k.*

Proof The loop property is trivial from Proposition 2.1.1, since $-v \in \Lambda(p, q)$, if $v \in \Lambda(p, q)$, for all $q \in cut(p)$.. All geodesic loops emanating from p return to p at time $2l = 2d(p, cut(p))$. They are simple until time $2l$, since a point of self-intersection would imply $Inj(p) < l$.

The last assertion is immediate from continuity of the index (see Proposition 1.98 [3]), which shows that this index is locally and thus globally constant (M is connected), once we show there are no conjugate points on the intervals $(0, l)$ and $(l, 2l)$. For the first interval this is immediate from $Inj(p) = l$, while for the second let J be a nonzero Jacobi field on γ, with $J(0) = J(t) = 0$ for some $t \in (l, 2l)$. The SL^p_{2l} property implies that $J(2l) = 0$ and $Inj(p) \leq 2l - t < l$, a contradiction. ∎

A Riemannian manifold M is called Allamigeon-Warner at a point p, named after the mathematicians who first studied them, if there exists $l > 0$ and an integer $k > 0$ such that for every unit speed geodesic emanating from p the first conjugate point to p at length l and has index k. So a Blaschke manifold at p is Allamigeon-Warner at the point p. Note that in case $k = 0$ this holds for $l = 2Inj(p)$ and the index is $n - 1$.

Let $k > 0$ and $S := \{v \in T_pM : \|v\| = l\}$ and $f := \exp_p|_S$. By Proposition 2.1.7, f is C^∞ and of constant rank $n - k - 1$.

Lemma 2.1.8 *For every $v \in S$ there is a neighbourhood V of v in S, such that $f(V)$ is a submanifold of dimension $n - k - 1$ of M.*

Proof This is immediate from the rank theorem. ∎

Now, we consider the k-dimensional subbundle of TS spanned by the k-planes $kerT_vf$, for $v \in S$, which is involutive and gives rise to a k-dimensional C^∞ foliation of S. We denote by $\phi(v)$ the maximal leaf containing $v \in S$ and let $\Phi := \{\phi(v) : v \in S)$ be the leaf space.

We first consider a given maximal leaf $\phi \in \Phi$. Its image $f(\phi)$ is a single point q in M and the inverse image $f^{-1}f(\phi) = f^{-1}(q)$ is compact being closed in S. Hence ϕ is compact as well, being a connected component of $f^{-1}(q)$. Note that if V is a neighbourhood of $v \in S$ in S, $f^{-1}f(V)$ is a union of maximal leaves, containing $\phi(v)$.

Lemma 2.1.9 *For $v \in \phi$, let $\gamma_v(t) = \exp_p(\frac{tv}{\|v\|})$. Then the map $\psi : \phi \to U_qM$ defined by $\psi(v) = \dot{\gamma}_v(l)$ maps ϕ diffeomorphically onto the great sphere $S' := (T_qf(V))^\perp \cap U_qM$, where $q = \gamma_v(l)$.*

Proof The map ψ is injective since a geodesic with given initial velocity vector is unique. Moreover, it is of maximal rank, because its tangent map can be described as follows: Let $w \in T_v\phi$, $\epsilon > 0$ and $v : (-\epsilon, \epsilon) \to \phi$ be any parametrised smooth curve with $v(0) = v$ and $\dot{v}(0) = w$. The Jacobi vector field J along γ_v with $J(0) = 0$ and $J'(0) = w$ is the variation field of the variation through geodesics of γ_v given by the formula $\Gamma(t, s) = \exp_p(tv(s))$ with $\Gamma(0, s) = \gamma_v$. Therefore, at $t = l$ we have

$$\frac{DJ}{dt}(l) = \frac{D}{dt}(\frac{\partial\Gamma}{\partial s})(l, 0) = \frac{D}{ds}(\frac{\partial\Gamma}{\partial t})(l, 0) = \left(\frac{D}{ds}\right)_{s=0}(\dot{\gamma}_{v(s)})(l) = \frac{d}{ds}\bigg|_{s=0}(\psi(v(s))) = \psi_{*v}(w).$$

But then $\psi_{*v}(w) \neq 0$ for $w \neq 0$ since a Jacobi field with $J(l) = 0$ and $J'(l) = 0$ must be identically zero. Note also that the image $\psi(\phi)$ is contained in $(T_q f(V))^{\perp}$ from the first variation formula. So, $\psi : \phi \to S'$ is a C^∞ injective map of maximal rank between compact manifolds, and is thus a diffeomorphism. ∎

We will now prove the following theorem due Allamigeon and Warner, which describes the structure of Allamigeon-Warner manifolds. We use only the Allamigeon-Warner condition and its implications, not assuming M to be necessarily a Blaschke manifold.

Theorem 2.1.10 *Let M be an Allamigeon-Warner n-manifold at p. Then the set Φ of maximal leaves has a natural structure of a C^∞ $(n-k-1)$-dimensional manifold. The canonical projection (quotient map) $\mathcal{P} : S \to \Phi$ which sends v to $\phi(v)$ makes S into a k-sphere bundle over Φ and f factors through an immersion $\mathcal{T} : \Phi \to M$ such that $f = \mathcal{T} \circ \mathcal{P}$. If on $\bar{B} = \overline{B(0,l)} := \{v \in T_pM : \|v\| \leq l\}$ we define the equivalence relation $\mathscr{R}$ by setting $v \mathscr{R} w$ if $v = w$ or $v, w \in S$ and are contained in the same maximal leaf, then $\hat{M} = \bar{B}/\mathscr{R}$ has a natural structure of a C^∞ n-dimensional manifold. Moreover, the topological structure of $\hat{M}$ can be described as follows: $\hat{M} = D \cup_a E$ where D is the n-dimensional disc, E is a C^∞ $(k+1)$-disc bundle over a C^∞ $(n-k-1)$-dimensional manifold, with boundary diffeomorphic to S^{n-1} and $a : \partial D \to \partial E$ an attaching diffeomorphism. Finally, the restriction $\exp_p\big|_{\bar{B}}$ factors through a map $\hat{\mathcal{T}} : \hat{M} \to M$ which is a covering map and satisfies $\exp_p\big|_{\bar{B}} = \hat{\mathcal{T}} \circ \hat{\mathcal{P}}$.*

$$\begin{tikzcd}
S \arrow[rr, hook] \arrow[dr, "f"] \arrow[dd, "\mathcal{P}"'] & & \bar{B} \arrow[dl, "\exp_p|_{\bar{B}}"'] \arrow[dd, "\hat{\mathcal{P}}"] \\
& M & \\
\Phi \arrow[ur, "\mathcal{T}"] \arrow[rr, hook] & & \hat{M} \arrow[ul, "\hat{\mathcal{T}}"']
\end{tikzcd}$$

Proof Since f is constant on the leaves, it factors as

$$\begin{tikzcd}
S \arrow[dr, "f"] \arrow[dd, "\mathcal{P}"'] & \\
& M \\
\Phi \arrow[ur, "\mathcal{T}"] &
\end{tikzcd}$$

Now let $v \in \phi$. Since ϕ is compact, $\|D\exp_p\|$ attains a positive minimum on the unitary normal bundle $UN_s\phi$ in S. The continuity of the exponential map implies that there is a neighbourhood V of v in S such that $\mathcal{T} : \mathcal{P}(V) \to f(V)$ is one-to-one. This gives the manifold structure of Φ.

