NOTES ON MEASURABLE COBOUNDARIES

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1 Preliminaries from Harmonic Analysis

Let $(G,+)$ be an abelian group. A function $f : G \to \mathbb{C}$ is said to be positive definite if

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k f(x_j - x_k) \geq 0$$

for every x_1,..., $x_n \in G$, λ_1,..., $\lambda_n \in \mathbb{C}$ and every $n \in \mathbb{N}$. Some basic properties of the positive definite functions are summarized in the following.

Proposition 1.1. *If G is an abelian group and $f : G \to \mathbb{C}$ is a positive definite function, then*
(a) $f(0) \geq 0$,
(b) $\overline{f(x)} = f(-x)$,
(c) $|f(x)| \leq f(0)$ *for every* $x \in G$, *and*
(d) $|f(x) - f(y)|^2 \leq 2f(0)[f(0) - \mathrm{Re}f(x - y)]$ *for every* x, $y \in G$.
(e) $|f(x + y) - f(y)|^2 \leq 2f(0)[f(0) - \mathrm{Re}f(x)]$ *for every* x, $y \in G$.

Proof. To prove (a) it suffices to take $n = 1$ and $x_1 = 0$, $\lambda_1 = 1$. The second property (b) follows from the elementary observation that if a, $b \in \mathbb{C}$ are such that $az + b\bar{z} \in \mathbb{R}$ for every $z \in \mathbb{C}$, then $a = \bar{b}$. Indeed, if we take $n = 2$, $x_1 = x$, $x_2 = 0$, then

$$(|\lambda_1|^2 + |\lambda_2|^2)f(0) + \lambda_1\bar{\lambda}_2 f(x) + \bar{\lambda}_1\lambda_2 f(-x) \geq 0$$

and in particular $\lambda_1\bar{\lambda}_2 f(x) + \bar{\lambda}_1\lambda_2 f(-x) \in \mathbb{R}$ for every λ_1, $\lambda_2 \in \mathbb{C}$. Taking the values

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{|f(x)|}{\overline{f(x)}}$$

in the above inequality yields (c), since if $f(x) = 0$, there is nothing to prove by (a). Property (d) is trivial if $f(x) - f(y) = 0$. Otherwise, we take $n = 3$, $x_1 = 0$, $x_2 = x$, $x_3 = y$ and then

$$(|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2)f(0) + \bar{\lambda}_1\lambda_2 f(x) + \lambda_1\bar{\lambda}_2\overline{f(x)} + \bar{\lambda}_1\lambda_3 f(y) + \lambda_1\bar{\lambda}_3\overline{f(y)}$$

$$+\lambda_2\bar{\lambda}_3 f(x - y) + \bar{\lambda}_2\lambda_3\overline{f(x - y)} \geq 0$$

for every λ_1, λ_2, $\lambda_3 \in \mathbb{C}$, by (b). Taking the special values

$$\lambda_1 = 1, \quad \lambda_2 = t\frac{|f(x) - f(y)|}{f(x) - f(y)}, t \in \mathbb{R}, \quad \lambda_3 = -\lambda_2,$$

we get

$$2[f(0) - \mathrm{Re} f(x-y)]t^2 + 2|f(x) - f(y)|t + f(0) \geq 0$$

for all $t \in \mathbb{R}$, and thus necessarily

$$4|f(x) - f(y)|^2 - 8f(0)[f(0) - \mathrm{Re} f(x-y)] \leq 0.$$

The last property (e) is a restatement of (d). $\square$

Corollary 1.2. *If G is an abelian group and $f : G \to \mathbb{C}$ is a positive definite function, then $G_0 = \{x \in G : f(x) = f(0)\}$ is a subgroup of G and f is constant on each of its cosets in G.* $\square$

Corollary 1.3. *If a positive definite function $f : \mathbb{R} \to \mathbb{C}$ is continuous at 0, then it is uniformly continuous.* $\square$

Examples 1.4. (i) A trivial example of a positive definite function $f : \mathbb{R} \to \mathbb{C}$ is $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$.
(ii) The cosine $\cos : \mathbb{R} \to \mathbb{R} \subset \mathbb{C}$ is a positive definite function because

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \cos(x_j - x_k) = \left| \sum_{j=1}^{n} \lambda_j \cos x_j \right|^2 + \left| \sum_{j=1}^{n} \lambda_j \sin x_j \right|^2 \geq 0.$$

