COHERENT MEASURES AND THE UNSTABLE MANIFOLD OF ISOLATED UNSTABLE ATTRACTORS

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ABSTRACT

In this note we give a statistical approximation of the unstable manifold of a connected isolated unstable attractor of a smooth flow using coherent measures relative to it. In the main result we show that almost all orbits in the support of a coherent measure relative to an isolated unstable attractor are contained in its unstable manifold.

1 Introduction

This note is concerned with the study of the negative limit sets of orbits in the region of attraction (the stable manifold) $W^+(A)$ of an isolated unstably attracting continuum A of a smooth flow $(\phi_t)_{t\in\mathbb{R}}$ on a connected smooth manifold M. It is well known that the first positive prolongation

$$D^+(A) = \{x \in M : L^-(x) \cap A \neq \varnothing\}$$

of A is asymptotically stable with the same region of attraction $W^+(D^+(A)) = W^+(A)$ (see Theorem 8.20 in [3]).

It is possible to define a finer topology on the region of attraction with respect to which A becomes asymptotically stable. This is called the intrinsic topology and was first introduced in [9] (for an alternative definition see [11]). The region of attraction endowed with the intrinsic topology is denoted by $W_i^+(A)$. The space $W_i^+(A)$ is locally compact, separable and metrizable.

The negative limit set of an orbit in $D^+(A) \setminus A$ may not be contained in A (see Example 4.4 in [2]). This is a case where the phenomenon explosion occurs (see [1], [2], [7], [10]). The flow explodes at a point $x \in W^+(A)$ if the identity map $id : W^+(A) \to W_i^+(A)$ is not continuous at x. Since A is not stable, the flow explodes at some point of A (Proposition 4.1 in [2]). Here we are interested in the detection of orbits whose negative limit sets are contained in A. In other words, we would like to have a tool in order to locate the unstable manifold (e.g. region of negative attraction)

$$W^-(A) = \{x \in M : \varnothing \neq L^-(x) \subset A\}$$

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of an isolated unstable attractor A. Note that $W^{-}(A) \setminus A$ may be nowhere dense in $D^{+}(A) \setminus A$ (see Example 4.4 in [2]).

We show that orbits in the unstable manifold can be located statistically through coherent measures, but it is impossible to locate explosive orbits in this way. The notion of coherent measure relative to an isolated invariant set of a smooth flow on a smooth closed manifold has been introduced in [6] in connection to the problem of the existence of a smooth Lyapunov 1-form for a general isolated invariant set of a smooth flow on a closed smooth manifold. First we give a preliminary characterization of coherent measures relative to a connected isolated unstable attractor. Precisely, we prove that a Radon measure μ on $W^{+}(A) \setminus A$ is coherent relative to A if and only if μ is invariant under the flow and its support is contained in $D^{+}(A) \setminus A$ (see Corollary 4.4). The main result is Theorem 4.8 below which says that if μ is a coherent measure relative to a connected isolated unstable attractor, then $M \setminus W^{-}(A)$ has μ-measure zero. So, actually, a Radon measure μ on $W^{+}(A) \setminus A$ is coherent relative to A if and only if it is invariant and $\mathrm{supp}\mu \subset \overline{W^{-}(A)} \setminus A$. It follows that

$$W^{-}(A) \subset A \cup \bigcup_{\mu \text{ coherent}} \mathrm{supp}\mu \subset \overline{W^{-}(A)}.$$

In case there are no external explosions, these three sets are equal.

2 Isolated unstable attractors

Let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a separable, locally compact, metrizable space M. The positive limit set of $x \in M$ is the closed, invariant set

$$L^{+}(x) = \{y \in M : \phi_{t_n}(x) \to y \quad \text{for some} \quad t_n \to +\infty\}.$$

The negative limit set $L^{-}(x)$ is defined analogously. Let $A \subset M$ be a compact invariant set. The invariant set

$$W^{+}(A) = \{x \in M : \varnothing \neq L^{+}(x) \subset A\}$$

is called the region of attraction (or stable manifold) of A. If $W^{+}(A)$ is an open neighbourhood of A, then A is called an attractor. A compact invariant set A is called (positively Lyapunov) stable if every neighbourhood of A contains a positively invariant open neighbourhood of A. A stable attractor is usually called asymptotically stable. An unstable attractor is an attractor which is not stable.

