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Divergence of C^1 vector fields and nontrivial minimal sets on 2-manifolds

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Abstract

We prove a Bendixson–Dulac type criterion for the nonexistence of nontrivial compact minimal sets of C^1 vector fields on orientable 2-manifolds. As a corollary we get that the divergence with respect to any volume 2-form of such a vector field must vanish at some point of any nontrivial compact minimal set. We also prove that all the nontrivial compact minimal sets of a C^1 vector field on an orientable 2-manifold are contained in the vanishing set of any inverse integrating factor. From this we get that if a C^1 vector field on an orientable 2-manifold has a nontrivial compact minimal set, then an infinitesimal symmetry is inessential on the minimal set.

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1. Introduction

A classical problem in the qualitative theory of 2-dimensional ordinary differential equations, with many applications in the physical sciences, is to examine the existence (or nonexistence) of limit cycles and describe their distribution in phase space. The Poincaré–Bendixson theorem and its generalizations can be used to prove the existence of periodic orbits, and in particular limit cycles, in various situations. One method to locate limit cycles that has appeared in the literature makes use of certain functions called inverse integrating factors (see Definition 2.1 below). More

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precisely, it has been proved in [5] that if a C^1 vector field X on an open subset U of $\mathbb{R}^2$ admits a C^1 inverse integrating factor $f : U \to \mathbb{R}$, then every limit cycle of X is contained in $f^{-1}(0)$. We refer the reader to [2] for further applications of inverse integrating factors in the qualitative study of planar C^1 vector fields.

In this work we are mainly concerned with the divergence and inverse integrating factors of C^1 vector fields on 2-manifolds. We were originally motivated by the question what kind of information can they give about the location and shape of compact limit sets, and in particular compact minimal sets, since a compact limit set always contains a compact minimal set. A C^1 vector field on a 2-manifold can have compact minimal sets which are not periodic orbits and are locally homeomorphic to the product of an open interval and a Cantor set. This kind of minimal sets are usually called nontrivial. The phenomenon of the existence of nontrivial compact minimal sets does not occur in C^2 vector fields on 2-manifolds (see [8]) and this is their great difference from C^1 vector fields. In Section 3 we prove a nonexistence result for nontrivial compact minimal sets of C^1 vector fields on orientable, connected, smooth 2-manifolds, which is analogous to the Bendixson–Dulac criterion on the nonexistence of periodic orbits for planar C^1 vector fields. More precisely, we prove that if the divergence with respect to some C^∞ volume 2-form is everywhere nonnegative (or everywhere nonpositive) then there are no nontrivial compact minimal sets (see Theorem 3.3). It follows from this that the divergence of a C^1 vector field with respect to any C^∞ volume 2-form always has a vanishing point on a nontrivial compact minimal set. This leads to the question whether for any nontrivial compact minimal set of a C^1 vector field on an orientable, connected, smooth 2-manifold there exists a C^∞ volume 2-form such that its divergence with respect to that volume 2-form vanishes identically on the minimal set. In the case of the Denjoy C^1 vector field on the 2-torus, which is concisely described in the Appendix of [9], such a C^∞ volume 2-form exists and is the riemannian volume element induced by the euclidean flat riemannian metric. An obstruction for a general positive answer is provided by the Gottschalk–Hedlund theorem (see [3]).

In Section 4 we prove that a nontrivial compact minimal set of a C^1 vector field X on an orientable, connected, smooth 2-manifold is contained in the vanishing set of every inverse integrating factor of X. Thus, in case a solution to the above question exists, it cannot have density (with respect to any given C^∞ volume 2-form) which is an inverse integrating factor of X on a neighborhood of the nontrivial compact minimal set. As a corollary we obtain that if X admits an inverse integrating factor for which 0 is a regular value, then X has no nontrivial compact minimal set. As one can get inverse integrating factors from infinitesimal symmetries, another corollary is that if X has a nontrivial compact minimal set, then an infinitesimal symmetry is inessential on the minimal set. Moreover, if it is C^2 it must vanish at some point of the nontrivial minimal set.

2. Divergence, inverse integrating factors and infinitesimal symmetries

Let M be an orientable, connected, smooth n-manifold, $n \geqslant 2$, oriented by a C^∞ volume n-form ω. If X is a C^1 vector field on M, then the divergence of X with respect to ω is defined as the unique continuous function $\operatorname{div}_\omega X : M \to \mathbb{R}$ such that $d(i_X\omega) = (\operatorname{div}_\omega X)\omega$. It is known (see Section 3 in Chapter I of [6]) that the (local) flow of X preserves ω if and only if $\operatorname{div}_\omega X = 0$. In this case X is called *ω-incompressible*.