(iii) For every $\xi \in \mathbb{R}$ the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = e^{i\xi x}$ is positive definite since

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k e^{i\xi(x_j - x_k)} = \left| \sum_{j=1}^{n} \lambda_j e^{i\xi x_j} \right|^2 \geq 0.$$

(iv) Let $\mathcal{H}$ be a Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. For every $x \in \mathcal{H}$ the double sequence $a_n = \langle U^n(x), x \rangle$, $n \in \mathbb{Z}$, is positive definite, since

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k a_{j-k} = \left\| \sum_{j=1}^{n} \lambda_j U^j(x) \right\|^2 \geq 0$$

for every λ_1,..., $\lambda_n \in \mathbb{C}$ and every $n \in \mathbb{N}$.
(v) Let μ be a non-negative finite Borel measure on S^1 and let

$$a_n = \int_{S^1} z^n d\mu, \quad n \in \mathbb{Z},$$

be the double sequence of its Fourier coefficients. The so defined function $a : \mathbb{Z} \to \mathbb{C}$ is positive definite, because

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \int_{S^1} z^{j-k} d\mu = \int_{S^1} \left| \sum_{k=1}^{n} \lambda_k z^k \right|^2 d\mu \geq 0.$$

(vi) Let μ be a non-negative finite Borel measure on $\mathbb{R}$. The Fourier-Stieltjies transform $\hat{\mu}$ of μ is the positive definite function

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} d\mu(x), \quad \xi \in \mathbb{R},$$

since
$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \hat{\mu}(\xi_j - \xi_k) = \int_{\mathbb{R}} \left| \sum_{k=1}^{n} \lambda_k e^{-i\xi_k x} \right|^2 d\mu(x) \geq 0.$$

The last two examples are the easy parts of the following two famous theorems of Harmonic Analysis, whose proofs can be found in [4]. The first one is due to G. Herglotz and its continuous version is due to S. Bochner.

Theorem 1.5. (Herglotz) *A double sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers is positive definite if and only if there exists a non-negative finite Borel measure μ on S^1 such that*

$$a_n = \int_{S^1} z^n d\mu, \quad n \in \mathbb{Z}. \quad \square$$

Theorem 1.6. (Bochner) *For a function $f : \mathbb{R} \to \mathbb{C}$ the following are equivalent:*
(i) f is positive definite and continuous.
(ii) f is the Fourier-Stieltjies transform of a non-negative finite Borel measure μ on $\mathbb{R}$, that is

$$f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} d\mu(x), \quad \xi \in \mathbb{R}. \quad \square$$

If $f : S^1 \to \mathbb{R}$ is in L^1 (with respect to Lebesgue measure), then f is said to be positive definite in the integral sense if

$$\iint_{S^1 \times S^1} f(x\bar{y})\overline{u(x)}u(y)dxdy \geq 0$$

for every continuous function $u : S^1 \to \mathbb{C}$. This implies that $\hat{f} \geq 0$. The following characterization is proved in [2].

Proposition 1.7. *A continuous function $f : S^1 \to \mathbb{C}$ is positive definite in the integral sense if and only if it is positive definite.* $\square$

A positive definite L^1 (locally bounded) function can be reconstructed from its Fourier transform according to the following theorem of S. Bochner whose proof can be found in [2].

Theorem 1.8. (Bochner) *If the L^1 function $f : \mathbb{R} \to \mathbb{C}$ is positive definite and essentially bounded in some neighbourhood of $1 \in S^1$, then*

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n) z^n$$

for almost every $z \in S^1$, so that f is equal almost everywhere to a continuous positive definite function. $\square$

In the above we take $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and a function $f : S^1 \to \mathbb{R}$ can be considered as a 2π-periodic function of a real variable. In the general real variable case, a bounded measurable function $f : \mathbb{R} \to \mathbb{C}$ is said to be positive definite in the integral sense if

$$\iint_{\mathbb{R}^2} f(x - y)u(x)\overline{u(y)}dxdy \geq 0$$

for every $u \in L^1$ (with respect to Lebesgue measure). A continuous positive definite function $f : \mathbb{R} \to \mathbb{C}$ is positive definite in the integral sense. This follows from Bochner's theorem, but it can also be proved directly although not so easily. Bochner's theorem was generalized by F. Riesz in [5] as follows.

