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The equipartition of curves

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ABSTRACT

In this paper we analyze the problem of partitioning a continuous curve into n parts with equal successive chords, the curve EquiPartition problem (EP). The goal is to locate $n-1$ consecutive curve points, so that the curve can be divided into n segments with equal chords under a distance function. We adopt a level set approach to prove that for any continuous injective curve in a metric space and any number n there always exists at least one n-equipartition (EP). A new approximate algorithm, that is the first EP algorithm, inspired from the level set approach is proposed for finding all solutions with high accuracy. Finally, EP based applications are presented and special properties of their solutions are discussed.

1. Introduction

The equipartition problem (EP) is the following. Let (X,ρ) be a metric space and let $c:[0,1]\to X$ be an injective continuous curve. We seek $0<t_1<\cdots<t_n<1$ such that $\rho(c(t_{i-1}),c(t_i))=\rho(c(t_i),c(t_{i+1}))$, where $t_0=0$ and $t_{n+1}=1$. This problem was posed by J.F. Pal in 1940. In 1954 K. Urbanik gave a proof of the existence of a solution to the problem based on the Brouwer fixed point theorem [20]. An alternative proof based again on the Brouwer fixed point theorem was presented recently in [11]. A problem of this kind is also the "square-peg" problem for which we refer the reader to [7].

Our approach is based on the topological study of the zero level set of a certain continuous function defined on the standard n-dimensional simplex Δ_n (see Fig. 1). As far as we know, this idea is new and has not been used before for the EP problem.

Of course the case $n=1$ is trivial, because if $F:[0,1]\to\mathbb{R}$ is the continuous function

$$F(t)=\rho\big(c(t),c(1)\big)-\rho\big(c(0),c(t)\big),$$

then $F(0)>0$ and $F(1)<0$ and by the Intermediate Value Theorem there exists some $0<t_1<1$ such that $F(t_1)=0$. Every point in $F^{-1}(0)$ is a solution to the problem. Our level set approach is an extension of this elementary idea. The fact that F changes signs on the boundary points of $[0,1]$ is equivalent to saying that the restriction of F to the boundary $\{0,1\}$ of

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$[0, 1]$ with values in $\mathbb{R} \setminus \{0\}$ is not homotopic to a constant. For $n > 1$ this is replaced by Proposition 2.1 below, which says that the restriction of a certain continuous function to the boundary of Δ_n is not homotopic to a constant.

In general, the number of solutions depends on the curve shape and n. There exist degenerate cases for which the number of solutions for a particular value of n is infinite. A brief exposition of the basic issues of the EP can be found in [16].

Generalizing the EP, one can take X to be a topological space and $\rho : X \times X \to \mathbb{R}^+$ a continuous function such that $\rho(x, y) = 0$ if and only if $x = y$ (not necessarily a distance function). One can ask then, given a positive integer n and real numbers $\lambda_0 > 0, \lambda_1 > 0, \dots, \lambda_n > 0$, if there exists a partition $0 < s_1 < s_2 < \cdots < s_n < 1$, such that

$$\lambda_0 \rho\big(c(0), c(s_1)\big) = \lambda_1 \rho\big(c(s_1), c(s_2)\big) = \cdots = \lambda_{n-1} \rho\big(c(s_{n-1}), c(s_n)\big) = \lambda_n \rho\big(c(s_n), c(1)\big).$$

This is equivalent to the following. Let $\Delta_2 = \{(s_1, s_2) \in \mathbb{R}^2 \colon\ 0 \leqslant s_1 \leqslant s_2 \leqslant 1\}$ be the standard 2-dimensional simplex. Let $d : \Delta_2 \to \mathbb{R}^+$ be a continuous function, such that $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$. Given a positive integer n and real numbers $\lambda_0 > 0, \lambda_1 > 0, \dots, \lambda_n > 0$, does there exist a partition $0 < s_1 < s_2 < \cdots < s_n < 1$ such that

$$\lambda_0 d(0, s_1) = \lambda_1 d(s_1, s_2) = \cdots = \lambda_{n-1} d(s_{n-1}, s_n) = \lambda_n d(s_n, 1)?$$

Curve segmentation is a challenging problem in computational geometry, particularly for pattern recognition applications. Object boundary representation and curve simplification are based on curve segmentation. Another example of such segmentation approach is the polygonal approximation [2], which is a well-known and widely studied problem. In some applications it could be interesting to have a uniform representation according to an appropriate quality measure. We have adopted such an approach for 3D modelling and non-articulated motion tracking [17], leading to the curve equipartition problem. The objective is the partition of the feature sequence into "homogeneous" segments with uniform characteristics according to a predefined error criterion.