If we have a C^1 vector field X on M and we want to reparametrize its flow so that it becomes ω-incompressible, then we must find an everywhere positive C^1 function $f : M \to \mathbb{R}$ such that $\frac{1}{f} \cdot X$ is ω-incompressible. Therefore,

$$0 = \operatorname{div}_\omega\left(\frac{1}{f} \cdot X\right) = X\left(\frac{1}{f}\right) + \frac{1}{f} \cdot \operatorname{div}_\omega X = -\frac{1}{f^2} \cdot Xf + \frac{1}{f} \cdot \operatorname{div}_\omega X.$$

In other words, we must find an everywhere positive C^1 solution of the linear partial differential equation $Xf = f \cdot \operatorname{div}_\omega X$. Thus we come to the following.

Definition 2.1. An *inverse integrating factor* (IIF) for X is a C^1 function $f : M \to \mathbb{R}$ satisfying the linear partial differential equation

$$Xf = f \cdot \operatorname{div}_\omega X. \tag{1}$$

The set H_X of C^1 solutions of (1) is a linear subspace of $C^1(M, \mathbb{R})$, which is closed with respect to the C^1 topology. It is clear that (1) may have no nowhere vanishing C^1 solution.

To describe the way that H_X depends on the choice of the volume form ω of M, let θ be another C^∞ volume n-form of M. There exists a unique nowhere vanishing C^∞ function $g : M \to \mathbb{R}$ such that $\theta = g\omega$. Since

$$d(i_X\theta) = dg \wedge i_X\omega + g(\operatorname{div}_\omega X)\omega$$

and

$$(Xg) \cdot \omega - dg \wedge i_X\omega = i_X(dg \wedge \omega) = 0,$$

we have

$$(\operatorname{div}_\theta X)\theta = (Xg) \cdot \omega + g(\operatorname{div}_\omega X)\omega$$

and therefore

$$\operatorname{div}_\theta X = \frac{1}{g} \cdot Xg + \operatorname{div}_\omega X.$$

If now f is an IIF with respect to ω, then

$$f \cdot \operatorname{div}_\theta X = \frac{f}{g} \cdot Xg + f \cdot \operatorname{div}_\omega X = \frac{f}{g} \cdot Xg + Xf$$

and so

$$X(gf) = (gf) \cdot \operatorname{div}_\theta X$$

or in other words gf is an IIF with respect to θ. This shows that the space of IIFs of X with respect to θ is $g \cdot H_X$, which is naturally isomorphic to H_X. Thus, H_X is essentially independent on the choice of the volume form on M.

The following two lemmas give basic properties of inverse integrating factors.

Lemma 2.2. *If $f : M \to \mathbb{R}$ is an IIF for X, then $f^{-1}(0)$ is invariant under the flow of X.*

Proof. Let $\gamma : I \to M$ be a maximal integral curve of X, where I is an open interval containing zero. The C^1 function $f \circ \gamma$ is a solution of the ordinary differential equation

$$x' = x \cdot \mathrm{div}_\omega X\big(\gamma(t)\big) \tag{2}$$

which is defined on $I \times \mathbb{R}$. It follows that

$$f\big(\gamma(t)\big) = f\big(\gamma(0)\big) \cdot \exp\left(\int_0^t (\mathrm{div}_\omega X)\big(\gamma(s)\big)\,ds\right) \tag{3}$$

for every $t \in I$. Hence $\gamma(I) \subset M \setminus f^{-1}(0)$, if $f(\gamma(0)) \neq 0$. $\square$

On the invariant open set $M \setminus f^{-1}(0)$ the C^1 function $\log|f|$ is a solution of the cohomological equation $Xu = \mathrm{div}_\omega X$.

Lemma 2.3. *For an IIF $f : M \to \mathbb{R}$ of X the following hold*:

(i) *The C^1 $(n-1)$-form $\frac{1}{f} i_X\omega$ on $M \setminus f^{-1}(0)$ is closed.*
(ii) *If $D \subset M \setminus f^{-1}(0)$ is an open set on which $\frac{1}{f} i_X\omega$ is exact and η is a C^2 $(n-2)$-form such that $\frac{1}{f} i_X\omega|_D = d\eta$, then $i_X(d\eta) = 0$ on D. In particular, if $n = 2$, then η is a C^2 function and a first integral of X on D.*

Proof. (i) Indeed, we have

$$\begin{aligned} d\left(\frac{1}{f} i_X\omega\right) &= -\frac{1}{f^2} \cdot df \wedge i_X\omega + \frac{1}{f} \cdot d(i_X\omega) = -\frac{1}{f^2} \cdot df \wedge i_X\omega + \frac{1}{f} \cdot (\mathrm{div}_\omega X)\omega \\ &= \frac{1}{f^2} \cdot \big[-df \wedge i_X\omega + (Xf) \cdot \omega\big] = \frac{1}{f^2} \cdot \big[-df \wedge i_X\omega + i_X(df) \wedge \omega\big] = 0. \end{aligned}$$

(ii) This is obvious. $\square$

The first assertion of Lemma 2.3 is equivalent to saying that the (local) flow of $\frac{1}{f} \cdot X$ on $M \setminus f^{-1}(0)$ preserves the C^∞ volume n-form ω (see [4,6]). The integral curves of $\frac{1}{f} \cdot X$ are reparametrizations of the integral curves of X and so both have the same (unoriented) orbits in $M \setminus f^{-1}(0)$.