Asymptotically stable compact invariant sets are very special cases of isolated invariant sets. A compact invariant set $A \subset M$ is called isolated if it has a compact neighbourhood V such that A is the maximal invariant set in V. Each such V is called an isolating neighbourhood of A and contains a smaller isolating neighbourhood N of A such that there are compact sets N^{+}, $N^{-} \subset \partial N$ with the following properties:

(i) $\partial N = N^{+} \cup N^{-}$.

(ii) For every $x \in N^{+}$ there exists $\epsilon > 0$ such that $\phi_t(x) \in M \setminus N$ for $-\epsilon \leq t < 0$, and for every $y \in N^{-}$ there exists $\delta > 0$ such that $\phi_t(y) \in M \setminus N$. for $0 < t \leq \delta$.

(iii) For every $x \in \partial N \setminus N^{+}$ there exists $\epsilon > 0$ such that $\phi_t(x) \in \mathrm{int}N$ for $-\epsilon \leq t < 0$, and for every $y \in \partial N \setminus N^{-}$ there exists $\delta > 0$ such that $\phi_t(y) \in \mathrm{int}N$ for $0 < t \leq \delta$.

The triad (N, N^+, N^-) is called an isolating block of A. The sets $A^\pm = \{x \in N : C^\pm(x) \subset N\}$ and $a^\pm = \partial N \cap A^\pm$ are compact and $A = A^+ \cap A^-$. Moreover, $\varnothing \neq L^+(x) \subset A$ for every $x \in A^+$ and $a^+ \subset N^+ \setminus N^-$.

If M is a smooth n-manifold and the flow is smooth, then every neighbourhood of an isolated invariant set A contains a smooth isolating block (N, N^+, N^-) of A. This means that N is a smooth compact n-manifold with boundary $\partial N = N^+ \cup N^-$, the sets N^+ and N^- are smooth compact $(n-1)$-manifolds with common boundary $N^+ \cap N^-$, which is a smooth compact $(n-2)$-manifold (without boundary) and on which the flow is externally tangent to N. Moreover, the flow is transverse to $N^+ \setminus N^-$ into N and transverse to $N^- \setminus N^+$ out of N (see [5], [13]).

Let $A \subset M$ be an isolated compact invariant set and let (N, N^+, N^-) be an isolating block of A. The final entrance time function $\tau : W^+(A) \to [-\infty, +\infty)$ defined by

$$\tau(x) = \sup\{t \in \mathbb{R} : \phi_t(x) \in M \setminus N\},$$

if $x \in W^+(A) \setminus A$ and $\tau(x) = -\infty$, if $x \in A$, is lower semicontinuous. This follows immediately from the definition and the continuity of the flow. Obviously, $\phi_{\tau(x)}(x) \in a^+$ and $\tau(\phi_t(x)) = \tau(x) - t$ for every $t \in \mathbb{R}$ and $x \in W^+(A) \setminus A$. The final entrance time function τ is discontinuous at $x \in W^+(A) \setminus A$ if and only if there is $x_n \to x$ such that $\tau(x_n) \to +\infty$ (see Lemma 3.1 in [1]). These are the points of the set $\phi(\mathbb{R} \times \partial_{\partial N} a^+)$.

It is clear from the above that if A is an isolated compact invariant set, then A is not necessarily asymptotically stable with respect to the restricted flow in $W^+(A)$. However, it is possible to define a new topology in $W^+(A)$, which is finer than the subspace topology, with respect to which the flow remains continuous and A becomes asymptotically stable. Roughly speaking, this new topology is obtained by cutting $W^+(A)$ along the discontinuity set of the final entrance time function with respect to any isolating block of A. It was originally defined in [9]. According to an alternative description given in [11] the intrinsic topology is the smallest topology which contains the subspace topology of $W^+(A)$ and the sets

$$W^+(A) \cap \bigcap_{t \geq 0} \phi_{-t}(V),$$

where V runs over the open neighbourhoods of A in M. We denote by $W_i^+(A)$ the region of attraction equipped with the intrinsic topology. The space $W_i^+(A)$ is locally compact, separable and metrizable and the topology of A remains unchanged (see [2]).