Shape representation [12] by polygonal approximation has become a popular technique for constructing a concise description of a boundary in the form of a sequence of straight lines. There are two main approaches to the problem: (1) Approximate the curve by a polygon minimizing an error criterion and (2) finding a subset of dominant points as vertices of the approximating polygon. Under the first approach, the goal is to capture the main characteristics of the boundary shape with the least number of line segments. The second approach works by locating the vertices of the approximating polygon directly through detecting points of high curvature. These vertices contain useful information concerning the curve shape and they can be used on image and shape analysis. On the other hand constraining the length of the line segments to be constant, as the equipartition does, gives a more concise representation, allowing more flexibility for studying deformations of the polygons, and finally of the original curve.

In Section 2 we adopt a level set approach to give a new geometric proof that the generalized EP has a solution for any n (see Theorem 2.5 below). An approximate algorithm inspired from the level set approach is described in Section 3 for finding all solutions with high accuracy. Finally, in Section 4, applications based to the EP are presented and special properties of their solutions are discussed.

2. A level set approach to the equipartition problem

In this section we give a new proof of existence of a solution to the general equipartition problem for any number of pieces. Our approach is based on the analysis of the connected components of the zero level set of a certain function using methods of Combinatorial and Algebraic Topology. For definitions and properties of homotopy and Brouwer degree we refer the reader to any standard text on Algebraic Topology, such as for instance [5].

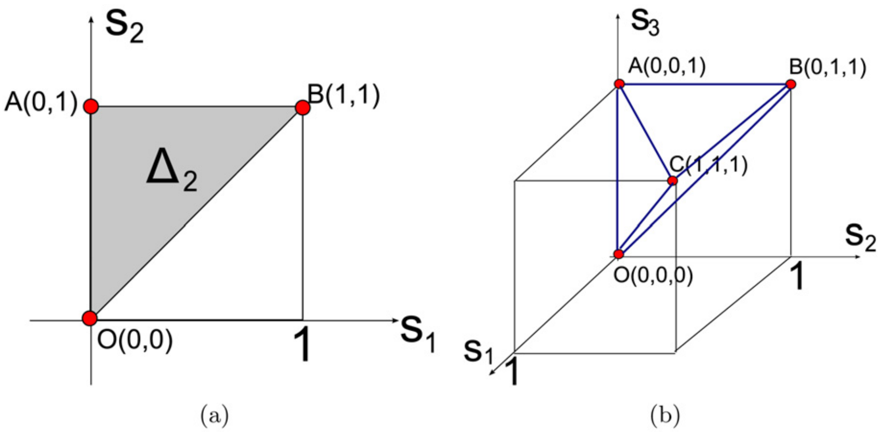

Fig. 1. (a) The triangle ABO is Δ_2. (b) The tetrahedron ABCO is Δ_3.

Let

$$\Delta_n = \{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n\colon\ 0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n \leqslant 1\}$$

be the n-dimensional simplex. Its boundary is $\partial\Delta_n = B_1 \cup B_2 \cup \cdots \cup B_{n+1}$, where

$$B_j = \{(s_1, s_2, \dots, s_n) \in \Delta_n\colon\ s_{j-1} = s_j\},$$

$1 \leqslant j \leqslant n+1$, are the $(n-1)$-faces of Δ_n, putting $s_0 = 0$ and $s_{n+1} = 1$. Fig. 1 illustrates Δ_n in cases $n = 2$ (triangle) and $n = 3$ (tetrahedron).