By a theorem of E. Hopf which generalizes the Poincaré Recurrence Theorem to σ-finite measures, there exists a Borel set $P \subset M \setminus f^{-1}(0)$ such that the volume of $M \setminus P \cup f^{-1}(0)$ is zero and if $x \in P$ then either $x \in L^+(x) \cap L^-(x)$ or $L^+(x) \cup L^-(x) \subset M \setminus f^{-1}(0)$, possibly empty, where $L^+(x)$ denotes the positive limit set and $L^-(x)$ denotes the negative limit set of the orbit of x (see [7, pp. 454–459]). Since the measure defined by a volume form is positive on nonempty open sets, P is a dense subset of $M \setminus f^{-1}(0)$.

Under certain circumstances it may be possible to describe the flow outside $f^{-1}(0)$. One such case is given by the following.

Proposition 2.4. *Let M be an orientable, connected, smooth n-manifold, $n \geqslant 2$, oriented by a C^∞ volume n-form ω and X be a C^1 vector field on M. Let M be noncompact and* $\operatorname{div}_\omega X > 0$ *everywhere on M. If $f : M \to \mathbb{R}$ is an IIF, then there exists a continuous function $g : M \to \mathbb{R}^+$ such that $g^{-1}(0) = f^{-1}(0)$ and g is strictly increasing along the orbits of X in $M \setminus f^{-1}(0)$. If in addition $\partial f^{-1}(0)$ is compact, then $f^{-1}(0)$ is globally negatively asymptotically stable.*

Proof. If N is a connected component of $M \setminus f^{-1}(0)$, then $f|_N > 0$ or $f|_N < 0$. So, if we define $g : M \to \mathbb{R}^+$ by

$$g(x) = \begin{cases} 0, & \text{if } f(x) = 0, \\ f(x), & \text{if } f(x) > 0, \\ -f(x), & \text{if } f(x) < 0, \end{cases}$$

then g is continuous and satisfies our requirements. In other words, it is a global Lyapunov function for the closed invariant set $f^{-1}(0)$ with respect to $-X$. It is well known that if $\partial f^{-1}(0)$ is compact, this implies that $f^{-1}(0)$ is globally negatively asymptotically stable with respect to X. □

Example 2.5. On $\mathbb{R}^2$ consider the volume element $\omega = dx \wedge dy$ and let X be the (complete) C^∞ vector field defined by

$$X(x, y) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Then $\operatorname{div}_\omega X = 1$ and the flow of X is given by $\phi_t(x, y) = (t + x, ye^t)$, $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is the C^∞ function $f(x, y) = y$, then f is an IIF and in this case $f^{-1}(0) = \mathbb{R} \times \{0\}$ does not have compact boundary.

Remarks 2.6.

(a) In case $\operatorname{div}_\omega X < 0$ everywhere on M, if $f : M \to \mathbb{R}$ is an IIF, then the closed invariant set $f^{-1}(0)$ admits a Lyapunov function and if $\partial f^{-1}(0)$ is compact, then $f^{-1}(0)$ is globally positively asymptotically stable.
(b) Proposition 2.4 is true under the weaker assumption $(\operatorname{div}_\omega X)^{-1}(0) \subset f^{-1}(0)$.
(c) If M is a closed manifold, then by Stoke's formula the continuous function $\operatorname{div}_\omega X$ cannot be everywhere strictly positive (or negative) on M.

If X is a C^1 vector field on an orientable, connected, smooth 2-manifold M with C^∞ volume 2-form ω, the IIFs of X are closely related to its infinitesimal symmetries. This observation is originally due to Sophus Lie. An *infinitesimal symmetry* of X is another C^1 vector field Y on M such that $[X, Y] = h \cdot X$ for some continuous function $h : M \to \mathbb{R}$. It is well known that if Y is a C^2 complete infinitesimal symmetry of X then the C^2 diffeomorphisms constituting its flow send oriented orbits of X onto oriented orbits of X. If Y is C^2 but not necessarily complete, this is true at least locally.

