

On the existence of absolutely continuous conformal measures for uniquely ergodic minimal Cantor homeomorphisms

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Abstract

We prove an existence criterion for absolutely continuous conformal measures in the case of uniquely ergodic minimal homeomorphisms of Cantor sets. It is applied to the case of a Denjoy C^1 diffeomorphism T of the circle in order to examine when $\log T'$ is a continuous coboundary on the unique Cantor minimal set of T .

1. *Introduction*

Let X be a compact metric space, $T : X \rightarrow X$ a continuous transformation and $f : X \rightarrow \mathbb{R}$ a continuous function. A Borel probability measure ν on X is called an e^f -conformal measure for T if

$$\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$. The function f is usually called the potential and e^f is called the weight. This kind of measure has been used without a particular name in [7] and appears naturally in the general framework of thermodynamic formalism. The existence of conformal measures has been studied in [2] for finite-to-one transformations, where they were obtained as weak* accumulation points of a certain parametrized family of probability measures. One can use alternatively the Schauder-Tychonoff fixed point theorem and this is our approach. Their existence in general is studied in [11] in the context of descriptive set theory.

This note is a first attempt to study the problem of the existence of absolutely continuous conformal measures with respect to a given ergodic T -invariant Borel probability measure in a particular case. More precisely, we assume that X is a Cantor set and T is a uniquely ergodic minimal homeomorphism of X with unique invariant Borel probability measure μ . Well known examples of uniquely ergodic minimal homeomorphisms on Cantor sets are adding machines, certain Toeplitz flows and the restrictions of orientation preserving homeomorphisms of the unit circle with irrational rotation numbers and without dense orbits to their unique minimal set. The problem is under what conditions does there exist an e^f -conformal measure ν which is absolutely continuous with respect to μ . This problem is closely related to the existence and regularity properties of solutions of the cohomological equation $f = u - u \circ T$, since if u is a continuous solution

then

$$\frac{e^{-u}}{\int_X e^{-u} d\mu} d\mu$$

is an e^f -conformal measure for T . Conversely, if ν is an e^f -conformal measure which is absolutely continuous with respect to μ , then $-\log\left(\frac{d\nu}{d\mu}\right)$ is a measurable solution. The relation is explained with details in section 2.

A continuous function $f : X \rightarrow \mathbb{R}$ is called a continuous coboundary (with respect to T) if there exists a continuous function $u : X \rightarrow \mathbb{R}$ such that $f = u - u \circ T$. The function u is usually called the transfer function. A necessary condition for f to be a continuous coboundary is of course $\int_X f d\mu = 0$, but the converse is not true. Since T is minimal and μ is ergodic, a continuous function f is a continuous coboundary on X if and only if it is a $L^\infty(\mu)$ coboundary, that is there exists $u \in L^\infty(\mu)$ such that $f = u - u \circ T$ μ -almost everywhere on X , by Proposition 4.2 on page 46 in [5] (see also [13]). According to the classical Gottschalk-Hedlund theorem (see page 102 in [6]) f is a continuous coboundary if and only if there exists $x_0 \in X$ such that

$$\sup\left\{\left|\sum_{k=0}^{n-1} f(T^k(x_0))\right| : n \in \mathbb{N}\right\} < +\infty.$$

It turns out that the condition we give for the existence of an absolutely continuous e^f -conformal measure with continuous Radon-Nikodym derivative is equivalent for f to be a continuous coboundary, in our case where X is a Cantor set and T is a uniquely ergodic minimal homeomorphism. It is weaker in general.

The main result of this work proved in section 3 can be formulated as follows.

THEOREM 1.1. *Let X be a Cantor set and $T : X \rightarrow X$ a uniquely ergodic minimal homeomorphism with unique invariant Borel probability measure μ . Let $f : X \rightarrow \mathbb{R}$ be a*

continuous function such that $\int_X f d\mu = 0$ and let $E_n(f) = \exp \sum_{k=1}^n f \circ T^{-k}$, if $n > 0$,

$E_0(f) = 1$ and $E_n(f) = \exp(-\sum_{k=0}^{|n|-1} f \circ T^k)$, if $n < 0$.