Let $d\colon \Delta_2 \to \mathbb{R}^+$ be a continuous function such that $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$. Let also $\lambda_0 > 0, \lambda_1 > 0, \dots, \lambda_n > 0$ be real numbers. For each $1 \leqslant j \leqslant n$ let $f_j\colon \Delta_n \to \mathbb{R}$ be the function defined by

$$f_j(s_1, s_2, \dots, s_n) = \lambda_j d(s_j, s_{j+1}) - \lambda_{j-1} d(s_{j-1}, s_j),$$

where again we have set $s_0 = 0$ and $s_{n+1} = 1$.

If $f = (f_1, f_2, \dots, f_{n-1})\colon \Delta_n \to \mathbb{R}^{n-1}$, then obviously

$$f^{-1}(0) \subset \{(0, 0, \dots, 0)\} \cup \operatorname{int} B_{n+1} \cup \operatorname{int} \Delta_n,$$

because $d^{-1}(0) = B_2$ in Δ_2, where int denotes topological interior.

If $F_n = (f_1, f_2, \dots, f_{n-1}, f_n)$, then F_n vanishes nowhere on $\partial\Delta_n$. A key property of F_n is the following.

Proposition 2.1. *For every integer $n \geqslant 2$, the continuous map $F_n|_{\partial\Delta_n}\colon \partial\Delta_n \to \mathbb{R}^n \setminus \{0\}$ has Brouwer degree $(-1)^n$.*

Proof. Examining the signs of f_j, $1 \leqslant j \leqslant n$, on $\partial\Delta_n$, we observe that the vector $F_n(s_1, s_2, \dots, s_n)$ at $(s_1, s_2, \dots, s_n)$ points inward Δ_n for every $(s_1, s_2, \dots, s_n) \in \partial\Delta_n$. Indeed, the $(n-1)$-face B_j, $1 \leqslant j \leqslant n+1$, is contained in the affine hyperplane $g_j^{-1}(0)$, where $g_j\colon \mathbb{R}^n \to \mathbb{R}$ is the affine map $g_j(s_1, s_2, \dots, s_n) = s_j - s_{j-1}$. As above, $s_0 = 0$ and $s_{n+1} = 1$. If $(s_1, s_2, \dots, s_n) \in B_j$, then

$$f_{j-1}(s_1, s_2, \dots, s_n) = -\lambda_{j-2} d(s_{j-2}, s_j) \leqslant 0,$$
$$f_j(s_1, s_2, \dots, s_n) = \lambda_j d(s_j, s_{j+1}) \geqslant 0$$

and therefore

$$\langle F_n(s_1, s_2, \dots, s_n), \nabla g_j(s_1, s_2, \dots, s_n)\rangle = \lambda_{j-2} d(s_{j-2}, s_j) + \lambda_j d(s_j, s_{j+1}),$$

for $1 < j < n+1$, while

$$\langle F_n(s_1, s_2, \dots, s_n), \nabla g_j(s_1, s_2, \dots, s_n)\rangle = \begin{cases} \lambda_1 d(0, s_2), & \text{if } j = 1, \text{ and} \\ \lambda_{n-1} d(s_{n-1}, 1), & \text{if } j = n+1. \end{cases}$$

Since $\langle F_n(s_1, s_2, \dots, s_n), \nabla g_j(s_1, s_2, \dots, s_n)\rangle \geqslant 0$ for every $(s_1, s_2, \dots, s_n) \in B_j$ and for every $1 \leqslant j \leqslant n+1$, we conclude that the vector $F_n(s_1, s_2, \dots, s_n)$ at $(s_1, s_2, \dots, s_n)$ points inward for every $(s_1, s_2, \dots, s_n) \in \partial\Delta_n$.

It follows now from Hopf's formula that

$$(-1)^n \deg F_n = \chi(\Delta_n) = 1,$$

where deg denotes the Brouwer degree and χ the Euler characteristic (see Example 4.8 on p. 269 and Proposition 4.9 on p. 270 of [5]). □

It follows from Proposition 2.1 that $F_n|_{\partial\Delta_n}\colon \partial\Delta_n \to \mathbb{R}^n \setminus \{0\}$ is not homotopic to a constant. This will be used in the proof of the main Lemma 2.4 below. In order to give the reader the idea of proof, we shall first treat the case $n = 2$, which corresponds to equipartition in three pieces and is relatively elementary.