If $g : M \to \mathbb{R}$ is a C^1 function, then $g \cdot X$ is an infinitesimal symmetry of X, but note that the orbits of $g \cdot X$ are subsets of orbits of X. So, in a way, $g \cdot X$ is not essential, since every C^1 diffeomorphism belonging to its flow induces the identity on the orbit space of X. In general, let $A \subset M$ be an invariant set with respect to X. An infinitesimal symmetry Y of X is called *inessential* (or *trivial*) on A if there exists a continuous function $g : A \to \mathbb{R}$ such that Xg exists, is continuous on A and $Y = g \cdot X$ on A. Then, $[X, Y] = (Xg) \cdot X$ on A.

Proposition 2.7. *If X, Y are C^1 vector fields on an orientable, connected, smooth 2-manifold M with C^∞ volume 2-form ω and $f = \omega(X, Y)$, then*

$$Xf = \omega\big(X, [X, Y]\big) + f \cdot \operatorname{div}_\omega X.$$

Proof. Suppose first that X is C^2. Then,

$$L_X i_Y \theta = i_{[X,Y]}\theta + i_Y L_X \theta$$

for every C^1 1-form θ, where L_X is the Lie derivative with respect to X and i denotes contraction. So,

$$\begin{aligned} Xf &= L_X i_Y i_X \omega = i_{[X,Y]} i_X \omega + i_Y L_X i_X \omega \\ &= \omega\big(X, [X, Y]\big) + i_Y i_X L_X \omega = \omega\big(X, [X, Y]\big) + f \cdot \operatorname{div}_\omega X. \end{aligned}$$

If X is only C^1, it can be approximated by a sequence of C^2 vector fields in the C^1 topology, and taking limits we arrive at the conclusion. □

The following corollary has its origins back to Sophus Lie.

Corollary 2.8. *Let M be an orientable, connected, smooth 2-manifold with a C^∞ volume 2-form ω and X be a C^1 vector field on M. If Y is a C^1 vector field on M such that $[X, Y] = h \cdot X$ for some continuous function $h : M \setminus F \to \mathbb{R}$, where F is the vanishing set of X, then $f = \omega(X, Y)$ is an IIF of X.*

If X is a C^∞ vector field on an orientable, connected, smooth 2-manifold with C^∞ volume 2-form ω, the set S_X of the C^∞ infinitesimal symmetries of X is a Lie subalgebra of the Lie algebra of C^∞ vector fields on M. The set I_X of the inessential C^∞ infinitesimal symmetries of X is an ideal of S_X. The map from S_X to the vector space of the C^∞ IIFs of X sending $Y \in S_X$ to $\omega(X, Y)$ provided by Corollary 2.8 is linear, and its kernel is precisely I_X, if X is nowhere vanishing on M.

Conversely, suppose that the C^1 function $f : M \to \mathbb{R}$ is an IIF of the C^1 vector field X and $\operatorname{div}_\omega X \neq 0$ everywhere on M. Let X_f be the continuous Hamiltonian vector field with Hamiltonian f defined by $i_{X_f}\omega = df$. Then,

$$f \cdot \operatorname{div}_\omega X = df(X) = \omega(X_f, X)$$

and therefore

$$f = \omega\left(X, -\frac{1}{\operatorname{div}_\omega X} \cdot X_f\right).$$

The vector field $Y = -(1/\operatorname{div}_\omega X) \cdot X_f$ is only continuous, but if X and f are C^∞, then Y is also C^∞ and there is a C^∞ function $h : M \setminus F \to \mathbb{R}$ such that $[X, Y] = h \cdot X$ on $M \setminus F$, since $[X, Y] \in \operatorname{Ker} i_X \omega$.

3. Divergence of vector fields and Poincaré maps

Let X be a C^1 vector field on an orientable, connected, smooth n-manifold M, $n \geqslant 2$, and $p \in M$ be such that $X(p) \neq 0$. There exist then $\epsilon > 0$ and a C^1 embedding $\psi : (-\epsilon, \epsilon) \times D^{n-1} \to M$, where D^{n-1} is the $(n-1)$-dimensional open unit disc, such that $\psi(0, 0) = p$, the set $V = \psi((-\epsilon, \epsilon) \times D^{n-1})$ is an open neighborhood of p, called a flow box around p, and

$$X|_V = \psi_*\left(\frac{\partial}{\partial t}\right).$$

For every $x \in D^{n-1}$ let $r(x) = \inf\{t > 0\colon\ \phi_t(\psi(0, x)) \in \psi(\{0\} \times D^{n-1})\}$, where ϕ is the flow of X. The continuity of the flow implies that the set $\{x \in D^{n-1}\colon\ r(x) < +\infty\}$ is open, but maybe empty. If $p \in L^+(p)$, then it is not empty. If moreover the orbit of p does not intersect $\overline{\psi(\{0\} \times D^{n-1})} \setminus \psi(\{0\} \times D^{n-1})$, there exists an open neighborhood U of 0 in D^{n-1} such that the function $r : U \to (\epsilon, +\infty)$ is C^1, because the flow is C^1. Consequently, the map $T : U \to D^{n-1}$ defined by

$$(\psi \circ j)\big(T(x)\big) = \phi\big(r(x), (\psi \circ j)(x)\big)$$

is a C^1 embedding of U onto an open subset of D^{n-1}, where $j : D^{n-1} \to (-\epsilon, \epsilon) \times D^{n-1}$ is the natural embedding $j(x) = (0, x)$. T is called the *first return map* or *Poincaré map* associated to the flow box.