(a) If there exists a constant $c \geq 1$ such that

$$E_n(f)(x) \leq c \int_X E_n(f) d\mu$$

for every $x \in X$ and $n \in \mathbb{Z}$, then there exists an e^f -conformal measure ν for T which is absolutely continuous with respect to μ such that $\frac{d\nu}{d\mu} \in L^\infty(\mu)$.

(b) The following assertions are equivalent:

(i) There exists a constant $c \geq 1$ such that

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f)(x) \leq c \int_X E_n(f) d\mu$$

for every $x \in X$ and $n \in \mathbb{Z}$.

(ii) There exists a continuous function $u : X \rightarrow \mathbb{R}$ such that $f = u - u \circ T$.

(iii) There exists a unique e^f -conformal measure ν for T such that ν is absolutely continuous with respect to μ and $\frac{d\nu}{d\mu}$ is continuous.

Actually the e^f -conformal measure ν for T in Theorem 1.1, which is absolutely continuous with respect to μ , is equivalent to μ . This is a consequence of our assumption that μ is the unique T -invariant Borel probability measure (see Remark 3.3).

Note that by Jensen's inequality we have $\int_X E_n(f) d\mu \geq 1$ for every $n \in \mathbb{Z}$.

The motivating example for this note was the case of Denjoy C^1 diffeomorphisms of the circle. Let $T : S^1 \rightarrow S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number which is not topologically conjugate to a rotation. It is well known that T has a unique Cantor minimal set K and a unique invariant Borel probability measure μ whose support is precisely K . These maps are usually called Denjoy C^1 diffeomorphisms. It was proved by A. Denjoy that $\int_K (\log T') d\mu = 0$ (see Proposition 1.1 on page 71 in [5]). However, using an argument due to M. Herman [5] and the main result of [9] one can show that $\log T'$ is never a continuous coboundary on S^1 . A T' -conformal measure is called in [3] automorphic measure for T of exponent 1. From Theorem 1.1, the chain rule and the unique ergodicity we get the following.

COROLLARY 1.2. *Let $T : S^1 \rightarrow S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K and unique T -invariant Borel probability measure μ . The following are equivalent:*

(i) *There exists a constant $c \geq 1$ such that*

$$\frac{1}{c} \int_K (T^n)' d\mu \leq (T^n)'(z) \leq c \int_K (T^n)' d\mu$$

or equivalently

$$\frac{1}{c} \leq \lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \frac{(T^j)'(T^n(z))}{(T^j)'(z)} \leq c$$

for every $z \in K$ and $n \in \mathbb{Z}$.

(ii) *There exists a continuous function $u : K \rightarrow \mathbb{R}$ such that $\log T' = u - u \circ T$ on K .*

(iii) *There exists a unique T' -conformal measure ν for T on K such that ν is absolutely continuous with respect (and actually equivalent) to μ and $\frac{d\nu}{d\mu}$ is continuous.*

A. Portela essentially gives in [12] a sufficient condition under which $\log T'$ is not a continuous coboundary on K . The question whether $\log T'$ is a continuous coboundary on K is closely related to a problem stated in [1].

2. Conformal measures for uniquely ergodic minimal homeomorphisms

Let X be a compact metric space and $T : X \rightarrow X$ a uniquely ergodic minimal homeomorphism with unique invariant Borel probability measure μ . Let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$. An e^f -conformal measure for T is a Borel

probability measure ν on X such that

$$\int_X \phi d\nu = \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$ (see also p. 188 in [7]). Note that if $A \subset X$ is a measurable set, we have by regularity

$$\nu(A) = \int_X \chi_A d\nu = \int_X (\chi_A \circ T) e^f d\nu = \int_{T^{-1}(A)} e^f d\nu.$$

Thus, an e^f -conformal measure ν for T is quasi-invariant with respect to T and the minimality of T implies that ν is positive on non-empty open sets.

It is easy to see that if $h : X \rightarrow X$ is a homeomorphism and $S = h \circ T \circ h^{-1}$, then $h_*\nu$ is an $e^{f \circ h}$ -conformal measure for S for every e^f -conformal measure ν for T .

Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X equipped with the weak* topology. Let $W : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ be the continuous map defined by

$$W(\nu)(\phi) = \frac{1}{\int_X e^f d\nu} \cdot \int_X (\phi \circ T) e^f d\nu$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$ (see p. 185 in [7]). Actually W is a homeomorphism and its inverse is given by the formula

$$W^{-1}(\nu)(\phi) = \frac{1}{\int_X e^{-f \circ T^{-1}} d\nu} \cdot \int_X (\phi \circ T^{-1}) e^{-f \circ T^{-1}} d\nu.$$

Since $\mathcal{M}(X)$ is compact and convex, it follows from the Schauder-Tychonoff theorem that W has a fixed point in $\mathcal{M}(X)$.

LEMMA 2.1. *If $\nu \in \mathcal{M}(X)$ is a fixed point of W , then $\int_X e^f d\nu = 1$.*

Proof. Since ν is a fixed point of W , for every $n \in \mathbb{N}$ we have

$$\left(\int_X e^f d\nu \right)^n = \int_X \exp\left(\sum_{k=0}^{n-1} f \circ T^k \right) d\nu,$$

as one easily verifies by induction. It follows that

$$n \left| \log \left(\int_X e^f d\nu \right) \right| = \left| \log \left(\int_X \exp\left(\sum_{k=0}^{n-1} f \circ T^k \right) d\nu \right) \right| \leq \left\| \sum_{k=0}^{n-1} f \circ T^k \right\|,$$

and therefore

$$\left| \log \left(\int_X e^f d\nu \right) \right| \leq \left\| \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right\| = 0,$$

since $\int_X f d\mu = 0$ and μ is the unique invariant Borel probability measure of T .

From the above follows immediately the existence of e^f -conformal measures for uniquely ergodic minimal homeomorphisms (see also Theorem 3.7 in [7]).

THEOREM 2.2. *Let X be a compact metric space and $T : X \rightarrow X$ a uniquely ergodic minimal homeomorphism with unique invariant Borel probability measure μ . Then for every continuous function $f : X \rightarrow \mathbb{R}$ such that $\int_X f d\mu = 0$ there exists an e^f -conformal measure for T .*

In the rest of this section we shall describe the connection between e^f -conformal measures for T and the cohomological equation $f = u - u \circ T$ (compare with [14]).

If there exists measurable solution u defined μ -almost everywhere on X such that $e^{-u} \in L^1(\mu)$, then there exists an e^f -conformal measure ν for T which is absolutely continuous with respect to μ defined by

$$\nu(A) = \frac{1}{\int_X e^{-u} d\mu} \cdot \int_A e^{-u} d\mu$$

for all Borel measurable sets $A \subset X$. If there exists a continuous function $u : X \rightarrow \mathbb{R}$ such that $f = u - u \circ T$ on X , then ν is the unique e^f -conformal measure for T . Indeed, if ν' is any e^f -conformal measure for T , then

$$\int_X \phi e^u d\nu' = \int_X (\phi \circ T) e^u d\nu'$$

for every continuous function $\phi : X \rightarrow \mathbb{R}$. The unique ergodicity implies now that

$$\int_X \phi d\mu = \frac{1}{\int_X e^u d\nu'} \cdot \int_X \phi e^u d\nu' = \left(\int_X e^{-u} d\mu \right) \cdot \int_X \phi e^u d\nu'.$$

Therefore,

$$d\nu' = \frac{e^{-u}}{\int_X e^{-u} d\mu} \cdot d\mu = d\nu.$$

PROPOSITION 2.3. *Let X be a compact metric space and $T : X \rightarrow X$ be a uniquely ergodic minimal homeomorphism with unique invariant Borel probability measure μ . Let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$ and $\nu \in \mathcal{M}(X)$ be an e^f -conformal measure for T . If ν is absolutely continuous with respect to μ and $g = \frac{d\nu}{d\mu}$, then*

(a) $g \circ T = ge^f$ and $g > 0$ μ -almost everywhere on X .