Theorem 1.9. *If a bounded measurable function $f : \mathbb{R} \to \mathbb{C}$ is positive definite in the integral sense, then there exists a non-negative, finite Borel measure μ on $\mathbb{R}$ such that*

$$f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} d\mu(x)$$

for almost every $\xi \in \mathbb{R}$. $\square$

Corollary 1.10. *A bounded measurable and positive definite function $f : \mathbb{R} \to \mathbb{C}$ is equal almost everywhere to a continuous positive definite function.* $\square$.

2 Distributions of multiplicative cocycles

Let X be a compact metric space and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points of X. A Borel measure μ on X is called the distribution of this sequence if

$$\int_X \phi d\mu = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(a_k)$$

for every $\phi \in C(X)$. Then μ is necessarily positive and a probability measure on X. As it is expected, most sequences in X have no distribution.

Let $(X, \mathcal{A}, \mu, T)$ be a measurable dynamical system, that is $(X, \mathcal{A})$ is a measurable space, μ is a probability measure on the σ-algebra $\mathcal{A}$ and $T : X \to X$ is an invertible bimeasurable map. A multiplicative cocycle is a function $F : \mathbb{Z} \times X \to S^1$ such that $F(n, .) : X \to S^1$ is measurable for every $n \in \mathbb{Z}$ and

$$F(m+n, x) = F(m, x)F(n, T^m(x))$$

for μ-almost every $x \in X$ and every $n \in Z$. The cocycle F is called a coboundary if there exists a measurable function $q : X \to S^1$ such that

$$F(n, x) = \frac{q(T^n(x))}{q(x)}$$

for μ-almost every $x \in X$ and every $n \in \mathbb{Z}$.

From a multiplicative cocycle F as above we can define a measurable dynamical systems $S : S^1 \times X \to S^1 \times X$ by

$$S(z, x) = (zF(1, x), T(x))$$

which is called the corresponding skew product of F. Inductively,

$$S^n(z, x) = (zF(n, x), T^n(x))$$

for every $(z,x) \in \S^1 \times X$ and $n \in \mathbb{Z}$. It is obvious that S is a bimeasurable bijection and preserves the product probability measure $\frac{1}{2\pi} dz \times \mu$ on $S^1 \times X$. Indeed, if $U \subset S^1$ is a Borel set and $V \in \mathcal{A}$, from Fubini's theorem we get

$$\iint_{S^1 \times X} (\chi_{U \times V} \circ S) dz d\mu = \iint_{S^1 \times X} \chi_U(zF(1,x)) \chi_V(T(x)) dz d\mu(x)$$

$$= \int_X \left(\int_{S^1} \chi_U(zF(1,x)) dz \right) \chi_V(T(x)) d\mu(x) = \int_X \left(\int_{S^1} \chi_U(z) dz \right) \chi_V(T(x)) d\mu(x)$$

$$= \left(\int_{S^1} \chi_U(z) dz \right) \cdot \left(\int_X \chi_V d\mu \right).$$

Using the corresponding skew product S we can show that the multiplicative cocycle F is uniquely determined by the measurable function $F(1,.) : X \to S^1$, since $F(0,x) = 1$ and inductively

$$F(n,x) = \prod_{k=0}^{n-1} F(1, T^k(x))$$

for μ-almost every $x \in X$ and $n \in \mathbb{N}$. Conversely, if $f : X \to S^1$ is a measurable function, then the formula

$$F(n,x) = \prod_{k=0}^{n-1} f(T^k(x))$$

defines a multiplicative cocycle. To see this, we consider the skew product transformation $S(z,x) = (zf(x), T(x))$, so that $S^n(z,x) = (zF(n,x), T^n(x))$, and now the formula $S^{n+m} = S^n \circ S^m$ yields the cocycle condition for F.

Theorem 2.1. *If F is a multiplicative cocycle for T, then for μ-almost every $x \in X$ the sequence $(F(n,x))_{n \in \mathbb{N}}$ in S^1 has a distribution.*

Proof. Firstly we observe that the Ergodic Theorem applied to $f(z,x) = z$ gives that the limit

$$f^*(z,x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(z,x)) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} zF(k,x)$$

exists for almost every $(z,x) \in S^1 \times X$ and hence the limit

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} F(k,x)$$

exists for μ-almost every $x \in X$.