If ω is a C^∞ volume n-form on M, it is easy to see that the continuous n-form $\psi^*\omega$ is given by the formula $\psi^*\omega = dt \wedge \psi^*(i_X\omega)$. Let $\Omega = (\psi \circ j)^*(i_X\omega)$.

Lemma 3.1. *For every $x \in U$ we have*

$$\big(T^*\Omega\big)_x = \exp\left(\int_0^{r(x)} (\operatorname{div}_\omega X)\big(\phi_s\big(\psi(0, x)\big)\big)\, ds\right) \cdot \Omega_x.$$

Proof. Since $\psi \circ j \circ T = \phi \circ (r, \psi \circ j)$, from the chain rule we have

$$\big(T^*\Omega\big)_x = (i_X\omega)_{\phi_{r(x)}(\psi(0,x))} \circ \left(\frac{\partial\phi}{\partial t}\big(r(x), \psi(0, x)\big) \circ r_{*x} + (\phi_{r(x)})_{*\psi(0,x)} \circ (\psi \circ j)_{*x}\right),$$

where the subscript $*$ means derivative. On the other hand,

$$\frac{\partial\phi}{\partial t}\big(r(x), \psi(0, x)\big) \circ r_{*x}(v) = r_{*x}(v) \cdot X\big(\phi_{r(x)}\big(\psi(0, x)\big)\big),$$

for every $v \in T_x D^{n-1}$ and $X(\phi_{r(x)}(\psi(0, x))) = (\phi_{r(x)})_{*\psi(0,x)}(X(\psi(0, x)))$ because ϕ is the flow of X. If now $v_1, v_2, \ldots, v_{n-1} \in T_x D^{n-1}$, then

$$\begin{aligned}&\big(T^*\Omega\big)_x(v_1, v_2, \ldots, v_{n-1})\\&\quad= \big((\phi_{r(x)})^*\omega\big)_{\psi(0,x)}\big(X\big(\psi(0,x)\big), (\psi\circ j)_{*x}(v_1), \ldots, (\psi\circ j)_{*x}(v_{n-1})\big).\end{aligned}$$

However, from Liouville's formula (see Theorem 3.2 in Chapter I of [6]) we get

$$\big((\phi_{r(x)})^*\omega\big)_{\psi(0,x)} = \exp\left(\int_0^{r(x)} (\operatorname{div}_\omega X)\big(\phi_s\big(\psi(0,x)\big)\big)\,ds\right)\cdot\omega_{\psi(0,x)}.$$

Substituting we arrive at the required formula. □

In case M is 2-dimensional there exists a continuous function $g : D^1 \to (0,+\infty)$ such that $\Omega = g\,dx$ on $D^1 = (-1,1)$. From Lemma 3.1 follows that the derivative of T is given by

$$T'(x) = \frac{g(x)}{g(T(x))}\cdot\exp\left(\int_0^{r(x)} (\operatorname{div}_\omega X)\big(\phi_s\big(\psi(0,x)\big)\big)\,ds\right) \tag{4}$$

for every $x \in U$. More generally, if $x \in U$ is such that the nth iterate T^n of T is defined at x, then from the chain rule and the group property of the flow we get

$$\big(T^n\big)'(x) = \prod_{k=0}^{n-1} T'\big(T^k(x)\big) = \frac{g(x)}{g(T^n(x))}\cdot\exp\left(\int_0^{S_n r(x)} (\operatorname{div}_\omega X)\big(\phi_s\big(\psi(0,x)\big)\big)\,ds\right), \tag{5}$$

where $S_n r(x) = \sum_{k=0}^{n-1} r(T^k(x)) > n\epsilon$.

It is well known that the phase portrait of a C^1 vector field X on a connected, orientable, smooth 2-manifold M can contain 1-dimensional compact minimal sets that are not periodic orbits (see the Appendix in [9]). Minimal sets of this kind are called *nontrivial*, and are locally homeomorphic to the cartesian product of an open interval with a Cantor set. However, a C^2 vector field on a 2-manifold cannot have nontrivial compact minimal sets (see [8]).