(b) If $\log g \in L^\infty(\mu)$, there exists a continuous function $u : X \rightarrow \mathbb{R}$ such that $f = u - u \circ T$ everywhere on X .

Proof. By regularity, for every measurable set $A \subset X$ we have

$$\nu(A) = \int_X (\chi_A \circ T) e^f d\nu = \int_X (\chi_A \circ T) e^f g d\mu = \int_A (ge^f) \circ T^{-1} d\mu.$$

It follows from the uniqueness of the Radon-Nikodym derivative that $g = (ge^f) \circ T^{-1}$ μ -almost everywhere and therefore $g \circ T = ge^f$ μ -almost everywhere, since μ is T -invariant. To prove $g > 0$, let $B = g^{-1}(0)$. Then B is measurable and $T(B) = B$, because for every $x \in B$ we have $g(T(x)) = g(x)e^{f(x)} = 0$ and $g(T^{-1}(x))e^{f(T^{-1}(x))} = g(x) = 0$. The ergodicity of μ implies now that $\mu(B) = 0$, since ν is a probability measure. This means that $g > 0$ μ -almost everywhere. Now $u = -\log g$ is a measurable solution of the cohomological equation $f = u - u \circ T$ and (b) follows from Proposition 4.2 on page 46 in [5].

3. Absolutely continuous conformal measures

Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. The set

$$E_\mu = \{\nu \in \mathcal{M}(X) : \nu \ll \mu\}$$

is not empty, since it contains μ , and is convex. In general, E_μ is not a closed subset of $\mathcal{M}(X)$ with respect to the weak* topology.

In the rest of this section $T : X \rightarrow X$ will always be a uniquely ergodic minimal homeomorphism with unique invariant Borel probability measure μ and $f : X \rightarrow \mathbb{R}$ a continuous function such that $\int_X f d\mu = 0$. Also, $W : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ will be the homeomorphism defined in section 2. One can prove easily by induction that

$$W^n(\nu)(\phi) = \frac{1}{\int_X (\exp \sum_{k=0}^{n-1} f \circ T^k) d\nu} \cdot \int_X (\phi \circ T^n) (\exp \sum_{k=0}^{n-1} f \circ T^k) d\nu$$

and

$$W^{-n}(\nu)(\phi) = \frac{1}{\int_X (\exp(-\sum_{k=1}^n f \circ T^{-k})) d\nu} \cdot \int_X (\phi \circ T^{-n}) (\exp(-\sum_{k=1}^n f \circ T^{-k})) d\nu$$

for every $n \in \mathbb{N}$.

If we put

$$E_n(f) = \begin{cases} \exp \sum_{k=1}^n f \circ T^{-k}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \exp(-\sum_{k=0}^{|n|-1} f \circ T^k), & \text{if } n < 0, \end{cases}$$

then from the invariance of μ we get

$$W^n(\mu)(\phi) = \frac{1}{\int_X E_n(f) d\mu} \cdot \int_X \phi E_n(f) d\mu$$

for every $n \in \mathbb{Z}$. Note also that

$$\int_X (\phi \circ T)^f E_n(f) d\mu = \int_X \phi E_{n+1}(f) d\mu$$

for every continuous $\phi : X \rightarrow \mathbb{R}$.

If now $A \subset X$ is a measurable set, it follows from regularity that

$$W^n(\mu)(A) = \frac{1}{\int_X E_n(f) d\mu} \cdot \int_A E_n(f) d\mu$$

which implies that $W^n(\mu) \in E_\mu$ and

$$\frac{dW^n(\mu)}{d\mu} = \frac{E_n(f)}{\int_X E_n(f) d\mu}$$

for every $n \in \mathbb{Z}$.