Since for every $m \in \mathbb{Z}$ the function F^m is again a multiplicative cocycle, the above observation implies that the limit

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} (F(k,x))^m$$

exists for every $m \in \mathbb{Z}$ and for μ-almost every $x \in X$. For each fixed $n \in \mathbb{N}$ and $x \in X$ the double sequence $((F(n,x))^m)_{m\in\mathbb{Z}}$ is positive definite, because for every $N \in \mathbb{N}$ and $\lambda_1,\ldots, \lambda_N \in \mathbb{C}$ we have

$$\sum_{j,k=1}^{N} \lambda_j \bar{\lambda}_k (F(n,x))^{j-k} = \left| \sum_{j+1}^{N} \lambda_j (F(n,x))^j \right|^2 \geq 0.$$

Thus, if $x \in X$ is a point for which the limit

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (F(k,x))^m$$

exists for every $m \in \mathbb{Z}$, then the sequence

$$\left(\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (F(k,x))^m \right)_{m\in\mathbb{Z}}$$

is positive definite. It follows now from Theorem 1.5 that the exists a non-negative finite Borel measure ν_x on S^1 such that

$$\int_{S^1} z^m d\nu_x(z) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (F(k,x))^m$$

for all $m \in \mathbb{Z}$. This means that ν_x is a distribution of the sequence $(F(n,x))_{n\in\mathbb{N}}$ of points of S^1. $\square$

3 Schmidt's criterion for additive cocycles

An additive cocycle with respect to a measurable dynamical system $(X, \mathcal{A}, \mu, T)$ is a function $A : \mathbb{Z} \times X \to \mathbb{R}$ such that $A(n, .) : X \to \mathbb{R}$ is measurable for every $n \in \mathbb{Z}$ and

$$A(m + n, x) = A(m, x) + A(n, T^m(x))$$

for μ-almost every $x \in X$ and every $n \in Z$. The cocycle a is called a coboundary if there exists a measurable function $u : X \to \mathbb{R}$ such that

$$A(n, x) = u(T^n(x)) - u(x)$$

for μ-almost every $x \in X$ and every $n \in \mathbb{Z}$.

An additive cocycle A is determined by the measurable function $f = A(1, .) : X \to \mathbb{R}$, because $A(0, x) = 0$ and

$$A(n, x) = \sum_{k=0}^{n-1} A(1, T^k(x)) = S_n f(x)$$

for every $x \in X$ and $n \in \mathbb{N}$. A similar formula holds for negative integers. More precisely,

$$A(-n, x) = \sum_{k=0}^{n-1} A(-1, T^{-k}(x))$$

and $A(1,x)+A(-1,T(x)) = A(1-1,x) = 0$, which means that $A(-1,x) = A(1,T^{-1}(x))$ for all $x \in X$. Hence

$$A(-n,x) = \sum_{k=1}^{n} f(T^{-k}(x))$$

for all $n \in \mathbb{N}$ and $x \in X$.

Obviously, the additive cocycle A is a coboundary if and only if there exists a measurable function $u : X \to \mathbb{R}$ such that $f(x) = A(1,x) = u(T(x)) - u(x)$ for μ-almost every $x \in X$. We shall make no distinction between f and the corresponding cocycle $A(n,x) = S_n f(x)$. A necessary condition for an integrable function to be a coboundary is the following ([1]).

Proposition 3.1. *Let $(X, \mathcal{A}, \mu, T)$ be a measurable dynamical system and $f \in L^1(\mu)$. If f is a coboundary, then*

$$\int_X f d\mu = 0.$$

Proof. Suppose that $\displaystyle\int_X f d\mu > \epsilon > 0$ and $u : X \to \mathbb{R}$ be a measurable function such that $f(x) = u(T(x)) - u(x)$ for μ-almost every $x \in X$. If

$$M = \{x \in X : \lim_{n\to+\infty} \frac{1}{n} S_n f(x) > \epsilon\}.$$

Then, $M \in \mathcal{A}$ and $\mu(M) > \epsilon$ from the Ergodic Theorem. For each $n \in \mathbb{N}$ we put

$$M_n = \bigcap_{m=n}^{\infty} \{x \in X : \frac{1}{m} S_m f(x) > \epsilon\}.$$

Obviously, $M_n \in \mathcal{A}$ and $M \subset \displaystyle\bigcup_{n=1}^{\infty} M_n$. Thus, there exists $n \in \mathbb{N}$ such that $\mu(M_n) > 0$. If $M_{n,l} = M_n \cap u^{-1}([l, l+1))$, $l \in \mathbb{Z}$, then $M_n = \displaystyle\bigcup_{l\in\mathbb{Z}} M_{n,l}$ and so there exists $l \in \mathbb{Z}$ such that $\mu(M_{n,l}) > 0$. By Poincaré Recurrence, there exist infinitely many $m \in \mathbb{N}$ such that