Lemma 2.2. *If $n = 2$, the connected component C of $f^{-1}(0)$ which contains $(0, 0, \dots, 0)$ has non-empty intersection with $\operatorname{int} B_3$.*

Proof. Let K be the connected component of $Y = f^{-1}(0) \cup [0, 1] \times \{1\}$ which contains $[0, 1] \times \{1\}$. Suppose that the conclusion is not true. Then $C \cap K = \emptyset$ and there exists a polygonal simple arc I which separates C form K in Δ_2 with endpoints $(0, t)$ and (s, s), for some $0 < t < 1$ and $0 < s < 1$, whose any other point is contained in the interior of $\Delta_2 \setminus Y$. This follows directly from Theorem 3.3 on p. 143 in [14], which says that any two connected components of a compact subset of the plane are separated by a simple closed polygonal curve in its complement. It is also proved in any dimension in the course of proof of Lemma 2.4 below. Since $f = f_1$ and $f(0, t) > 0$, $f(s, s) < 0$, it follows from the Intermediate Value Theorem that $I \cap f^{-1}(0) \neq \emptyset$, contradiction. □

Proposition 2.3. *Let $d : \Delta_2 \to \mathbb{R}^+$ be a continuous function such that $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$. For any set of real numbers $\lambda_0 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, there exist $0 < s_1 < s_2 < 1$ such that*

$$\lambda_0 d(0, s_1) = \lambda_1 d(s_1, s_2) = \lambda_2 d(s_2, 1).$$

Proof. We denote as in the proof of Lemma 2.2 by C the connected component of $f^{-1}(0)$ which contains $(0, 0)$. We have $f_2(0, 0) = \lambda_2 d(0, 1) > 0$. Lemma 2.2 says that there exists some $0 < s < 1$ such that $(s, 1) \in C$. Since $f_2(s, 1) = -\lambda_1 d(s, 1) < 0$, there exists some $(s_1, s_2) \in C$ such that $f_2(s_1, s_2) = 0$, by the Intermediate Value Theorem. Obviously, $(s_1, s_2) \in F_2^{-1}(0)$. □

Lemma 2.4. *If $n \geqslant 3$, the connected component C of $f^{-1}(0)$ which contains $(0, 0, \dots, 0)$ has non-empty intersection with $\operatorname{int} B_{n+1}$.*

Proof. Suppose that $C \cap B_{n+1} = \emptyset$. Then there exists an embedded polyhedral, compact, connected $(n-1)$-manifold $D \subset \Delta_n \setminus f^{-1}(0)$ whose boundary ∂D is a topological $(n-2)$-dimensional sphere S^{n-2} such that $\partial D \subset \partial \Delta_n \setminus B_{n+1}$ and $D \setminus \partial D \subset \operatorname{int} \Delta_n$. One way to construct D is the following. Let $Y = B_{n+1} \cup f^{-1}(0)$ and let K be the connected component of Y which contains B_{n+1}. Then $C \cap K = \emptyset$ and so there are two disjoint compact sets $A_1, A_2 \subset Y$ such that $C \subset A_1$, $K \subset A_2$ and $A_1 \cup A_2 = Y$ (see Theorem 5.6 on p. 82 in [14]). Obviously, $A_1 \cap \partial \Delta_n = \{(0, 0, \dots, 0)\}$. Let $0 < \delta < \frac{1}{4} \operatorname{dist}(C, K)$ and $I_0 = [0, \delta/\sqrt{n}]^n$, where $\operatorname{dist}(C, K) = \inf\{\|x - y\|\colon x \in C \text{ and } y \in K\}$. Then I_0 has diameter δ and therefore is disjoint from A_2. Let now

$$0 < \epsilon < \frac{1}{\sqrt{n}} \min\left\{\delta, \frac{1}{4} \operatorname{dist}(A_1 \setminus \operatorname{int} I_0, \partial \Delta_n)\right\}$$