Let C denote the convex hull of the orbit $\{W^n(\mu) : n \in \mathbb{Z}\}$ of μ under W . Then $W(C) = C$. Indeed, let

$$t_n = \frac{\int_X E_{n+1}(f) d\mu}{\int_X E_n(f) d\mu}$$

for all $n \in \mathbb{Z}$. If $a_1, \dots, a_n \geq 0$ are such that $a_1 + \dots + a_n = 1$ and $j_1, \dots, j_n \in \mathbb{Z}$, then

$$W\left(\sum_{k=1}^n a_k W^{j_k}(\mu)\right) = \sum_{k=1}^n \frac{a_k t_{j_k}}{a_1 t_{j_1} + \dots + a_n t_{j_n}} \cdot W^{j_k+1}(\mu).$$

This shows that $W(C) \subset C$ and similarly $W^{-1}(C) \subset C$. Since \bar{C} is a compact convex subset of $\mathcal{M}(X)$, it follows from the Schauder-Tychonoff theorem that W has a fixed point in \bar{C} or in other words there is an e^f -conformal measure for T in \bar{C} , by Lemma 2.1.

THEOREM 3.1. *Let X be a Cantor set and $T : X \rightarrow X$ a uniquely ergodic minimal homeomorphism with unique invariant Borel probability measure μ . Let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $\int_X f d\mu = 0$.*

(a) *If there exists a constant $c \geq 1$ such that $E_n(f)(x) \leq c \int_X E_n(f) d\mu$ for every $x \in X$ and $n \in \mathbb{Z}$, then $\bar{C} \subset E_\mu$ and $\frac{d\nu}{d\mu} \in L^\infty(\mu)$ for every $\nu \in \bar{C}$.*

(b) *If moreover there exists a constant $c \geq 1$ such that*

$$\frac{1}{c} \int_X E_n(f) d\mu \leq E_n(f)(x) \leq c \int_X E_n(f) d\mu$$

for every $x \in X$ and $n \in \mathbb{Z}$, then $\log\left(\frac{d\nu}{d\mu}\right) \in L^\infty(\mu)$ for every $\nu \in \bar{C}$.

Proof. Let $\nu \in \overline{C}$. There exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ in C such that $\nu = \lim_{n \rightarrow +\infty} \nu_n$ in the weak* topology. Let $f_n = \frac{d\nu_n}{d\mu}$, $n \in \mathbb{N}$. Note that f_n is continuous and $f_n \leq c$ for every $n \in \mathbb{N}$, by our assumption. Since X is assumed to be a Cantor set, for each $k \in \mathbb{N}$ there exists a finite cover \mathcal{P}_k of X consisting of disjoint, open-closed sets in X of diameter less than $1/k$. Then, $\bigcup_{k=1}^{\infty} \mathcal{P}_k$ is a basis for the topology of X and generates its Borel σ -algebra. In particular, the Borel σ -algebra of X is generated by the algebra of open-closed sets. In order to prove that $\nu \in E_\mu$, let $A \subset X$ be a Borel set with $\mu(A) = 0$. There exists an open-closed set $A_0 \subset X$ such that $\mu(A \Delta A_0) < \epsilon$ and $\nu(A \Delta A_0) < \epsilon$. Thus $0 < \mu(A_0) < \epsilon$ and $|\nu(A) - \nu(A_0)| < \epsilon$. Since A_0 is open and closed in X , we have $\nu(A_0) = \lim_{n \rightarrow +\infty} \nu_n(A_0)$, by weak* convergence. There exists now some $n_0 \in \mathbb{N}$ such that $|\nu_n(A_0) - \nu(A_0)| < \epsilon$ for $n \geq n_0$, and so

$$0 < \nu(A_0) < \nu_n(A_0) + \epsilon = \int_{A_0} f_n d\mu + \epsilon \leq c \cdot \mu(A_0) + \epsilon < (c+1)\epsilon.$$

Therefore $0 \leq \nu(A) < (c+2)\epsilon$ for every $\epsilon > 0$, which means that $\nu(A) = 0$. This shows that $\nu \in E_\mu$.