$$\mu(T^m(M_{n,l}) \cap M_{n,l}) = \mu(M_{n,l}) > 0.$$

We pick such a m large enough so that $\dfrac{1}{m} < \epsilon$. Thus, there exists $x \in X$ such that $u(x)$, $u(T^m(x)) \in [l.l+1)$ and

$$\frac{1}{m}(u(T^m(x)) - u(x)) > \epsilon.$$

This means that $|u(x) - u(T^m(x))| < 1$ and at the same time $u(T^m(x)) - u(x) > m\epsilon > 1$. This contradiction proves the assertion. □

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. A non-negative finite Borel measure μ on $\mathbb{R}$ is said to be the distribution of the sequence if

$$\int_{\mathbb{R}} e^{-i\xi x} d\mu(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi a_k}$$

for almost all $\xi \in \mathbb{R}$ (with respect to the Lebesque measure). Unlike the compact case, the distribution of a real sequence may be the zero measure.

Example 3.2. Let $a_n = n$, $n \in \mathbb{N}$. If $\xi \in \mathbb{R}$ is such that $\frac{\xi}{2\pi}$ is irrational, then

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi k} = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} dt = 0.$$

This implies that the sequence does have a distribution, but it is the zero measure.

Suppose that the limit of the right hand side exists for almost all $\xi \in \mathbb{R}$. Then, the so defined function $f : \mathbb{R} \to \mathbb{C}$ is bounded, measurable and positive definite in the integral sense, because

$$\iint_{\mathbb{R}^2} f(x-y)u(x)\overline{u(y)}dxdy = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} |\hat{u}(a_k)|^2 \geq 0$$

for every $u \in L^1(\mathbb{R})$. It follows from Theorem 1.9 that there exists a non-negative, finite Borel measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{-i\xi x} d\mu(x) = f(\xi) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi a_k}$$

for almost all $\xi \in \mathbb{R}$. Moreover, for every $g \in L^1(\mathbb{R})$, if we multiply and integrate, from Fubini's theorem and dominated convergence we get

$$\int_{\mathbb{R}} \hat{g} d\mu = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \hat{g}(a_k).$$

Since the image of the Fourier transform is uniformly dense in $C_0(\mathbb{R})$, we conclude that

$$\int_{\mathbb{R}} h d\mu = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} h(a_k)$$

for every $h \in C_0(\mathbb{R})$.

Proposition 3.3. *For every measurable function $f : X \to \mathbb{R}$ the sequence $(S_n f(x))_{n\in\mathbb{N}}$ has a distribution for μ-almost every $x \in X$.*

Proof. For each $\xi \in \mathbb{R}$ the function $F_\xi : \mathbb{Z} \times X \to S^1$ defined by $F_\xi(n, x) = e^{-i\xi S_n f(x)}$ is a multiplicative cocycle. By Theorem 2.1, for μ-almost every $x \in X$ the sequence $(F_\xi(n, x))_{n\in\mathbb{N}}$ has a distribution. In particular, for μ-almost every $x \in X$ the limit

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi S_k f(x)}$$

exists. Let

$$E = \{(\xi, x) \in \mathbb{R} \times X : \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi S_k f(x)} \quad \text{does not exist}\}.$$

If $E_x = \{\xi \in \mathbb{R} : (\xi, x) \in E\}$ and $E_\xi = \{x \in X : (\xi, x) \in E\}$, form Fubini's theorem we have
$$\int_X \left(\int_{\mathbb{R}} \chi_{E_x}(\xi) d\xi \right) d\mu = \iint_{\mathbb{R}\times X} \chi_E d\xi d\mu = \int_{\mathbb{R}} \left(\int_X \chi_{E_\xi}(x) d\mu \right) d\xi = 0.$$
Therefore, there exists a measurable set $M \subset X$ with $\mu(M) = 1$ such that

$$\int_{\mathbb{R}} \chi_{E_x}(\xi) d\xi = 0$$

for all $x \in M$. Thus, for every $x \in M$ we have $\chi_E(\xi, x) = \chi_{E_x}(\xi) = 0$ for almost every $\xi \in \mathbb{R}$. In other words, for μ-almost every $x \in X$ the limit