Let ψ be the map of Lemma 2.1.9. We consider the diffeomorphism

$$(f,\psi) : \psi^{-1}\psi(V) \to UN_M f(V) = (Tf(V))^{\perp} \cap UM.$$

The commutative diagram

$$\begin{tikzcd}
S \supset \psi^{-1}\psi(V) \arrow[r, "{(f,\psi)}"] \arrow[d, "\mathcal{P}"] & UN_M f(V) \arrow[d, "\mathcal{P}^{\perp}"] \\
\Phi \supset \mathcal{P}\psi^{-1}\psi(V) \arrow[r, "\mathcal{T}"] & f(V)
\end{tikzcd}$$

gives the smooth k-sphere bundle structure of $\mathcal{P} : S \to \Phi$.

To pass to $\hat{M}$, let us consider the normal geodesic neighbourhood of radius ϵ of $f(V)$ in M, denoted $(f(V))_\epsilon$. We suppose that ϵ is small, so that $(f(V))_\epsilon = B(f(V), \epsilon)$ and the exponential map restricted on the open normal disc bundle of radius ϵ of $f(V)$ in M

$$\exp : \{v \in N_M f(V) : \|v\| < \epsilon\} \to (f(V))_\epsilon$$

is a diffeomorphism.

By compactness there are some $\epsilon > 0$ and a finite open cover of Φ consisting of open sets V such that $\bigcup\limits_V f^{-1}f(V) = S$ and we have diffeomorphisms onto $(f(V))_\epsilon$ as above. We then set $D = \bar{B}(0, l-\epsilon)$ and $E = (\bar{B} \setminus B(0, l-\epsilon))/\mathscr{R}$ which are the ones as claimed in the statement, as it is seen noting that $\hat{\mathcal{P}} : E \to \Phi$ is actually the mapping cone for the projection of the bundle $\mathcal{P} : S \to \Phi$.

The map $\hat{\mathcal{T}}$ is a covering since M is compact and $\hat{\mathcal{T}}$ has maximal rank everywhere. This holds on D from the Allamigeon-Warner condition and at Φ, it is implied by its manifold structure and Lemma 2.1.9. ∎

We now return to Blaschke manifolds. Note that in Proposition 2.1.7 it may happen that $\gamma(l)$ is not a conjugate point along γ (for example this is the case for the real projective spaces). In this case the first conjugate point appears at distance $2l = 2Inj(p)$ on all geodesics emanating from p, since M is SL^p_{2l} and M is Allamigeon-Warner for the index is $n-1$. We will nevertheless call this the $k = 0$ case. This is mainly to distinguish from the $k = n-1$ case, where the first conjugate point appears at distance $l = Inj(p)$. From now on the integer $k \geq 0$ will always denote the common index of unit speed geodesics emanating from p on the interval $[0, Inj(p)]$.

Corollary 2.1.11 *Let M be a Blaschke n-manifold at p and $k > 0$. Let also $l = Inj(p)$. Then, the cut locus $cut(p)$ is a $n-k-1$ dimensional submanifold of M and there exists a smooth k-sphere fibration $S \to cut(p)$ where S is the sphere of radius l in T_pM as denoted. If $k = n-1$, then M is homeomorphic to a sphere.*

Proof Since the exponential map $\exp_p$ is injective on $B = B(0, l)$, the proof of Theorem 2.1.10 shows that the covering $\hat{\mathcal{T}} : \hat{M} \to M$ constructed there is actually a diffeomorphism, also showing $cut(p) = f(s)$. The other assertions follow immediately from Theorem 2.1.10, while for the last $M \approx \bar{B}/\mathscr{R} \approx S^n$. ∎

Now we treat the case $k = 0$.

Proposition 2.1.12 *Let M be a Blaschke manifold at p and $k = 0$. Consider the Riemannian universal cover $\tilde{\pi} : \tilde{M} \to M$ and $\tilde{p} \in \tilde{M}$ be a lift of p. Then, $\tilde{M}$ is a Blaschke manifold at $\tilde{p}$ with $Inj(\tilde{p}) = 2l$, where $l = Inj(\phi)$. Moreover, M is homeomorphic to $\mathbb{RP}^n$ and its cut locus is a $n-1$ dimensional submanifold diffeomorphic to $\mathbb{RP}^{n-1}$.*

Proof First we see that $\Lambda(p, q)$ consists of two antipodal points, since it is a great sphere and $dim\Lambda(p, q) \leq k = 0$. As noticed earlier our manifold is an Allamigeon-Warner manifold at p for the distance $2l$ and for $k = n-1$. We observe that M is not simply connected. If this was the case then $\hat{\mathcal{T}} : \hat{M} \to M$ would be a diffeomorphism and $cut(p) = p$, a contradiction.

We will show that all cut values of $\tilde{M}$ at $\tilde{p}$ are exactly $2l$, establishing the first assertion. A unit speed geodesic $\tilde{\gamma}$ emanating from $\tilde{p}$ projects down to M on a geodesic

γ and since M is SL^p_{2l}, there is a nonzero Jacobi field J along γ with $J(0) = J(2l) = 0$. We can lift it to a Jacobi field along $\tilde{\gamma}$, showing that all cut valus of $\tilde{M}$ at $\tilde{p}$ are $\leq 2l$. For the other inequality we claim that all unit speed geodesics $\tilde{\gamma}$ emanating from $\tilde{p}$ are minimizing for $t < 2l$, which will prove that all the cut points appear at distance $\geq 2l$ and therefore the cut locus of $\tilde{p}$ is indeed spherical and $d(\tilde{p}, cut(\tilde{p})) = 2l$. Indeed, let $q \in M \setminus \{p\}$. Since M is SL^p_{2l}, any geodesic of length $r \leq 2l$ joining p and q in M gives rise to a second geodesic joining those points, of nonzero length $2l - r < 2l$. Now we conclude that there are exactly two geodesics of length $\geq 2l$ joining p and q in M, namely any segment of length $r = d(p, q) \leq l$ and the corresponding geodesic of length $2l - r$. Indeed if q is not in $cut(p)$, a third geodesic would imply the existence of two different geodesics joining p and q of length less than $l = Inj(p)$, a contradiction. For $q \in cut(p)$ any such geodesic must be of length l for otherwise the SL^p_{2l} property gives one with length $< l$. So we get exactly two geodesics in this case as well, since $\Lambda(p, q)$ consists of two antipodal points. However, different lifts of q in $\tilde{M}$ correspond to different segments joining $\tilde{p}$ to each of these lifts. These are necessarily of length $\leq 2l$ and therefore project down to M to geodesics of length $\leq 2l$ joining p and q, which we counted to be exactly two. Thus there can be at most two such lifts and exactly two since M if not simply connected. Moreover there is exactly one geodesic of length $\leq 2l$ joining $\tilde{p}$ to each of these lifts, showing the minimality of all geodesics of length $< 2l$ emanating from p and establishing the Blaschke property of $\tilde{M}$ at $\tilde{p}$.