In order to prove that $\frac{d\nu}{d\mu} \in L^\infty(\mu)$, let $\mathcal{P}_k(x)$ denote the element of the partition \mathcal{P}_k which contains the point $x \in X$. If $g_k : X \rightarrow \mathbb{R}$ is the function defined by

$$g_k(x) = \frac{\nu(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))},$$

then $\lim_{k \rightarrow +\infty} g_k(x) = \frac{d\nu}{d\mu}(x)$ μ -almost everywhere on X and in $L^1(\mu)$ (see page 8 in [10]). On the other hand, by the weak* convergence, for every $k \in \mathbb{N}$ and $x \in X$ there exists some $n_k \in \mathbb{N}$ such that

$$|\nu(\mathcal{P}_k(x)) - \nu_{n_k}(\mathcal{P}_k(x))| < \frac{1}{k} \mu(\mathcal{P}_k(x)).$$

It follows that

$$0 \leq g_k(x) < \frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = \frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \leq \frac{1}{k} + c$$

for every $x \in X$ and $k \in \mathbb{N}$. Therefore, $0 \leq \frac{d\nu}{d\mu}(x) \leq c$ μ -almost everywhere on X , and so $\frac{d\nu}{d\mu} \in L^\infty(\mu)$. This proves (a). If the assumption of (b) holds, then $\frac{1}{c} \leq f_n \leq c$ for every $n \in \mathbb{N}$ and in addition to the above we have

$$g_k(x) > -\frac{1}{k} + \frac{\nu_{n_k}(\mathcal{P}_k(x))}{\mu(\mathcal{P}_k(x))} = -\frac{1}{k} + \frac{1}{\mu(\mathcal{P}_k(x))} \int_{\mathcal{P}_k(x)} f_{n_k} d\mu \geq -\frac{1}{k} + \frac{1}{c}$$

for every $x \in X$ and $k \in \mathbb{N}$. Therefore, $0 \leq \frac{1}{c} \leq \frac{d\nu}{d\mu}(x) \leq c$ μ -almost everywhere on X , which means that $\log\left(\frac{d\nu}{d\mu}\right) \in L^\infty(\mu)$.

Proof of Theorem 1.1. (a) This is the combination of Proposition 2.3, Theorem 3.1(a) and the Schauder-Tychonoff theorem.

(b) If (i) holds, then (ii) holds, by Theorem 3.1(b), Proposition 2.4(b) and Proposition 4.2 on page 46 in [5]. Moreover, (iii) is true from the discussion preceding Proposition 2.3. It is trivial to verify that (ii) implies (i). Finally, if (iii) holds, then from Proposition 2.3(b) we get (ii).

Remark 3.2. The reader may have noticed some similarity of the condition imposed in (a) of Theorem 3.1 with the condition appearing in Theorem 1 of [8].

Remark 3.3. Let (X, \mathcal{B}, ν) be a probability space and $T : X \rightarrow X$ be a bimeasurable bijection such that if $B \in \mathcal{B}$ then $\nu(B) = 0$ if and only if $\nu(T^{-1}(B)) = 0$. If μ is a T -invariant probability measure on \mathcal{B} such that ν is absolutely continuous with respect to μ , then there exists a T -invariant probability measure μ' on \mathcal{B} which is equivalent to ν . Such a measure can be constructed as follows. Let $g = \frac{d\nu}{d\mu}$ and $A = g^{-1}(0)$. Then, $A \in \mathcal{B}$ and $\nu(A) = 0$. If $S = \bigcup_{n \in \mathbb{Z}} T^n(A)$, then $S \in \mathcal{B}$ is T -invariant and $\nu(S) = 0$. If we put $\mu'(B) = \mu(B \setminus S)$ for every $B \in \mathcal{B}$, then μ' is a T -invariant probability measure on \mathcal{B} which is equivalent to ν .

It follows from this observation that the absolutely continuous e^f -conformal measure ν in Theorem 1.1 is actually equivalent to μ , since μ is assumed to be the unique T -invariant Borel probability measure and ν is quasi-invariant with respect to T .

4. Automorphic measures for Denjoy C^1 diffeomorphisms

Let $T : S^1 \rightarrow S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number $\rho(T)$. It is well known [5] that T is uniquely ergodic and there exists a unique minimal set $K \subset S^1$ which is the support of the unique T -invariant Borel probability measure μ , and either $K = S^1$, in which case T is topologically conjugate to the rotation by the angle $2\pi\rho(T)$ or K is a Cantor set and T is only topologically semi-conjugate to the rotation by the angle $2\pi\rho(T)$. In the latter case T is a Denjoy C^1 diffeomorphism and the semi-conjugation is never C^1 . In both cases, K is the non-wandering set of T and

$$\int_{S^1} \log(T^n)' d\mu = 0$$

for every $n \in \mathbb{Z}$.