The final entrance time function $\tau : W_i^+(A) \to [-\infty, +\infty)$ is continuous for any isolating block (N, N^+, N^-) of A (see Lemma 3.1 in [2]). The function $F : W_i^+(A) \to [0, +\infty)$ defined by

$$F(x) = \begin{cases} e^{\tau(x)}, & \text{if } x \in W_i^+(A) \setminus A, \\ 0, & \text{if } x \in A \end{cases}$$

is then continuous. It is obvious that $A = F^{-1}(0)$ and $F(\phi_t(x)) = e^{-t}F(x)$ for every $t \in \mathbb{R}$ and $x \in W_i^+(A) \setminus A$. This shows that A is globally asymptotically stable in $W_i^+(A)$. Moreover, the restricted flow on $W_i^+(A) \setminus A$ is parallelizable and each level set $F^{-1}(c)$, $c > 0$, is a compact global section. In particular the set $a^+ = F^{-1}(1)$ is a global section to the flow on $W_i^+(A) \setminus A$ and thus $W_i^+(A) \setminus A$ is homeomorphic to $\mathbb{R} \times a^+$.

Note that $F : W^+(A) \to [0, +\infty)$ is a lower semicontinuous Lyapunov function for A, which by no means implies that A is stable with respect to the restricted flow in

$W^+(A)$. The identity map $id : W^+(A) \to W_i^+(A)$ is continuous at a point $x \in W^+(A)$ if and only if F is continuous at x. If $W^+(A)$ is locally compact, it follows that the set of points where the identity map $id : W^+(A) \to W_i^+(A)$ is continuous contains an invariant, open and dense set $D \subset W^+(A) \setminus A$. If M is a smooth manifold, the flow is smooth and A is a connected isolated unstable attractor, then for every smooth isolating block (N, N^+, N^-) of A the set a^+ has nonempty interior in ∂N, by transversality. This implies that D is also dense in $W_i^+(A) \setminus A$ (and of course open).

3 Invariant Radon measures in the region of attraction

Let M be a connected, locally compact, metric space and $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on M. Let $A \subset M$ be a connected isolated unstable attractor of the flow. Note that the chain recurrent set of the restricted flow on $W^+(A)$ is contained in $D^+(A)$. In the particularly interesting case where A is a minimal set, then $D^+(A)$ is a chain component of the flow. This very mild kind of recurrence is the only one that points of $W^+(A) \setminus A$ may present.

If (N, N^+, N^-) is an isolating block for A with $N \subset W^+(A)$, then the flow defines an equivariant continuous bijection $h : \mathbb{R} \times a^+ \to W^+(A) \setminus A$, where on $\mathbb{R} \times a^+$ we consider the parallel flow with section a^+. Recall that h is not a homeomorphism in general. It becomes a homeomorphism if on the region of attraction we consider the intrinsic topology. However, it is Borel bimeasurable, because it maps closed subsets of $\mathbb{R} \times a^+$ to Borel subsets of $W^+(A) \setminus A$. So, h induces a one-to-one correspondence between the Borel measures on $\mathbb{R} \times a^+$ and the ones on $W^+(A) \setminus A$. Note that by Poincaré Recurrence there is no non-trivial invariant finite Borel measure on $W^+(A) \setminus A$, nor on $D^+(A) \setminus A$.

If μ is a flow invariant Borel measure on $W^+(A) \setminus A$, then $(h^{-1})_*\mu$ is a Borel measure on $\mathbb{R} \times a^+$, which is invariant under the parallel flow. Therefore, $\mu = h_*(\lambda \times \nu)$, where λ is the Lebesgue measure on $\mathbb{R}$ and ν is the Borel measure on a^+ defined by the formula

$$\nu(K) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu(h([0, \epsilon) \times K)) = \mu(h((0, 1) \times K))$$

for every Borel set $K \subset a^+$. If μ is locally finite, hence a positive Radon measure, then ν is a finite measure, because a^+ is compact. The converse is not true. If we begin with a finite Borel measure ν on a^+, the formula $\mu = h_*(\lambda \times \nu)$ defines a σ-finite invariant Borel measure on $W^+(A) \setminus A$, which may be not locally finite. Indeed, there may exist a point $x \in a^+ \cap D^+(A)$ such that $L^-(x) \cap (D^+(A) \setminus A) \neq \varnothing$. Let ν_x be the Dirac point measure at x and $\mu = h_*(\lambda \times \nu_x)$. If $y \in L^-(x) \cap (D^+(A) \setminus A)$, so that the flow explodes at y, and B is a flow box corresponding to some local section S at y of some extent $\epsilon > 0$, there are $t_n \to -\infty$ such that $\phi_{t_n}(x) \in S$ and $\phi_{t_n}(x) \to y$. Then,

$$\mu(B) \geq \sum_{n=1}^{\infty} (\lambda \times \nu_x)((t_n - \epsilon, t_n + \epsilon) \times \{x\}) = \sum_{n=1}^{\infty} 2\epsilon = +\infty.$$

In the sequel we shall assume that M is a connected smooth manifold and the flow is smooth with infinitesimal generator X. So, in every neighbourhood of A there is a smooth isolating block of A. Let $\mu = h_*(\lambda \times \nu)$ be an invariant, locally finite Borel measure on $W^+(A) \setminus A$, which is equivalent to saying that μ is a (positive) Radon measure.