For this Blaschke manifold, we have $k = n - 1$ and is therefore homeomorphic to S^n by Corollary 2.1.11. In fact, points $q \in cut(p)$ correspond to points $\tilde{q}$ with $d(\tilde{p}, \tilde{q}) = l$ and this shows $cut(p)$ is a smooth, free $\mathbb{Z}_2$ quotient of the $n-1$ sphere $\exp_{\tilde{p}}^{-1}\{\tilde{q} : d(\tilde{p}, \tilde{q}) = l\}$ and is therefore diffeomorphic to a projective space $\mathbb{RP}^{n-1}$. ∎

In fact much more can be proved concerning pointed Blaschke manifolds and we quote without proof the following important theorem of Bott and Samelson (see [4], [16]).

Theorem 2.1.13 *Let (M^n, g) be a pointed Blaschke manifold for $n \geq 2$ and $k \geq 0$ be as before. Then if $k > 0$, M is simply connected and the integral cohomology ring of M is generated by a single element. More precisely, one has only the following possibilities:*

(i) $k = 1$, $n \in 2\mathbb{Z}$, and M has the homotopy type of $\mathbb{C}P^{n/2}$.

(ii) $k = 3$, $n \in 4\mathbb{Z}$ and M has the integral cohomology ring of the quaternionic projective space $\mathbb{H}P^{n/4}$.

(iii) $k = 7$, $n = 16$, and M has the integral cohomology ring of the Cayley projective plane $\mathbb{C}aP^2$.

(iv) $k = n - 1$, any n and M is homemorphic to a sphere.

(v) $k = 0$, any n and M is diffeomorphic to $\mathbb{RP}^n$.

The Riemannian manifolds appearing in the Bott-Samelson theorem are actually Blaschke, i.e. Blaschke manifolds at all their points. This motivates us to say that a Blaschke manifold is modelled on one of the above manifolds if it fits the corresponding description in terms of k. The above spaces are thus the models for Blaschke manifolds.

Note that for $n = 2, 4$ case (iv) above has the same description in terms of n and k, with the cases (i), (ii) respectively. This causes no problem however, since the corresponding manifolds also coincide (up to isometry with respect to the standard metrics).

2.2 Blaschke's conjecture

We move now to Blaschke manifolds and the Blaschke Conjecture. First we will prove that Blaschke manifolds are exactly the ones that are pointed Blaschke manifolds at all their points. We need a proposition, which is of independent interest. We continue to use the notations of the last section 2.1.

Proposition 2.2.1 *Let (M, g) be a Blaschke manifold at each point. Then, for every $p \in M$, the fibers of the fibration*

$$\exp_p : S \to cut(p)$$

are great spheres. Moreover, each geodesic loop γ of length $2l$ is closed at $p = \gamma(0)$, i.e. $\dot\gamma(0) = \dot\gamma(2l)$ where l is the cut value at p.

Proof For every $q \in cut(p)$ we have

$$\exp_p^{-1}(q) = \{lv : v \in U_pM, \exp_p lv = q\} = l\Lambda(q,p)$$

which is a great sphere in U_pM since M is Blaschke at q. For the other assertion, let γ be a unit speed geodesic with $\gamma(0) = p$ and $\dot\gamma(0) \neq \dot\gamma(2l)$. The continuity of the cut value implies continuity of l as a function of p. Now for $\epsilon > 0$ small we consider $q = \gamma(\epsilon)$. By Proposition 2.1.7, γ is simple and thus $\gamma(t) \neq q$ for $t \in [0, 2l] \setminus \{\epsilon\}$. The fact that M is Blaschke at q and the continuity of the cut value, imply the existence of a small $\delta > 0$ such that $\gamma(2l + \delta) = \gamma(\epsilon)$, since the geodesic $\gamma(t + \epsilon)$ emanating from q must be a loop with length $2l(q)$. But this is a contradiction, since $\dot\gamma(0) \neq \dot\gamma(2l)$ implies $\gamma|_{[0,\epsilon]}$ and $\gamma|_{[2l,2l+\delta]}$ are two geodesic segments joining p and q, of small length and with different initial vectors. ∎

Proposition 2.2.2 *A Riemannian manifold is Blaschke if and only if it is Blaschke at all points.*

Proof Only the converse statement needs proof. Let M be Blaschke at all points and p, $q \in M$. There is a segment from p to q which extends to a simple closed geodesic on $[0, 2l(p)]$, by Proposition 2.2.1, where $l(p)$ denotes the cut value at p. Shifting the origin of this geodesic to be q rather than p we obtain $l(p) = l(q)$. This shows that the cut value is constant. Therefore,

$$InjM = \inf_{p \in M} l(p) = \sup_{p \in M} l(p) = diamM$$

which shows M is a Blaschke manifold. ∎

The Blaschke Conjecture states that a Blaschke manifold should be isometric to its model with its standard Riemannian metric (up to constant rescaling).

Since the main subject of this survey is to present in detail the proof of the spherical case, let us first show that this case also implies the $\mathbb{RP}^n$ case.

Proposition 2.2.3 *If the S^n case of the Blaschke conjecture is true, then so is the $\mathbb{RP}^n$ case.*

Proof Let g be a Blaschke metric on $\mathbb{RP}^n$. From Proposition 2.1.12, the Riemannian universal cover is a Blaschke manifold at all points, and is a Blaschke manifold, by Proposition 2.2.2. So, if the spherical Blaschke Conjecture is true, it is isometric to the euclidean n-sphere (up to a constant factor). This implies that g is isometric to the standard metric on $\mathbb{RP}^n$. ∎

Chapter 3

The volume of manifolds with closed geodesics

In this chapter we shall prove the result of A. Weinstein (in the even dimensional case) and C.T. Yang (in the odd dimensional case) that the volume of a Blaschke manifold modelled on the sphere equals the volume of the standard sphereof the same dimension and diameter. We will follow and analyse the original papers [21] and [22]. Historically, this was the first step, topological in nature as we will see below, towards the resolution of the Blaschke conjecture in the spherical case. The proof of the affirmative answer consists of the above result combined with the analytic in nature Berger-Kazdan isoembolic inequality presented in chapter 1, from which we get that equality of volumes is possible only in the case of isometry.

3.1 The manifold of geodesics

A Blaschke manifold (M, g) has the property that all unit speed geodesics are simple closed loops of length $2InjM$. This condition as well as a weaker version of it are important in the topological study of Blaschke manifolds which follows and therefore deserves a special name.

Definition 3.1.1 *A C_{2l}^p manifold is a Riemannian manifold such that all unit speed geodesics through the point p are closed and with length $2l$. A SC_{2l}^p manifold is a C_{2l}^p manifold but we also demand that all geodesics are also simple on* $[0, 2l]$. *We say that (M, g) is a C_{2l} or SC_{2l} manifold if it is C_{2l}^p or SC_{2l}^p for all $p \in M$, respectively.*

Let (M, g) be a $C_{2\pi}$-manifold. The geodesic vector field Z generates a free action of $SO(2)$ on the unit tangent bundle UM, defined by

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot v = \zeta^t v, \qquad v \in UM,$$

where $(\zeta^t)_{t \in \mathbb{R}}$ is the geodesic flow.

The quotient $UM/SO(2)$ is a compact $2n-2$ dimensional manifold and the projection

$$\pi : UM \to UM/SO(2)$$

is a principal fiber bundle with structure group $SO(2)$ (see [6], Chapter II, Theorem 5.4). The manifold $CM = UM/SO(2)$ is called the manifold of oriented geodesics.