Let A be a nontrivial compact minimal set of X, let $p \in A$ and let V be a flow box around p as above. Then $K = \{x \in D^1 \colon \psi(0,x) \in A\}$ is a Cantor set and shrinking V, if necessary, we can choose so that $A \cap (\overline{\psi(\{0\}\times D^1)} \setminus \psi(\{0\}\times D^1)) = \emptyset$. Thus, there is an open neighborhood U of K in D^1 such that the function $r : U \to (\epsilon,+\infty)$ is defined and is C^1. Moreover, K is minimal under the corresponding Poincaré map T.

Lemma 3.2. *For every open neighborhood W of K in U there exists a connected component I of $U \setminus K$ at least one of whose endpoints is contained in K such that T^n is defined on $\bar I$ and $T^n(\bar I) \subset W$ for every $n \geqslant 0$.*

Proof. Since K is a Cantor set, there exists some $a \in K$ which is an endpoint of some connected component of $U \setminus K$. Then $T^n(a)$ is defined and is endpoint of a connected component of $U \setminus K$ for every $n \geqslant 0$. Let $\delta = \inf\{\operatorname{dist}(x, U \setminus W)\colon x \in K\}$. Then $\delta > 0$ and since the sum of the lengths of these connected components must be at most 2, there exists some n_0 such that the connected component of $U \setminus K$ with one endpoint $T^n(a)$ has length less that δ for every $n \geqslant n_0$. It suffices now to take I to be the connected component of $U \setminus K$ with one endpoint $T^{n_0}(a)$. $\square$

The above calculations lead to the following nonexistence result of Poincaré–Bendixson type, which is analogous to the Bendixson–Dulac criterion on the nonexistence of periodic orbits for planar C^1 vector fields.

Theorem 3.3. *Let X be a C^1 vector field on an orientable, connected, smooth 2-manifold M. If there exists a C^∞ volume 2-form ω on M such that* $\operatorname{div}_\omega X \geqslant 0$ *everywhere on M, then X has no nontrivial compact minimal set.*

Proof. Suppose that A is a nontrivial compact minimal set. Using the preceding notations, let $I = (a, b)$ be the connected component of $U \setminus K$ given by Lemma 3.2, starting with any open neighborhood W of K having compact closure contained in U. By the Mean Value Theorem, for every $n \geqslant 1$ there exists some $a < \zeta_n < b$ such that $T^n(b) - T^n(a) = (T^n)'(\zeta_n) \cdot (b - a)$. Since $\overline{W}$ is compact, there exists some $c > 1$ such that $\frac{1}{c} < g(x) < c$ for every $x \in \overline{W}$. From Eq. (5) giving $(T^n)'$ and our hypothesis follows now that $(T^n)'(\zeta_n) > \frac{1}{c^2}$ for every $n \geqslant 1$. Therefore,

$$\frac{2}{b-a} \geqslant \sum_{n=1}^{\infty} \frac{T^n(b) - T^n(a)}{b-a} = \sum_{n=1}^{\infty} \big(T^n\big)'(\zeta_n) > \sum_{n=1}^{\infty} \frac{1}{c^2} = +\infty.$$

This contradiction proves the conclusion. $\square$

Since the vector fields X and $-X$ have the same unoriented orbits, Theorem 3.3 is also true under the assumption $\operatorname{div}_\omega X \leqslant 0$ everywhere on M. Thus, if A is a nontrivial compact minimal set of a C^1 vector field X on an orientable, connected, smooth 2-manifold M with a fixed C^∞ volume 2-form, then in every connected open neighborhood of A the divergence of X takes positive and negative values, and so there are points at which it vanishes. Since A is a continuum and M is a manifold, A has a neighborhood basis consisting of connected open subsets of M, and so we get the following corollary which can also be proved directly using a similar argument as in the proof of Theorem 3.3.

Corollary 3.4. *Let X be a C^1 vector field on an orientable, connected, smooth 2-manifold M. If $A \subset M$ is a nontrivial compact minimal set of X, then for every C^∞ volume 2-form ω on M the divergence* $\operatorname{div}_\omega X$ *of X with respect to ω vanishes at some point of A.*

4. Nontrivial compact minimal sets and inverse integrating factors

Let M be an orientable, connected, smooth n-manifold, $n \geqslant 2$, oriented by a C^∞ volume n-form ω and let X is a C^1 vector field on M with (local) flow ϕ. Let also A be a compact minimal set of X. Suppose that $\theta = \frac{1}{f} \cdot \omega$ is another C^∞ volume n-form for some C^∞ function $f : M \to \mathbb{R} \setminus \{0\}$. If the divergence of X with respect to both volume forms ω and θ vanishes on A, then $Xf = 0$ on A and so f must be constant on A, because A is minimal. By Corollary 3.4, if M is 2-dimensional and A is a nontrivial compact minimal set, then for every C^∞ volume 2-form θ on M there exists some point of A at which $\operatorname{div}_\theta X$ vanishes. The above observation shows that it is not true that the divergence of X vanishes everywhere on A for every C^∞ volume 2-form on M. The question now arises whether there exists a C^∞ volume 2-form $\theta = \frac{1}{f} \cdot \omega$ on M such that $\operatorname{div}_\theta X = 0$ on A. As we saw in the beginning of Section 2, this question is equivalent to the problem of the existence of a C^∞ function $f : M \to \mathbb{R} \setminus \{0\}$ which satisfies the linear partial differential equation $Xf = f \cdot \operatorname{div}_\omega X$ on A. Recall that by the Gottschalk–Hedlund theorem for compact minimal sets, there exists a (necessarily unique up to constant) continuous function $f : A \to \mathbb{R} \setminus \{0\}$ such that $X(\log |f|) = \operatorname{div}_\omega X$ on A if and only if there exists a point $p \in A$ and $c > 0$ such that