Proposition 3.1. *Let $f : W^+(A) \to \mathbb{R}$ be a smooth function with compact support such that*
(i) $Xf = 0$ on some open neighbourhood of A and
(ii) f is constant on A (which follows from (i) if A is a minimal set).

Then the limit $\lim_{T\to+\infty} f(\phi_{-T}(x))$ exists for μ-almost all $x \in W^+(A) \setminus A$ and

$$\int_{W^+(A)\setminus A} Xf d\mu = (f|_A) \cdot \nu(a^+ \cap (W^+(A) \setminus D^+(A)).$$

Proof. Since Xf has compact support contained in $W^+(A) \setminus A$, we use Fubini's theorem to compute

$$\int_{W^+(A)\setminus A} Xf d\mu = \int_{W^+(A)\setminus A} Xf dh_*(\lambda \times \nu) = \int_{\mathbb{R}\times a^+} (Xf) \circ h dt d\nu$$

$$= \int_{a^+} \int_{-\infty}^{+\infty} \frac{\partial (f \circ h)}{\partial t}(t, x) dt d\nu = \int_{a^+} \lim_{T\to+\infty} (f(\phi_T(x)) - f(\phi_{-T}(x))) d\nu.$$

Our assumptions imply that for every $x \in W^+(A)$ there exists some $T_0 > 0$ such that $f(\phi_T(x)) = f|_A$ for every $T \geq T_0$. Hence $\lim_{T\to+\infty} f(\phi_{-T}(x))$ exists for ν-almost every $x \in a^+$, and so for μ-almost every $x \in W^+(A) \setminus A$, and

$$\int_{W^+(A)\setminus A} Xf d\mu = (f|_A) \cdot \nu(a^+) - \int_{a^+} \lim_{T\to+\infty} f(\phi_{-T}(x)) d\nu.$$

Since f has compact support, if $\lim_{T\to+\infty} f(\phi_{-T}(x))$ exists, then it must necessarily vanish in case $x \notin D^+(A)$ or it must be equal to $f|_A$ in case $x \in D^+(A) \setminus A$. Therefore,

$$\int_{a^+} \lim_{T\to+\infty} f(\phi_{-T}(x)) d\nu = (f|_A) \cdot \nu(a^+ \cap D^+(A))$$

from which follows that

$$\int_{W^+(A)\setminus A} Xf d\mu = (f|_A) \cdot [\nu(a^+) - \nu(a^+ \cap (D^+(A) \setminus A))] = (f|_A) \cdot \nu(a^+ \cap (W^+(A) \setminus D^+(A)). \quad \square$$

Corollary 3.2. *Let $\mu = h_*(\lambda \times \nu)$ be an invariant Radon measure on $W^+(A) \setminus A$ whose support is contained in $D^+(A) \setminus A$. If $f : W^+(A) \to \mathbb{R}$ is a smooth function with compact support such that $Xf = 0$ on some open neighbourhood of A and f is constant on A, then*

$$\int_{W^+(A)\setminus A} Xf d\mu = 0$$

and the limit $\lim_{T\to+\infty} f(\phi_{-T}(x))$ exists for μ-almost all $x \in D^+(A) \setminus A$.

Proof. Since the support of μ is assumed to be contained in $D^+(A) \setminus A$, the support of ν must be contained in $a^+ \cap D^+(A)$. Therefore, $\nu(a^+ \cap (W^+(A) \setminus D^+(A)) = 0$ and the assertion follows immediately from Proposition 3.1. $\square$

4 Coherent measures and the unstable manifold

Let M be a connected smooth manifold and X be a complete smooth vector field on M with flow $(\phi_t)_{t\in\mathbb{R}}$. Throughout this section $A \subset M$ will be a connected isolated unstable attractor of the flow.