A T' -conformal measure ν for T on K will be called automorphic for T and is a Borel probability measure on K such that

$$\int_K \phi d\nu = \int_K (\phi \circ T) T' d\nu$$

for every continuous function $\phi : K \rightarrow \mathbb{R}$. If $\log T'$ is a continuous coboundary on K , then there exists a unique automorphic measure for T absolutely continuous with respect to μ .

Let now $h : S^1 \rightarrow S^1$ be an orientation preserving C^1 diffeomorphism and $S = h \circ T \circ h^{-1}$. Then S is a Denjoy C^1 diffeomorphism with unique minimal set $h(K)$ and

unique S -invariant Borel probability measure $h_*\mu$. If ν is an automorphic measure for T , then

$$\nu' = \frac{h'}{\int_K h' d\nu} \cdot h_*\nu$$

is automorphic for S . It follows that if $\nu \ll \mu$, then $\nu' \ll h_*\mu$.

In the known examples of Denjoy C^1 diffeomorphisms T constructed by Denjoy and Herman we have $T'(z) = 1$ for every z in the unique minimal set K of T . In this case the unique T -invariant Borel probability measure μ is trivially the unique automorphic measure for T . So, all the C^1 conjugates of the examples of Denjoy-Herman have absolutely continuous unique automorphic measures on their unique minimal sets.

From Theorem 1.1(a), Remark 3.3 and the chain rule we obtain a sufficient condition for a Denjoy C^1 diffeomorphism to have an automorphic measure which is absolutely continuous with respect to its invariant Borel probability measure.

COROLLARY 4.1. *Let $T : S^1 \rightarrow S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K and unique T -invariant Borel probability measure μ . If there exists a constant $c \geq 1$ such that*

$$(T^n)'(z) \leq c \int_K (T^n)' d\mu$$

or equivalently

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \frac{(T^j)'(T^n(z))}{(T^j)'(z)} \geq \frac{1}{c}$$

for every $z \in K$ and $n \in \mathbb{Z}$, then there exists an automorphic measure for T which is equivalent to μ .

Let $T : S^1 \rightarrow S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K and unique T -invariant Borel probability measure μ . The set \mathcal{C} of the connected components of $S^1 \setminus K$ is countable and $\sum_{I \in \mathcal{C}} \lambda(I) \leq 2\pi$, where λ denotes the Lebesgue measure on S^1 . So, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of real numbers such that

$$(i) \sum_{n=1}^{\infty} \lambda_n \leq 2\pi,$$

$$(ii) \lambda_{n+1} < \lambda_n \text{ for every } n \in \mathbb{N}, \text{ and}$$

(iii) for every $n \in \mathbb{N}$ there exists some (possibly not unique) $I \in \mathcal{C}$ such that $\lambda_n = \lambda(I)$.

The sequence $(\lambda_n)_{n \in \mathbb{N}}$ is called the spectrum of K . It has been proved by D. McDuff that the sequence of ratios $(\frac{\lambda_n}{\lambda_{n+1}})_{n \in \mathbb{N}}$ is bounded and has 1 as a nontrivial limit point (see [4]). We do not know in general whether it actually converges to 1 or not.

PROPOSITION 4.2. *Let $T : S^1 \rightarrow S^1$ be a Denjoy C^1 diffeomorphism with unique Cantor minimal set K . Let $(\lambda_n)_{n \in \mathbb{N}}$ be the spectrum of K and $\sigma = \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\lambda_{n+1}}$.*

(i) *There exists a point $z_0 \in K$ such that $(T^n)'(z_0) \leq \frac{1}{\sigma}$ for every $n \in \mathbb{N}$.*

(ii) *There exists a point $z_1 \in K$ such that $T'(z_1) \geq \sigma$.*

(iii) There exists a point $z_2 \in K$ such that $(T^n)'(z_2) \geq 1$ for every $n \in \mathbb{N}$.