$$\left| \int_0^t (\operatorname{div}_\omega X)\big(\phi_s(p)\big)\, ds \right| < c$$

for every $t \in \mathbb{R}$ (see [3]). So an obvious obstruction for a positive answer is the following. If there exists a C^∞ volume 2-form ω on M, a point $p \in A$ and a sequence $t_n \to +\infty$ such that

$$\lim_{n \to +\infty} \int_0^{t_n} (\operatorname{div}_\omega X)\big(\phi_s(p)\big)\, ds = +\infty \quad (\text{or } -\infty)$$

then there is no C^∞ volume 2-form θ on M such that $\operatorname{div}_\theta X = 0$ on A.

In the rest of this section we shall prove that such a (even C^1) solution on A, if it exists, cannot be the restriction to A of an IIF of X on a neighborhood of A and derive some corollaries.

In [1] we have given the following description of the flow near a nontrivial compact minimal set A (of a general continuous flow) on an orientable 2-manifold M. There exists a connected, open, invariant neighborhood E of A with the following properties:

(a) The restricted flow on $E \setminus A$ is completely unstable.
(b) If $x \in E$, then $L^+(x) \cup L^-(x) \subset A \cup \partial E$ and $L^+(x) = A$ or $L^-(x) = A$.
(c) Every connected component of $E \setminus A$ contains at least one point x such that $L^+(x) = L^-(x) = A$.
(d) ∂E contains no nontrivial compact minimal set, and if M is compact, it contains no periodic orbit also.

Proposition 4.1. *Let X be a C^1 vector field on a connected, orientable, smooth 2-manifold M, which is oriented by a C^∞ volume 2-form ω. If $f : M \to \mathbb{R}$ is an IIF for X, then every nontrivial compact minimal set of X is contained in $f^{-1}(0)$.*

Proof. Let $A \subset M$ be a nontrivial compact minimal set of X and E be the connected, open, invariant neighborhood of A as above. Suppose that A is not contained in $f^{-1}(0)$. Then $A \cap f^{-1}(0) = \emptyset$, because A is minimal and $f^{-1}(0)$ is closed and invariant, by Lemma 2.2. Property (b) of E implies that $E \cap f^{-1}(0) = \emptyset$ also. Since $\frac{1}{f} \cdot X$ restricted to E preserves the volume 2-form ω, by Lemma 2.3, it follows from E. Hopf's generalization of the Poincaré Recurrence Theorem that there exists a dense subset P of E such that if $x \in P$ then either $x \in L^+(x) \cap L^-(x)$ or $L^+(x) \cup L^-(x) \subset M \setminus E$, possibly empty. Therefore $P \cap (E \setminus A) \neq \emptyset$ and we get a contradiction with property (b) of E. $\square$

The above proposition has a number of corollaries. First we obtain the following analytic criterion of Poincaré–Bendixson type.

Corollary 4.2. *Let X be a C^1 vector field on a connected, orientable, smooth 2-manifold M, which is oriented by a C^∞ volume 2-form ω. If X admits an IIF $f : M \to \mathbb{R}$ such that 0 is a regular value of f, then X has no nontrivial compact minimal set.*

If C is a limit cycle of a C^1 vector field X on a connected, orientable, smooth 2-manifold M, then $C \subset f^{-1}(0)$ for every IIF $f : M \to \mathbb{R}$ of X. This is proved in [5, Theorem 9] in case M is an open subset of $\mathbb{R}^2$, but the proof works for any orientable 2-manifold. So we arrive at the following.

Corollary 4.3. *Let X be a C^1 vector field on a connected, orientable, smooth 2-manifold M and $x \in M$. If $L^+(x)$ is a 1-dimensional compact minimal set, then $L^+(x) \subset f^{-1}(0)$ for every IIF $f : M \to \mathbb{R}$ of X.*

Finally, we obtain the following corollary concerning infinitesimal symmetries.