Definition 4.1. A Radon measure μ on $W^+(A) \setminus A$ is called *coherent relative to A* if it has the following property:
(C) For every smooth function $f : W^+(A) \to \mathbb{R}$ with compact support such that $df = 0$ on some open neighbourhood of A we have

$$\int_{W^+(A)\setminus A} Xf d\mu = 0.$$

This definition is a slight generalization of the one introduced in [6].

Lemma 4.2. *Every coherent measure relative to A is invariant under the flow on $W^+(A) \setminus A$.*

Proof. If $f : W^+(A) \setminus A \to \mathbb{R}$ is a smooth function with compact support, then for every $t \in \mathbb{R}$ we have

$$\int_{W^+(A)\setminus A} (f \circ \phi_t - f)d\mu = \int_{W^+(A)\setminus A} \int_0^t X(f \circ \phi_s)ds d\mu$$

$$= \int_0^t \int_{W^+(A)\setminus A} X(f \circ \phi_s)d\mu ds = 0,$$

since μ is assumed coherent. If $B \subset W^+(A)\setminus A$ is a compact set, there exists a decreasing sequence of smooth functions with compact supports $f_n : W^+(A) \setminus A \to [0, 1]$, $n \in \mathbb{N}$, which converges pointwise to χ_B and then

$$\mu(B) = \int_{W^+(A)\setminus A} \chi_B d\mu = \lim_{n\to+\infty} \int_{W^+(A)\setminus A} f_n d\mu = \lim_{n\to+\infty} \int_{W^+(A)\setminus A} (f_n \circ \phi_t)d\mu$$

$$= \int_{W^+(A)\setminus A} \chi_B \circ \phi_t d\mu = \int_{W^+(A)\setminus A} \chi_{\phi_{-t}(B)} d\mu = \mu(\phi_{-t}(B))$$

for every $t \in \mathbb{R}$. It follows by regularity that this holds for every Borel subset of $W^+(A) \setminus A$. □

Lemma 4.3. *The support of any coherent measure relative to A is contained in $D^+(A) \setminus A$.*

Proof. Let $x_0 \in W^+(A) \setminus D^+(A)$. Since $D^+(A)$ is asymptotically stable with region of attraction $W^+(D^+(A)) = W^+(A)$, there exists a smooth, uniformly unbounded Lyapunov function $f : W^+(A) \to [0, +\infty)$ for $D^+(A)$ (see [12], [8]). More precisely, f is continuous on $W^+(A)$, smooth on $W^+(A) \setminus D^+(A)$, $f^{-1}(0) = D^+(A)$ and $Xf < 0$ on $W^+(A) \setminus D^+(A)$. Moreover, $\lim_{t\to-\infty} f(\phi_t(x)) = +\infty$ for every $x \in W^+(A) \setminus D^+(A)$ and $f^{-1}([0, c])$ is compact for every $c > 0$.

Let $\psi : \mathbb{R} \to [0,1]$ be a smooth function such that $\psi^{-1}(1) = (-\infty, \frac{f(x_0)}{2}]$, $\psi^{-1}(0) = [2f(x_0), +\infty)$ and $\psi'(t) < 0$ for $\frac{f(x_0)}{2} < t < 2f(x_0)$. Let $g = \psi \circ f$. Then g is smooth, $dg = 0$ on the open neighbourhood $f^{-1}[0, \frac{f(x_0)}{2})$ of A and g has compact support contained in $f^{-1}[0, 2f(x_0)]$, the latter being compact. Also, $Xg > 0$ on the open neighbourhood $f^{-1}(\frac{f(x_0)}{2}, 2f(x_0))$ of x_0. If now μ is a Radon measure on $W^+(A) \setminus A$ such that $x_0 \in \mathrm{supp}\mu \cap (W^+(A) \setminus D^+(A))$, there is a compact neighbourhood W of x_0 contained in $f^{-1}(\frac{f(x_0)}{2}, 2f(x_0))$ and so

$$\int_{W^+(A)\setminus A} Xg d\mu \geq \int_W Xg d\mu > 0.$$

Therefore μ cannot be coherent relative to A. $\square$

The combination of Corollary 3.2 and Lemma 4.3 yields the following characterization of coherent measures.

Corollary 4.4. *For a Radon measure μ on $W^+(A) \setminus A$ the following assertions are equivalent:*
(i) μ is a coherent measure relative to A.
(ii) The support of μ is contained in $D^+(A) \setminus A$ and μ is invariant.