Proposition 3.1.2 *The contact form $\eta \in \Lambda^1 UM$ is a connection form for the principal $SO(2)$-bundle $\pi : UM \to CM$ and $d\eta$ is the curvature of this connection.*

Proof We define a horizontal distribution Q on TUM. For $v \in UM$ let

$$Q_v := \{\xi \in T_vUM | \eta(\xi) = 0\} = \{\xi \in T_v(UM) | \bar{g}(\xi, Z) = 0\}$$

where $\bar{g}$ is the Sasaki metric.

Then differentiability of $\bar{g}$ and Z implies a differentiable $\bar{g} - orthogonal$ splitting $TUM = \mathbb{R}Z \oplus Q$. Moreover, $Q = ker\eta_v$ is $SO(2)$ invariant.

Thus Q is a connection on this principal bundle, and since $\eta(Z) = 1$, η is the associated connection form. Then the curvature form of this connection is $d\eta$ since $SO(2)$ is abelian and thus $d\eta$ is horizontal. ∎

The tangent space of the manifold of oriented geodesics at a point $\gamma \in CM$ is identified with the horizontal space Q_v, with $\gamma = \pi(v)$. We now prove that tangent vectors at γ naturally correspond to normal Jacobi fields along γ. We first prove a lemma.

Lemma 3.1.3 *Let $v, u \in UM$ and $\pi(u) = \pi(v) = \gamma$. Then $u = \zeta^t v$ for some $t \in \mathbb{R}$, so that $\gamma_v(t) = \gamma_u(0)$. Shifting time by t, we get a map $\sigma^t : \mathcal{J}_v \to \mathcal{J}_u$ which restricts to $\sigma^t : \mathcal{J}_v^\perp \to \mathcal{J}_u^\perp$. There is a commutative diagram of isomorphisms.*

$$\begin{tikzcd} Q_v \arrow[r, "j_v"] \arrow[d, "\zeta_*^t"] & \mathcal{J}_v^\perp \arrow[d, "\sigma^t"] \\ Q_u \arrow[r, "j_u"] & \mathcal{J}_u^\perp \end{tikzcd}$$

In particular, σ^t is independent of t, and we have a natural identification $\mathcal{J}_v^\perp \cong \mathcal{J}_u^\perp$. In view of this we will write $\mathcal{J}_\gamma^\perp$ to denote the space of normal Jacobi fields along γ.

Proof By definition, $Q_v = T_vUM \cap ker\eta_v$ and the horizontal isomorphisms are thus a consequence of Proposition 1.2.1. Let now $\xi \in Q_v$. We compute (with the notation of section 1.2):

$$J_\xi(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \tau \circ \zeta^t(z(s)) = \tau_*(\zeta_*^t(\dot{z}(0)) = \tau_*\zeta_*^t\xi = J_{\zeta_*^t\xi}(0)$$

and

$$J'_\xi(t) = \frac{D}{dt}\frac{\partial F}{\partial s}(0,t) = \frac{D}{ds}\frac{\partial F}{\partial t}(0,t) = \frac{D}{ds}\bigg|_{s=0} \tau_* Z(\zeta^t z(s))$$
$$= \frac{D}{ds}\bigg|_{s=0} (\zeta^t z(s)) = K(J_*^t\xi) = J'_{\zeta_*^t\xi}(0).$$

Therefore, the Jacobi fields J_ξ and $J_{\zeta_*^t\xi}$ have the same initial conditions at the point $\gamma_u(0) = \gamma_v(t)$, which implies commutativity of the diagram. Now let $u = \zeta^s v$ for some $s \in \mathbb{R}$. Since M is $C_{2\pi}$ we must have $s = t + 2k\pi, k \in \mathbb{Z}$ and therefore $\zeta^t = \zeta^s : UM \to$

UM for the same reason, and the left arrow of the diagram is independent of t. The last statement follows and therefore we get $\mathcal{J}_\gamma^\perp$, the space of (nonparametrized) normal Jacobi fields along γ. ∎

Proposition 3.1.4 *Let $\gamma \in CM$. With the notation above, there is a natural identification:*

$$T_\gamma CM \cong \mathcal{J}_\gamma^\perp$$

Proof Let $v \in \pi^{-1}(\gamma)$. The obvious fact $\pi_* \zeta_*^t = \pi_*$ and the previous lemma imply that the sequence of isomorphisms

$$T_\gamma CM \xrightarrow{\pi_*^{-1}} Q_v \xrightarrow{j_v} \mathcal{J}_v^\perp \to \mathcal{J}_\gamma^\perp$$

does not depend on the choice of $v \in UM$ and identifies the tangent space $T_\gamma CM$ to the space $\mathcal{J}_\gamma^\perp$, as stated. ∎

3.2 The volume of spheres with closed geodesics

Let (M,g) be a $C_{2\pi}$-manifold. In what follows we shall use the notations and conventions of the previous section 3.1.

From Proposition 3.1.2, we see that $d\eta$, or $\begin{pmatrix} 0 & d\eta \\ -d\eta & 0 \end{pmatrix}$ in $\mathfrak{so}(2)$ notation, is horizontal and invariant under the $SO(2)$ action $\mathcal{L}_Z d\eta = d\mathcal{L}_Z \eta = 0$ on UM. Thus it is the pullback of some two form in CM. By the standard theory of the Euler class, if we set

$$\pi^*(\Omega) = \frac{1}{2\pi} Pf \begin{pmatrix} 0 & d\eta \\ -d\eta & 0 \end{pmatrix} = \frac{1}{2\pi} d\eta$$

then Ω is closed and represents a real cohomology class in CM which is the image of the Euler class

$$e \in H^2(CM, \mathbb{Z})$$

of the principal $SO(2)$-bundle $\pi : UM \to CM$ under the coefficient homomorphism

$$H^2(CM, \mathbb{Z}) \to H^2(CM, \mathbb{R}).$$

Proposition 3.2.1 *The 2-form Ω is nondegenerate on CM and since it is also closed, it defines a symplectic structure on CM. In particular, CM is orientable and Ω^{n-1} is a nowhere vanishing $2n-2$ form. Moreover,*

$$j(M,g) := \int_{CM} \Omega^{n-1} = \langle e^{n-1}, [CM] \rangle$$

where $e \in H^2(CM, \mathbb{Z})$ is the Euler class of the fibration $\pi : UM \to CM$ and $\langle \cdot, \cdot \rangle$ is the duality pairing.

Proof The nondegeneracy of Ω is immediate since $d\eta$ is nondegenerate on the horizontal distribution. Thus Ω is a closed and nondegenerate two-form on the $2n-2$ dimensional manifold CM, and thus a symplectic structure, which also implies orientability. The commutative diagram:

$$\begin{tikzcd} H^{2n-2}(CM,\mathbb{Z}) \arrow[r] \arrow[d, "\cap[CM]"', "\simeq"] & H^{2n-2}(CM,\mathbb{R}) \arrow[d, "\int_{CM}"', "\simeq"] \\ H_0(CM,\mathbb{Z}) = \mathbb{Z} \arrow[r] & \mathbb{R} \end{tikzcd}$$

where the upper horizontal arrow stands for the coefficient homomorphism followed by the de Rham isomorphism, yields that

$$\int_{CM} \Omega^{n-1} = \langle e^{n-1}, [CM] \rangle$$

and thus $j(M,g)$ is an integer topological invariant of the fibration $\pi : UM \to CM$. ∎

We next relate the volume of (M,g) to the integer $j(M,g)$.