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi S_k f(x)}$$

exists for almost every $\xi \in \mathbb{R}$. From the preceding considerations, this implies the conclusion. $\square$

The above theorem says that for μ-almost every $x \in X$ there exists a non-negative finite Borel measure μ_x on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} g d\mu_x = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} g(S_k f(x))$$

for every $g \in C_0(\mathbb{R})$. Since

$$\int_{\mathbb{R}} g d\mu_{T(x)} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} g(S_{k+1} f(x) - f(x))$$

or equivalently

$$\int_{\mathbb{R}} g(t + f(x)) d\mu_{T(x)}(t) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} g(S_{k+1} f(x)) = \int_{\mathbb{R}} g(t) d\mu_x(t)$$

for every $g \in C_0(\mathbb{R})$, it follows that $\mu_{T(x)}(A - f(x)) = \mu_x(A)$ for every Borel set $A \subset \mathbb{R}$. In particular, $\mu_{T(x)}(\mathbb{R}) = \mu_x(\mathbb{R})$ and $T^{-1}(\{x \in X : \mu_x = 0\}) = \{x \in X : \mu_x = 0\}$. This implies that if T is ergodic, then either $\mu_x = 0$ for μ-almost every $x \in X$ or $\mu_x \neq 0$ for μ-almost every $x \in X$.

Theorem 3.4. *A measurable function $f : X \to \mathbb{R}$ is a coboundary if and only if $\mu_x \neq 0$ for μ-almost every $x \in X$.*

Proof. Suppose that there exists a measurable function $u : X \to \mathbb{R}$ such that $f = u - u \circ T$ μ-almost everywhere on X. For convenience we put

$$h(\xi, x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-i\xi S_k f(x)}.$$

We know that for μ-almost every $x \in$ the limit $h(\xi, x)$ exists for almost every $\xi \in \mathbb{R}$. Obviously,

$$h(\xi, x) = e^{-i\xi u(x)} \cdot \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} e^{i\xi u(T^k(x))}.$$

Let $G = \{x \in X : h(\xi, x) = 0 \quad \text{for almost all} \quad \xi \in \mathbb{R}\}$. An easy computation shows that $h(\xi, T(x)) = e^{-i\xi f(x)}h(\xi, x)$, which implies that $T(G) = G$. If now $g \in L^1(\mathbb{R})$ is such that $\hat{g} > 0$, then

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \hat{g}(u(T^k(x))) = 0$$

for all $x \in G$ and integrating over G we arrive at

$$\int_G (\hat{g} \circ u)d\mu = 0.$$

This means that necessarily $\mu(G) = 0$.

Conversely, if $\mu_x \neq 0$ for μ-almost every $x \in X$, then $\mu_{T(x)}(\mathbb{R}) = \mu_x(\mathbb{R}) > 0$ for μ-almost every $x \in X$. Let $u : X \to \mathbb{R}$ be defined by

$$u(x) = \sup\{t \in \mathbb{R} : \mu_x((-\infty, t)) \leq \frac{1}{2}\mu_x(\mathbb{R})\}.$$

We have

$$u(T(x)) = \sup\{t \in \mathbb{R} : \mu_x((-\infty, t + f(x))) \leq \frac{1}{2}\mu_x(\mathbb{R})\} = u(x) + f(x)$$

and it only remains to prove that u is measurable. Let $t \in \mathbb{R}$ be fixed. There exists a non-decreasing sequence $(g_n)_{n\in\mathbb{N}}$ in $C_0(\mathbb{R})$ converging pointwise to $\chi_{(-\infty,t)}$ and so

$$\mu_x((-\infty, t)) = \sup\{\int_{\mathbb{R}} g_n d\mu_x : n \in \mathbb{N}\} = \lim_{m\to+\infty}\left(\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} g_m(S_k f(x))\right).$$

Hence the function $\psi : X \to \mathbb{R}$ defined by

$$\psi(x) = \mu_x((-\infty, t)) - \frac{1}{2}\mu_x(\mathbb{R})$$

is measurable and $u^{-1}([t, +\infty)) = \psi^{-1}((-\infty, 0])$ is a measurable set. $\square$

The following characterization of measurable coboundaries through a growth criterion is due to K. Schmidt.