Proof. (i) There exist positive integers $n_k \rightarrow +\infty$ such that $\sigma = \lim_{k \rightarrow +\infty} \frac{\lambda_{n_k}}{\lambda_{n_k+1}}$. For every $k \in \mathbb{N}$ there exists some $I_k \in \mathcal{C}$ such that $\lambda_{n_k} \leq \lambda(I_k)$ and $\lambda(T^n(I_k)) \leq \lambda_{n_k+1}$ for every $n \in \mathbb{N}$. Since $\lim_{k \rightarrow +\infty} \lambda(T(I_k)) = 0$, there exists an accumulation point $y \in K$ of $\bigcup_{k=1}^{\infty} T(I_k)$. So, $z_0 = T^{-1}(y)$ is an accumulation point of $\bigcup_{k=1}^{\infty} I_k$, and passing to a subsequence, if necessary, we may assume that $\lim_{k \rightarrow +\infty} \text{dist}(z_0, I_k) = 0$. On the other hand, from the mean value theorem we have

$$(T^n)'(z_0) = \lim_{k \rightarrow +\infty} \frac{\lambda(T^n(I_k))}{\lambda(I_k)}$$

for every $n \in \mathbb{N}$, because T is C^1 . For every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sigma - \epsilon < \frac{\lambda_{n_k}}{\lambda_{n_k+1}} \leq \frac{\lambda(I_k)}{\lambda(T^n(I_k))}$$

for every $k \geq k_0$ and every $n \in \mathbb{N}$. Therefore, $(T^n)'(z_0) \leq \frac{1}{\sigma - \epsilon}$ for every $\epsilon > 0$ and $n \in \mathbb{N}$, and the conclusion follows.

(ii) Let $I \in \mathcal{C}$. For every $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that

$$\lambda(T^{-m_k}(I)) \leq \lambda_{n_k+1} < \lambda_{n_k} \leq \lambda(T^{-m_k+1}(I)).$$

By the mean value theorem, there exists $z_k \in T^{-m_k}(I)$ such that

$$\lambda(T^{-m_k+1}(I)) = T'(z_k)\lambda(T^{-m_k}(I)).$$

For every $\epsilon > 0$ there exists some $k_0 \in \mathbb{N}$ such that

$$T'(z_k) = \frac{\lambda(T^{-m_k+1}(I))}{\lambda(T^{-m_k}(I))} \geq \frac{\lambda_{n_k}}{\lambda_{n_k+1}} > \sigma - \epsilon$$

for every $k \geq k_0$. Since $\lim_{k \rightarrow +\infty} \lambda(T^{-m_k}(I)) = 0$, the sequence $(z_k)_{k \in \mathbb{N}}$ has an accumulation point $z \in K$. It follows that for every $\epsilon > 0$ there exists $z \in K$ such that $T'(z) \geq \sigma - \epsilon$, because T is C^1 . For the same reason and the compactness of K we conclude that there exists a point $z_1 \in K$ such that $T'(z_1) \geq \sigma$.

(iii) This is Lemma 3.1 in [12].

A. Portela proved in [12] that if the sequence of ratios $(\frac{\lambda_n}{\lambda_{n+1}})_{n \in \mathbb{N}}$ does not converge to 1, that is $\sigma > 1$, then $\inf\{(T^n)'(z_0) : n \in \mathbb{N}\} = 0$. Therefore $\{\log(T^n)'(z) : n \in \mathbb{Z}\}$ is unbounded for every $z \in K$, by minimality of K . It follows from the Gottschalk-Hedlund theorem that in this case $\log T'$ is not a continuous coboundary on K . From Corollary 1.2 and Proposition 4.2 we get the following nonexistence result.

COROLLARY 4.3. *Let $T : S^1 \rightarrow S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K and unique T -invariant Borel probability measure μ . If the sequence*

$$\{\log \int_K (T^n)' d\mu : n \in \mathbb{N}\}$$

is unbounded, then $\log T'$ is not a continuous coboundary on K .

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