Proposition 3.2.2 $j(M,g) = \dfrac{2Vol(M,g)}{Vol(S^n, can)}$.

Proof The Liouville measure on UM is induced by the contact 1-form η and is represented by the $2n-1$-form

$$dUM_{\bar{g}} = \frac{1}{(n-1)!}\eta \wedge d\eta^{n-1}.$$

We compute:

$$\begin{aligned} Vol(UM,\bar{g}) &= \frac{1}{(n-1)!}\int_{UM} \eta \wedge d\eta^{n-1} \\ = \frac{(2\pi)^{n-1}}{(n-1)!}\int_{UM} \eta \wedge \pi^*\Omega^{n-1} &= \frac{(2\pi)^{n-1}}{(n-1)!}\int_{\gamma\in CM} \left(\int_{\pi^{-1}\gamma} \eta\right)\Omega^{n-1} \\ &= \frac{(2\pi)^n}{(n-1)!}\int_{CM} \Omega^{n-1} = \frac{(2\pi)^n j(M,g)}{(n-1)!} \end{aligned}$$

where in the third equality we have used the Fubini formula for integration over the fiber, while for the fourth we have

$$\int_{\pi^{-1}\gamma} \eta = \int_0^{2\pi} \eta(Z_{\dot{\gamma}(t)})dt = 2\pi.$$

Now we use $Vol(UM,\bar{g}) = Vol(S^{n-1}, can)Vol(M,g)$ and obtain:

$$j(M,g) = \frac{(n-1)!Vol(S^{n-1}, can)}{(2\pi)^n} Vol(M,g) = \frac{2Vol(M,g)}{Vol(S^n, can)}$$

where for the last equality we have used the well known formula

$$Vol(S^n, can)Vol(S^{n-1}, can) = 2\frac{(2\pi)^n}{(n-1)!}. \quad ∎$$

We have thus proved that the volume of a $C_{2\pi}$-manifold is actually a half-integer multiple of the volume of the standard sphere:

$$Vol(M,g) = \frac{j(M,g)}{2} Vol(S^n, can).$$

If we prove that $j(M,g) = 2$, then the spherical case of Blaschke's conjecture will follow. In other words, the proof of Blaschke's conjecture for the spheres is reduced to proving the following theorem to which the rest of this section is devoted.

Theorem 3.2.3 *If M is a Blaschke manifold homeomorphic to the n-sphere, then $j(M,g) = 2$.*

First we need to compute the (integral) cohomology of UM, when M is a n-sphere. A standard reference for the topological tools used below is [5].

Since M is orientable, we have a Gysin sequence for the fibration $\tau : UM \to M$:

$$\cdots \to H^{k-1}(UM) \to H^{k-n}(M) \xrightarrow{\cup \bar{e}} H^k(M) \xrightarrow{\tau *} H^k(UM) \to H^{k-n+1}(M) \to \cdots$$

where $\bar{e}$ is the Euler class of the bundle $\tau : UM \to M$. Now $\langle \bar{e}, [M] \rangle = \chi(M)$, the Euler characteristic, and thus $\bar{e}$ is zero or the double of the generator of $H^n(M)$, depending on whether n is odd or even.

Now, UM is a compact and connected $(2n-1)-$ dimensional orientable manifold and thus we have

$$H^0(UM) \cong H^{2n-1}(UM) \cong \mathbb{Z}$$

We consider the even and odd dimensional cases separately.

Let n be even. For $k = n$ the Gysin sequence gives the short exact sequence

$$\mathbb{Z} \xrightarrow[\text{by } 2]{\text{multiplication}} \mathbb{Z} \xrightarrow{\tau^*} H^n(UM) \to 0$$

from which follows that $H^n(UM) \cong \mathbb{Z}_2$. If $n < k < 2n-1$, then $1 < k-(n-1) < n$ and so $H^k(M) = H^{k-(n-1)}(M) = 0$. Therefore $H^k(UM) = 0$. Similarly, we set $H^k(UM) = 0$ for $0 < k < n-1$. For $k = n$, the Gysin sequence gives the exact sequence

$$0 \to H^{n-1}(UM) \to \mathbb{Z} \xrightarrow[\text{by } 2]{\text{multiplication}} \mathbb{Z}$$

and so $H^{n-1}(UM) = 0$.

Let n be odd. For $k = n-1$ we have the exact sequence

$$0 \to H^{n-1}(UM) \to \mathbb{Z} \to 0$$

which means that $H^{n-1}(UM) \cong \mathbb{Z}$. For $k = n$ we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\tau^*} H^n(UM) \to 0$$

and so $\tau^* : \mathbb{Z} \cong H^n(UM)$. In all other cases, it is obvious that $H^k(UM)$ vanishes.

Summarizing, we conclude that if n is even, then

$$H^k(UM) = \begin{cases} \mathbb{Z}, & \text{for } k = 0,\ 2n-1 \\ \mathbb{Z}_2, & \text{for } k = n \\ 0, & \textit{otherwise}, \end{cases}$$

and if n is odd, then

$$H^k(UM) = \begin{cases} \mathbb{Z}, & \text{for } k = 0,\ n-1,\ n,\ 2n-1 \\ 0, & \text{otherwise.} \end{cases}$$

We are now ready to study the cohomology of the circle bundle $\pi : UM \to CM$.

Proposition 3.2.4 *Let M be a $C_{2\pi}$-manifold homeomorphic to the n-sphere, $n \geq 2$.*
(a) If n even, then

$$H^k(CM) = \begin{cases} \mathbb{Z}, & \textit{for } k = 0, 2, 4, \ldots, 2n-2 \\ 0, & \textit{otherwise} \end{cases}$$

Moreover, the homomorphism $H^{k-2}(CM) \xrightarrow{\cup e} H^k(CM)$ appearing in the Gysin sequence for $\pi : UM \to CM$ is an isomorphism for $k = 2, \ldots, n-2, n+2, \ldots, 2n-2$ and a monomorphism of cokernel $\mathbb{Z}_2$ for $k = n$.
(b) If n is odd, then

$$H^k(CM) = \begin{cases} \mathbb{Z}, & \textit{for } k = 0, 2, 4, \ldots, 2n-2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \textit{for } k = n-1 \\ 0, & \textit{otherwise} \end{cases}$$

Moreover there are exact sequences:

$$0 \to H^{n-3}(CM) \xrightarrow{\cup e} H^{n-1}(CM) \xrightarrow{\pi^*} H^{n-1}(UM) \to 0$$
$$0 \to H^n(UM) \to H^{n-1}(CM) \to H^{n+1}(CM) \to 0$$

which are parts of the Gysin sequence of the fibration π.