Corollary 4.5. *If $f : W^+(A) \to \mathbb{R}$ is a smooth function such that $Xf = 0$ on some open neighbourhood of A and f is constant on A, then $Xf \in L^1(\mu)$ and*

$$\int_{W^+(A)\setminus A} Xf d\mu = 0$$

for every coherent measure μ relative to A. $\square$

Combining Proposition 3.1 with Lemma 4.2 and Lemma 4.3 we get also the following.

Corollary 4.6. *If μ is a coherent measure relative to A then for every smooth function $f : W^+(A) \to \mathbb{R}$ with compact support such that $df = 0$ on some open neighbourhood of A there exists a Borel set $B \subset D^+(A) \setminus A$ such that $\mu(W^+(A) \setminus (A \cup B)) = 0$ and the limit $\lim_{T \to +\infty} f(\phi_{-T}(x))$ exists for every $x \in B$.* $\square$

The following lemma guarantees the existence of coherent measures seemingly of a rather special kind. As the simple proof shows, for every $x \in W^-(A)$ there exists a coherent measure μ relative to A such that such that $x \in \mathrm{supp}\mu$.

Lemma 4.7. *There exists a coherent measure on $W^+(A) \setminus A$ relative to A.*

Proof. Since A is an unstable attractor, there exists at least one point $x \in W^+(A) \setminus A$ such that $\varnothing \neq L^{\pm}(x) \subset A$, and so necessarily $x \in D^+(A) \setminus A$ (see Corollary 1.2 of Chapter VI in [4]). The parametrization $\gamma_x : \mathbb{R} \to D^+(A) \setminus A$ of the orbit of x defined by $\gamma_x(t) = \phi_t(x)$ is a homeomorphism onto its image. If μ is the push forward of the Lebesgue measure on $\mathbb{R}$ by γ_x, then μ is locally finite. We shall prove that it satisfies condition (C). Let $f : W^+(A) \to \mathbb{R}$ be a smooth function with compact support such that $df = 0$ on some open neighbourhood V of A. Since A is connected, V can be chosen to be also connected. Then $f|_V$ is constant and

$$\int_{W^+(A)\setminus A} Xf d\mu = \int_{-\infty}^{+\infty} (f \circ \gamma_x)'(t)dt = f|_V - f|_V = 0. \quad \square$$

The simple coherent measure relative to A constructed in Lemma 4.7 has the property that the negative limit set of almost every point is contained in A. Actually, this is a property that every coherent measure has.

Theorem 4.8. *If μ is a coherent measure relative to A, then $\varnothing \neq L^-(x) \subset A$ for μ-almost every $x \in D^+(A) \setminus A$.*

Proof. Suppose that μ is coherent relative to A. Let (N, N^+, N^-) be an isolating block of A with $N \subset W^+(A)$. Since μ is invariant under the flow, by Lemma 4.2, it has the form $\mu = h_*(\lambda \times \nu)$, where λ is the Lebesgue measure on $\mathbb{R}$ and ν is a finite Borel measure on a^+. Let V be a relatively compact open neighbourhood of a^+ such that $\overline{V} \subset W^+(A) \setminus A$. There exists a smooth function $f : W^+(A) \to [0,1]$ such that $f^{-1}(1) = a^+$ and $f^{-1}(0) = W^+(A) \setminus V$. According to Corollary 4.6, there exists a Borel set $B \subset D^+(A) \setminus A$ such that $\mu(W^+(A) \setminus (A \cup B)) = 0$ and $\lim_{T\to+\infty} f(\phi_{-T}(x))$ exists for every $x \in B$. If $x \in B$ is such that $L^-(x) \cap (D^+(A) \setminus A) \neq \varnothing$, there exist $y \in a^+$ and $s_n \to +\infty$ such that $\phi_{-s_n}(x) \to y$. On the other hand, $L^-(x) \cap A \neq \varnothing$, because $x \in D^+(A) \setminus A$, and so there are $z \in A$ and $t_n \to +\infty$ such that $\phi_{-t_n}(x) \to z$. Then $f(\phi_{-t_n}(x)) = 0$ eventually for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} f(\phi_{-s_n}(x)) = f(y) = 1$, which contradicts the existence of $\lim_{T\to+\infty} f(\phi_{-T}(x))$. This proves that $\varnothing \neq L^-(x) \subset A$ for every $x \in B$. $\square$

Corollary 4.9. *If μ is a coherent measure relative to A, then* $\mathrm{supp}\mu \subset \overline{W^-(A)} \setminus A$. $\square$

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