Corollary 4.4. *Let X be a C^1 vector field on a connected, orientable, smooth 2-manifold M. If $A \subset M$ is a nontrivial compact minimal set of X, then every infinitesimal symmetry of X is inessential on A.*

Proof. Let Y be an infinitesimal symmetry of X and $h : M \to \mathbb{R}$ be a continuous function such that $[X, Y] = h \cdot X$. If ω is any C^∞ volume 2-form on M, then $f = \omega(X, Y)$ is an IIF of X, by Corollary 2.8. Therefore, $A \subset f^{-1}(0)$ from Proposition 4.1, which means that there is a continuous function $g : A \to \mathbb{R}$ such that $Y = g \cdot X$ on A, since X nowhere vanishes on A. Moreover, Xg exists on A and $h|_A = Xg$. So, $Xg : A \to \mathbb{R}$ is continuous. Hence Y is inessential on A. □

Remark 4.5. Note that a C^2 infinitesimal symmetry Y of X in the situation of Corollary 4.4 must vanish at some point of A. Indeed, if Y nowhere vanishes on A, then A is also a nontrivial compact minimal set of Y and so Y cannot be C^2 by [8].

Proposition 4.1 or Corollary 4.3 can be used as a substitute to Poincaré–Bendixson theory to exclude the existence of nontrivial compact minimal sets for C^1 vector fields on orientable 2-manifolds. As an illustration we give a specific example.

Example 4.6. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic C^1 function given by the formula

$$\psi(x) = \begin{cases} \frac{1}{2}\sin^2 x, & \text{if } (2k-1)\pi \leqslant x \leqslant 2k\pi, \\ \frac{2}{3}\sin^{3/2} x, & \text{if } 2k\pi \leqslant x \leqslant (2k+1)\pi, \end{cases}$$

where $k \in \mathbb{Z}$. Let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ be the $2\pi\mathbb{Z}^2$-invariant C^1 function with

$$\tilde{f}(x, y) = \psi(x) + \psi(y).$$

The vanishing set of $\tilde{f}$ consists of the points

$$(2n\pi, 2m\pi),\ \big((2n+1)\pi, 2m\pi\big),\ \big(2n\pi, (2m+1)\pi\big),\ \big((2n+1)\pi, (2m+1)\pi\big), \quad n, m \in \mathbb{Z}.$$

If $\lambda, \mu : \mathbb{R} \to \mathbb{R}$ are 2π-periodic C^1 functions (take for instance $\lambda = \mu = \psi$), the C^1 vector field

$$\tilde{X}(x, y) = \mu(y) f(x, y) \frac{\partial}{\partial x} + \lambda(x) f(x, y) \frac{\partial}{\partial y}$$

is $2\pi\mathbb{Z}^2$-invariant. So $\tilde{X}$ projects to a C^1 vector field X on the 2-torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. The vanishing set F of X consists of the four points $p(0,0)$, $p(\pi,0)$, $p(\pi,0)$ and $p(\pi,\pi)$, where $p : \mathbb{R}^2 \to T^2$ is the quotient map. Also $\tilde{f}$ induces a C^1 function $f : T^2 \to \mathbb{R}$ with $f^{-1}(0) = F$. Let ω denote the euclidean volume 2-form on T^2. This means that $p^*\omega = dx \wedge dy$. An easy calculation shows that f is an IIF for X with respect to ω. Now Proposition 4.1 ensures that X has no nontrivial compact minimal set in $T^2 \setminus F$, and therefore on T^2, since F is finite. Note that the main result of [8] cannot be applied here, as X is not twice differentiable. The classical Poincaré–Bendixson theory cannot be applied either, since F is finite.

References

[1] K. Athanassopoulos, The flow near nontrivial minimal sets on 2-manifolds, Math. Proc. Cambridge Philos. Soc. 108 (1990) 569–573.
[2] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, On the integrability of two-dimensional flows, J. Differential Equations 157 (1999) 163–182.
[3] R. McCutcheon, The Gottschalk–Hedlund theorem, Amer. Math. Monthly 106 (1999) 670–672.
[4] I.A. Garcia, D. Shafer, Integral invariants and limit sets of planar vector fields, J. Differential Equations 217 (2005) 363–376.
[5] H. Giacomini, J. Llibre, M. Viano, On the nonexistence, existence and uniqueness of limit cycles, Nonlinearity 9 (1996) 501–516.
[6] R. Mañé, Ergodic Theory and Differentiable Dynamics, Springer-Verlag, New York, 1987.
[7] V.V. Nemitskij, V.V. Stepanov, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, NJ, 1960.
[8] A.J. Schwartz, A generalization of a Poincaré–Bendixson theorem to closed two dimensional manifolds, Amer. J. Math. 85 (1963) 453–458.
[9] P.A. Schweitzer, Counterexamples to the Seifert conjecture and opening closed leaves of foliations, Ann. of Math. 100 (1974) 386–400.