Proof We use the Gysin sequence of the oriented circle bundle $\pi : UM \to CM$:

$$\cdots H^{k-1}(UM) \to H^{k-2}(CM) \xrightarrow{\cup e} H^k(CM) \xrightarrow{\pi^*} H^k(UM) \to \cdots$$

and the previous calculations.
First of all we have:

$$H^0(CM) \cong H^{2n-2}(CM) \cong \mathbb{Z}$$

(a) Suppose that n is even. If $0 < k < n$, from the Gysin sequence we get the exact sequence

$$0 \to H^{k-2}(CM) \xrightarrow{\cup e} H^k(CM) \to 0.$$

It follows inductively that $H^k(CM) \cong \mathbb{Z}$ for even $0 \leq k < n$ and $H^k(CM) = 0$ for odd $0 < k < n$.

If $n < k < 2n-2$ is even, then we have a exact sequence

$$0 \to H^k(CM) \xrightarrow{\cup e} H^{k+2}(CM) \to 0$$

and so $H^k(CM) \cong \mathbb{Z}$, inductively. Obviously, $H^{n+1}(CM) = H^{2n-1}(CM) = 0$, from the Gysin sequence and what we have proved already. Also, $H^{2n-3}(CM) = H^1(CM) = 0$, by duality and the Universal Coefficient Theorem. Inductively, as above, we see that $H^k(CM) = 0$ for odd $n < k < 2n-2$.

Finally, in the case $k = n$ we get the exact sequence

$$0 \to H^{n-2}(CM) \xrightarrow{\cup e} H^n(CM) \to \mathbb{Z}_2 \to 0$$

where $H^{n-2}(CM) \cong \mathbb{Z}$. Thus, we have two possibilities: $H^n(CM) \cong \mathbb{Z}$ or $H^n(CM) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. But $H^n(CM) \cong H_{n-2}(M)$, by duality, and the torsion part of $H_{n-2}(CM)$ is isomorphic to the torsion part of $H^{n-1}(M) = 0$, by Universal Coefficients. So the latter is excluded and $H^n(CM) \cong \mathbb{Z}$.

(b) Let now n be odd. As in case (a), inductively we see that $H^k(CM) \cong \mathbb{Z}$ for even $0 \le k \le 2n$ with $k \ne n-1$ and $H^n(CM) = 0$ for k odd.

For $k = n-1$ we have the split short exact sequence

$$0 \to H^n(UM) \to H^{n-1}(CM) \xrightarrow{\cup e} H^{n+1}(CM) \to 0$$

and therefore $H^{n-1}(CM) \cong \mathbb{Z} \oplus \mathbb{Z}$, because $H^{n-3}(CM) \cong \mathbb{Z}$, since n is odd. ∎

Proof of Theorem 3.2.4, when n is even. The previous lemma shows that:

$$(\cup e)^{n-1} : H^0(CM) \to H^{2n-2}(CM)$$

is a monomorphism of cokernel $\mathbb{Z}_2$ and thus $\langle e^{n-1}, [CM] \rangle = 2$. ∎

Now we move on to investigate the case when n is odd. Recall our assumptions that M is an $n-$dimensional $C_{2\pi}$ manifold, with $n = 2m+1$, homeomorphic to the sphere. We make a further assumption on (M, g) which is certainly satisfied for Blaschke manifolds. Namely that there is a point $p \in M$, such that for all $\gamma \in CM$, p is not a point of self-intersection for γ.

Since $\cup e : H^{2k}(CM) \cong H^{2k+2}(CM)$ for $0 \le k < m-1$, it follows that e^{m-1} is a generator of $H^{2m-2}(CM)$. From the exact sequence

$$0 \to H^{2m-2}(CM) \xrightarrow{\cup e} H^{2m}(CM) \xrightarrow{\pi^*} H^{2m}(UM) \to 0$$

there exists $b \in H^{2m}(CM)$ such that $\pi^*(b)$ is a generator of $H^{2m}(UM) \cong \mathbb{Z}$ and $\{b, e^m\}$ is a basis of $H^{2m}(CM)$. We will construct a suitable element b which which make the calculation of $j(M, g)$ easy.

First we need some preparation in order to work geometrically with homology and intersection numbers rather than cohomology and cup products.

We derive a homology Gysin sequence which is Poincare dual to the one in cohomology. Let $\pi_D : DM \to CM$ be the disc bundle associated to the principal circle bundle $\pi : UM \to CM$, so that $UM = \partial DM$. We write $i : CM \to DM$ for the zero section and $j : UM \to DM$ and $k : DM \to (DM, UM)$ for the inclusions. Let also

$u \in H^2(DM, UM)$ be the Thom class of the bundle and $e \in H^2(CM)$ be its Euler class as before. Using the homology and cohomology versions of the Thom isomorphism

$$\begin{aligned} T_C : H^{2m}(CM) &\to H^{2m+2}(DM, UM) \\ b &\mapsto (\pi_D^* b) \cup u \end{aligned}$$

$$\begin{aligned} T_H : H_{2m}(DM, UM) &\to H_{2m-2}(CM) \\ z &\mapsto \pi_{D*}(u \cap z) \end{aligned}$$

we derive the commutative (modulo sign) diagram

$$\begin{tikzcd} & & H^{2m}(CM) \arrow[r, dashed, "\cup e"] \arrow[d, "\cong"', "T_C"] & H^{2m+2}(CM) \arrow[d, "\cong"', "\pi_D^*"] & \\ 0 \arrow[r] & H^{2m+1}(UM) \arrow[r] \arrow[d, "\cong"', "\cap[UM]"] & H^{2m+2}(DM, UM) \arrow[r, "k^*"] \arrow[d, "\cong"', "\cap[DM]"] & H^{2m+2}(DM) \arrow[r] \arrow[d, "\cong"', "\cap[DM]"] & 0 \\ 0 \arrow[r] & H_{2m}(UM) \arrow[r, "j_*"'] \arrow[dr, "\pi_*"'] & H_{2m}(DM) \arrow[r] \arrow[d, "\cong"', "\pi_{D*}"] & H_{2m}(DM, UM) \arrow[r] \arrow[d, "\cong"', "T_H"] & 0 \\ & & H_{2m}(CM) \arrow[r, "\psi"'] & H_{2m-2}(CM) & \end{tikzcd}$$

Thus, since the upper part of the diagram is the standard derivation of the Gysin sequence, we will have the sequence we need once we show that the two vertical compositions of isomorphisms represent Poincare duality. Let $b \in H^*(CM)$. On the one hand,

$$\begin{aligned} \pi_{D*}(T_C b \cap [DM]) &= \pi_{D*}\Big((\pi_D^*(b) \cup u) \cap [DM]\Big) \\ &= \pi_{D*}\Big(\pi_D^* b \cap (u \cap [DM])\Big) \\ &= \pi_{D*}(\pi_D^* b \cap i_*[CM]) \\ &= b \cap [CM] \end{aligned}$$

and on the other hand,

$$\begin{aligned} T_H(\pi_D^* b \cap [DM]) &= \pi_{D*}\Big(u \cap (\pi_D^* b \cap [DM])\Big) \\ &= \pi_{D*}\Big(\pi_D^* b \cap (u \cap [DM])\Big) \\ &= b \cap [CM]. \end{aligned}$$

Note that since the upper horizontal arrow is cup product with the Euler class e, by duality the lower horizontal arrow must represent cap product with e. Thus we have derived the following Poincare duality commutative (modulo sign) diagram

$$\begin{tikzcd} 0 \arrow[r] & H^{2m+1}(UM) \arrow[r, "\text{Gysin map}"] \arrow[d, "\cong"', "\cap[UM]"] & H^{2m}(CM) \arrow[r] \arrow[d, "\cong"', "\cap[CM]"] & H^{2m+2}(CM) \arrow[r] \arrow[d, "\cong"', "\cap[CM]"] & 0 \\ 0 \arrow[r] & H_{2m}(UM) \arrow[r, "\pi_*"] & H_{2m}(CM) \arrow[r] & H_{2m-2}(CM) \arrow[r] & 0 \end{tikzcd}$$

It follows that the Gysin map $H^{2m+1}(UM) \to H^{2m}(CM)$ corresponds under Poincare duality to the induced homomorphism $\pi_* : H_{2m}(UM) \to H_{2m}(CM)$.