Theorem 3.5. *A measurable function $f : X \to \mathbb{R}$ is a coboundary if and only if for every $\epsilon > 0$ there exists $M > 0$ such that for every $n \in \mathbb{N}$ there exists a measurable set $F_n \subset X$ with $\mu(F_n) > 1 - \epsilon$ and such that $|S_n f(x)| \leq M$ for every $x \in F_n$.*

Proof. Suppose that there exists a measurable function $u : X \to \mathbb{R}$ such that $f = u \circ T - u$ μ-almost everywhere on X and let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $0 \leq 1 - \mu(u^{-1}((-n, n))) < \frac{\epsilon}{2}$ for every $n \geq N$. Thus, if we put

$$F_n = u^{-1}((-n, n)) \cap T^{-n}(u^{-1}((-n, n))),$$

then $\mu(X \setminus F_n) < \epsilon$ and $|S_n f(x)| = |u(T^n(x)) - u(x)| \leq 2N$ for every $x \in F_n$.

For the converse we assume that f is not a coboundary. From Theorem 3.4, there exists a measurable set $F \subset X$ with $\mu(F) > 0$ such that for every $x \in F$ we have

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} e^{-i\xi S_k f(x)} = 0$$

for almost every $\xi \in \mathbb{R}$. Applying our assumption for the choice $\epsilon = \dfrac{1}{2}\mu(F)$, there exists some $M > 0$ such that for every $n \in \mathbb{N}$ there exists a measurable set $F_n \subset X$ with $\mu(F_n) > 1 - \dfrac{1}{2}\mu(F)$ and $|S_n f(x)| \leq M$ for every $x \in F_n$. Then, $\mu(F \cap F_n) > \dfrac{1}{2}\mu(F)$. The Fourier transform of the Fejer kernel K_{2M} is the continuous function

$$\hat{K}_{2M}(t) = \max\left\{0, 1 - \frac{|t|}{2M}\right\}$$

(see page 139 in [4]). Multiplying with $K_{2M}(\xi)$ and integrating with respect to ξ we arrive at

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \max\left\{0, 1 - \frac{|S_k f(x)|}{2M}\right\} = 0.$$

However, the integral of the average in the limit is

$$\frac{1}{n}\sum_{k=1}^{n}\int_X \max\left\{0, 1 - \frac{|S_k f(x)|}{2M}\right\} d\mu \geq \frac{1}{n}\sum_{k=1}^{n}\int_{F\cap F_n} \max\left\{0, 1 - \frac{|S_k f(x)|}{2M}\right\} d\mu$$

$$\geq \frac{1}{n}\sum_{k=1}^{n}\int_{F\cap F_k} \frac{1}{2} d\mu \geq \frac{1}{4}\mu(F) > 0$$

for every $n \in \mathbb{N}$. This contradiction concludes the proof. □

In the ergodic case the above growth condition can be somewhat weakened.

Theorem 3.6. *Let T be ergodic and let $f : X \to \mathbb{R}$ be a measurable function. For $n \in \mathbb{N}$ and $M > 0$ let $E_{n,M} = \{x \in X : |S_n f(x)| > M\}$. Then, f is a coboundary if and only if there exists some $M > 0$ such that*

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \mu(E_{k,M}) < 1.$$

Proof. The necessity is obvious from Schmidt's criterion because if f is a coboundary, then for every $\epsilon > 0$ there exists $M > 0$ such that $\mu(E_{n,M}) < \epsilon$ for every $n \in \mathbb{N}$. For the sufficiency suppose that f is not a coboundary and le $M > 0$ be any. There exists a measurable set $F \subset X$ such that $\mu(F) > 0$ and $\mu_x = 0$ for all $x \in F$, by Theorem 3.4. Since T is assume to be ergodic, we have $\mu_x = 0$ for μ-almost every $x \in X$ and thus

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} g(S_k f(x)) = 0$$

for every $g \in C_0(\mathbb{R})$. We choose any $g \in C_0(\mathbb{R})$ such that $g \geq 0$ and $g(t) = 1$ for $|t| \leq M$. Integrating,

$$0 = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \int_X g(S_k f(x)) d\mu = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\int_{E_{k,M}} g(S_k f(x)) d\mu + 1 - \mu(E_{k,M}) \right)$$

and therefore

$$1 - \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \mu(E_{k,M}) = 0. \quad \square$$

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