Lemma 3.2.5 *Consider the exact sequence*

$$0 \to H^{2m+1}(UM) \xrightarrow[map]{Gysin} H^{2m}(CM) \xrightarrow{\cup e} H^{2m+2}(CM) \to 0$$

and let $a \in H^{2m}(CM)$ be a generator of the image of the Gysin map. Then $a \cup a = 2g$ for some generator $g \in H^{4m}(CM)$.

Proof We will work in homology, so that we can make use of geometric data to compute intersection rather than cup products. Since cup and intersection product correspond under duality, we have $(a \cup a) \cap [CM] = a \cap a^*$ where a^* is the corresponding generator of the image of $\pi_* : H_{2m}(UM) \to H_{2m}(CM)$ to a, i.e. $a \cap [CM] = a^*$.

We are going to prove that the intersection product $a \cap a^* = 2$ which is of course equivalent to the original statement. By hypothesis, there is a point $p \in M$ such that any closed geodesic in M doesn't have p as a point of self-intersection.

Now, an easy compactness argument shows that there is a neighbourhood (geodesic ball) V of p in M, satisfying:

1. for all $q \in V$, q is not a point of self-intersection of any geodesic in M.

2. The intersection of V with each geodesic emanating from p to a single open arc.

We consider the homology Gysin sequence for the bundle $\tau : UM \to M$, derived as above,

$$H_n(UM) \to H_n(M) \xrightarrow{\cap \bar{e}} H_0(M) \to H_{n-1}(UM) \to 0$$

and we use that the map $\cap \bar{e} : H_0(M) \to H_{n-1}(UM)$ is zero in the odd dimensional case, where $\bar{e}$ is the Euler class of UM. This gives the isomorphism of the upper horizontal arrow of the commutative diagram:

$$\begin{tikzcd} H_0(M) \arrow[r] \arrow[d, "T_H"', "\cong"] & H_{2m}(UM) \arrow[d, "\cong"] \\ H_n(TM, TM \setminus M) \arrow[r, "\partial"] & H_{n-1}(TM \setminus M) \end{tikzcd}$$

and we get that choosing $q \in M$ representing a generator of $H_0(M)$, we can represent a generator of $H_{2m}(UM)$ by $\tau^{-1}q$ and therefore $\pi\tau^{-1}q$ represents a generator of the image of:

$$\pi_* : H_{2m}(UM) \to H_{2m}(CM)$$

Next we consider the map:

$$\pi|_{\tau^{-1}q} : \tau^{-1}q \to CM$$

and prove that it is injective, and of maximal rank for all $q \in V$ and since $\tau^{-1}q = U_qM$ is compact it follows that it is an imbedding, i.e. $\tau^{-1}q$ is a submanifold of CM diffeomorphic to S^{2m}.

That $\tau^{-1}q$ is $1-1$ follows from property (1) of V which assumes that distinct vectors in U_qM correspond to geometrically distinct geodesics.

Next recall that $T_\gamma CM$ is identified with the space of vertical Jacobi fields along γ, denoted by $\mathcal{J}_\gamma^\perp$. Now, let $\gamma \in CM$ with $\gamma(0) = q$ and $w \in T_{\dot\gamma(0)}\tau^{-1}q$. Then, under the above identification, $\pi_* w$ is the Jacobi field along γ with $J(0) = 0$ and $J'(0) = w$. This shows that the map:

$$\pi_* : T_{\dot\gamma(0)}\tau^{-1}q \to \{J \in \mathcal{J}_\gamma^\perp : J(q) = 0\}$$

is onto, and the latter space is $2m$ dimensional. Thus we finally have $\pi\tau^{-1}q \approx S^{2m}$ for all $q \in V$.

Let now $p \in M$ be as chosen and choose another $q \in M$ and $\gamma \in seg(p, q)$. Let $\tau^{-1}p$ and $\tau^{-1}q$ be thus oriented, so that they represent the same generator of $H_{2m}(UM)$. Property (2) of V implies that the intersection of the sets $\pi\tau^{-1}p$ and $\pi\tau^{-1}q$ consists of two points exactly, namely

$$\pi\tau^{-1}p \cap \pi\tau^{-1}q = \{\gamma, -\gamma\}.$$

We claim that this intersection is transversal at both points and that the intersection number is equal. This will imply $a \cap a^* = 2$.

First we observe that the intersection is transversal. Under the identification $T_\gamma CM \cong \mathcal{J}_\gamma^\perp$ mentioned above, we have:

$$T_\gamma\pi\tau^{-1}p = \{J \in \mathcal{J}_\gamma^\perp : J(p) = 0\} \qquad \text{and}$$

$$T_\gamma\pi\tau^{-1}q = \{J \in \mathcal{J}_\gamma^\perp : J(q) = 0\}$$

Thus $T_\gamma\pi\tau^{-1}p \cap T_\gamma\pi\tau^{-1}q = 0$ for q close enough to p, since then a Jacobi field with $J(p) = J(q) = 0$ should be zero.

In order to show that the intersection numbers are equal, we consider the maps

$$\begin{array}{lcl} \lambda : UM \to UM & , & \lambda' : CM \to CM \\ v \mapsto -v & , & \gamma \mapsto -\gamma \end{array}$$

with the commutative diagram:

$$\begin{tikzcd} UM \arrow[r, "\lambda"] \arrow[d, "\pi"] & UM \arrow[d, "\pi"] \\ CM \arrow[r, "\lambda'"] & CM \end{tikzcd}$$

from the coordinate expression $\eta = g_{ij}v^i dx^j$ we immediately see that $\lambda^*\eta = -\eta$ and thus $\lambda^* d\eta^{n-1} = (-1)^{2m} d\eta^{n-1} = d\eta^{n-1}$ and since $d\eta$ is horizontal, we get $(\lambda')^*\Omega^{n-1} = \Omega^{n-1}$ and λ' is orientation preserving on CM. Combining with the fact that λ' reverses orientation on both $\pi\tau^{-1}p$ and $\pi\tau^{-1}(q)$, since a free $\mathbb{Z}_2$ action on S^{2m} is orientation reversing, shows the intersection numbers are equal, and the lemma is proved. ∎

Lemma 3.2.6 *There is a basis $\{b, e^m\}$ of $H^{2m}(CM)$ such that if a and g are as in the previous lemma, we have $a \cup b = g$ and $a = 2b - e^m$.*

Proof The exact sequences

$$0 \longrightarrow H^{2m-2}(CM) \xrightarrow{\cup e} H^{2m}(CM) \xrightarrow{\pi^*} H^{2m}(UM) \longrightarrow 0$$

$$0 \leftarrow H^{2m+2}(CM) \xleftarrow{\cup e} H^{2m}(CM) \xleftarrow[\text{map}]{\text{Gysin}} H^{2m+1}(UM) \leftarrow 0$$

are dual to each other and split. Since a is a generator of the image of the Gysin map, there exists $b \in H^{2m}(CM)$ such that $b \cup a = g$, by Poincaré duality. If $\sigma \in H^{2m+1}(UM)$ is the generator mapped to a by the Gysin map, then

$$\langle \pi^*(b) \cup \sigma, [UM] \rangle = \langle b \cup a, [CM] \rangle = \langle g, [CM] \rangle = 1.$$

Hence $\pi^*(b)$ is a generator of $H^{2m}(UM)$, again by Poincaré duality. Since e^{m-1} is a generator of $H^{2m-2}(CM)$, we conclude that $\{b, e^m\}$ is a basis of $H^{2m}(CM)$.

Now $a = kb + le^m$, for some k, $l \in \mathbb{Z}$, since $\{b, e^m\}$ is a basis and we get from the last lemma

$$2g = a \cup a = a \cup (kb + le^m) = kg$$

which means that $k = 2$. Moreover, $a \cup e^m = 0$, by exactness of

$$H^{2m+1}(UM) \to H^{2m}(CM) \xrightarrow{\cup e} H^{2m+2}(CM)$$

and thus $a = 2b + le^m$. Since

$$g = a \cup b = (2b + le^m) \cup b = 2b \cup b + le^m \cup b$$

is a generator, l must be odd, say $l = 2s - 1$, for some $s \in \mathbb{Z}$. If we set $b' = b + se^m$, then $\{b', e^m\}$ is a basis of $H^{2m}(CM)$, $a \cup b' = g$ and $a = 2b' - e^m$. Therefore the lemma holds for the basis $\{b', e^m\}$. ∎

We are now ready to complete the proof of Theorem 3.2.4.

Proof of Theorem 3.2.4, when n is odd Let $\{b, e^m\}$ be the basis given by the last lemma, and also $b \cup b = rg$, for some $r \in \mathbb{Z}$. Then

$$b \cup e^m = b \cup (2b - a) = (2r - 1)g$$

and

$$e^m \cup e^m = e^m \cup (2b - a) = 2b - e^m = (4r - 2)g.$$

Now Poincare duality implies:

$$\pm 1 = \det \begin{pmatrix} \langle b \cup b, [CM] \rangle & \langle b \cup e^m, [CM] \rangle \\ \langle e^m \cup b, [CM] \rangle & \langle e^m \cup e^m, [CM] \rangle \end{pmatrix} = \begin{pmatrix} r & 2r-1 \\ 2r-1 & 4r-2 \end{pmatrix} = 2r - 1$$

and thus $r = 0$ or 1. Hence $e^{2m} = \pm 2g$ and thus the theorem follows. ∎

3.3 Wiedersehen manifolds

In this final section we summarize the results that we have proved. We have been concerned with Blaschke manifolds homeomorphic to a sphere. This is precisely the class of the so called Wiedersehen manifolds.

A complete Riemannian n-manifold is called Wiedersehen if there exists some $R > 0$ such that the cut locus $cut(p)$ of any point $p \in M$ is a singleton at distance R from p. Obviously, such a manifold is compact and $diam(M) = Inj(M) = R$. Therefore, it is a Blaschke manifold modelled on the sphere.

The main result that we have proved can be summarized as follows.

Theorem 3.3.1 *A Wiedersehen Riemannian n-manifold, $n \geq 2$, of diameter $R > 0$ is isometric to the standard round n-sphere of radius R/π.*

Bibliography

[1] M. Berger, Une borne inférieure pour le volume d'une variété riemanniene en fonction du rayon d'injectivité, *Ann. Inst. Fourier, Grenoble* 30 (1980), 259-265

[2] M. Berger, Sur certaines variétés à géodésiques toutes fermées, *Bol. Soc. Brasil. Mat.* 9 (1978), no. 2, 89–96

[3] A.L. Besse, Manifolds all of whose geodesics are closed, Ergebnisse Math. Grenzbebiete vol. 93, Springer, 1978

[4] R. Bott, On manifolds all of whose geodesics are closed, *Ann. of Math.* 60 (1954), 375-382

[5] G. Bredon, Topology and Geometry, Springer, 1993

[6] G. Bredon, An introduction to compact transformation groups, Academic Press, 1972

[7] I. Chavel, Riemannian Geometry: A modern introduction, Cambridge Univ. Press, 1993

[8] J. Cheeger and D.G. Ebin, Comparison theorems in Riemannian Geometry, North-Holland, 1975

[9] H. Gluck, F. Warner, and C. T. Yang, Division algebras, fibrations of spheres by great spheres and the topological determination of space by the gross behavior of its geodesics, *Duke Math. J.* 50 (1983), no. 4, 1041–1076

[10] J.J. Hebda, Blaschke manifolds with taut geodesics, *Duke Math. J.* 48 (1981), no. 1, 85-91

[11] B. McKay, The Blaschke conjecture and great circle fibrations of spheres *Amer. J. Math.* 6 (2004), 1155-1191

[12] H. Nakagawa and K. Shiohama, On Riemannian manifolds with certain cut loci I, *Tohoku Math. J.* 22 (1970), 14-23

[13] H. Nakagawa and K. Shiohama, On Riemannian manifolds with certain cut loci II, *Tohoku Math. J.* 22 (1970), 357-361

[14] A.G. Reznikov, The weak Blaschke conjecture for $\mathbb{C}P^n$, *Invent. Math.* 117 (1994), 447-454

[15] T. Sakai, Riemannian Geometry, Translations of Mathematical Monographs, vol. 149, Amer. Math. Soc., 1997

[16] H. Samelson, On manifolds with many closed geodesics, *Portugaliae Math.* 22 (1963), 193-196

[17] H. Sato, On topological Blaschke Conjecture I, Geometry of Geodesics and Related Topics, Advanced Studies in Pure Math., vol. 3, Kinokuniya, Tokyo, 1984, pp. 231-238

[18] H. Sato and T. Mizutani, On topological Blaschke Conjecture II, *Tohoku Math. J.* 36 (1984), 159-174

[19] H. Sato, On topological Blaschke Conjecture III, Curvature and Topology of Riemannian Manifolds, Lecture Notes in Mathematics Vol.1201 (1986), pp 241-253, Springer

[20] K. Shankar, R. Spatzier and B. Wilking, Spherical rank rigidity and Blaschke manifolds, *Duke Math. J.* 128 (2005), 65-81

[21] A. Weinstein, On the volume of manifolds all of whose geodesics are closed, *J. Differential Geometry* 9 (1974), 513-517

[22] C.T. Yang, Odd-dimensional wiedersehen manifolds are spheres, *J. Differential Geometry* 15 (1980), 91-96

[23] C. T. Yang, Smooth great circle fibrations and an application to the topological Blaschke conjecture, *Trans. Amer. Math.* Soc. 320 (1990), no. 2, 507–524.