and let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_k = 1\}$ be a partition of $[0, 1]$ such that $\delta/\sqrt{n} \in \mathcal{P}$ and $t_j - t_{j-1} < \epsilon$, for all integers $1 \leqslant j \leqslant k$. Let $\mathcal{P}^n$ be the corresponding partition of $[0, 1]^n$ and let Q_0 be the union of all n-cubes I in $\mathcal{P}^n$ such that $I \cap A_1 \neq \emptyset$ and $I \cap \operatorname{int} I_0 = \emptyset$. Note that if I and J are n-cubes of $\mathcal{P}$ in Q_0, then $I \cap J$ is a common face of I and J, if non-empty, which can be thickened to an n-dimensional parallelepiped which does not intersect A_2. In case $I \cap J$ does not intersect Y we may thicken it to a n-dimensional parallelepiped with the same property. Adding these thickenings to Q_0 we obtain a set Q which is a polyhedral, connected, compact n-manifold with boundary contained in $\operatorname{int} \Delta_n$. Moreover, $A_1 \setminus I_0 \subset \operatorname{int} Q$, $Q \cap A_2 = \emptyset$ and

$$f^{-1}(0) \cap \partial Q = A_1 \cap \partial Q \subset I_0 \cap Q = I_0 \cap \partial Q = \operatorname{int}_F(F \cap \partial Q),$$

where $F = [0, \delta/\sqrt{n}]^{n-1} \times \{\delta/\sqrt{n}\}$ is the top $(n-1)$-face of I_0, and int_F denotes topological interior relative to the set F. The set $G = F \cap \Delta_n \setminus \operatorname{int}_F(F \cap \partial Q)$ is homeomorphic to a $(n-1)$-dimensional disc with a finite number of holes. Note that the $(n-2)$-simplex $F \cap \partial \Delta_n$ is contained in G. Let now D be the connected component of $(\partial Q \setminus \operatorname{int}_F(F \cap \partial Q)) \cup G$ which contains $F \cap \partial \Delta_n$. Then D is a polyhedral, connected, compact $(n-1)$-manifold with boundary, whose boundary is precisely $F \cap \partial \Delta_n$, hence homeomorphic to S^{n-2}, and therefore $\partial D \subset \partial \Delta_n \setminus B_{n+1}$. Also, $D \setminus \partial D \subset \operatorname{int} \Delta_n$ and $D \subset \Delta_n \setminus f^{-1}(0)$, by construction.

Since f vanishes nowhere on $\partial \Delta_n \setminus \{(0, 0, \dots, 0)\} \cup \operatorname{int} B_{n+1}$, it vanishes nowhere on D. Consequently, the continuous map $f|_{\partial D} : \partial D \to \mathbb{R}^{n-1} \setminus \{0\}$ has Brouwer degree zero and so does the continuous map $f|_{\partial B_{n+1}}$ for the same reason, that is because it is the restriction of a continuous map, namely f, from the compact connected $(n-1)$-dimensional manifold $D \cup (\partial \Delta_n \setminus (I_0 \cup \operatorname{int} B_{n+1}))$ with boundary ∂B_{n+1} to $\mathbb{R}^{n-1} \setminus \{0\}$. However, this contradicts Proposition 2.1, because $f|_{\partial B_{n+1}}$ is identified with $F_{n-1}|_{\partial \Delta_{n-1}}$. □

We can now state and prove the main result of this section from which the existence of solution to the general equipartition problem follows.

Theorem 2.5. *Let $d : \Delta_2 \to \mathbb{R}^+$ be a continuous function such that $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$. Then for every positive integer n and real numbers $\lambda_0 > 0$, $\lambda_1 > 0, \dots, \lambda_n > 0$, there exists a partition $0 < s_1 < s_2 < \cdots < s_n < 1$ such that*

$$\lambda_0 d(0, s_1) = \lambda_1 d(s_1, s_2) = \cdots = \lambda_{n-1} d(s_{n-1}, s_n) = \lambda_n d(s_n, 1).$$

Proof. This follows from Lemma 2.4 in the same manner as Proposition 2.3 follows from Lemma 2.2, since on one hand $f_n(0, 0, \dots, 0) = \lambda_n d(0, 1) > 0$ and on the other $f_n(s_1, s_2, \dots, s_{n-1}, 1) = -\lambda_{n-1} d(s_{n-1}, 1) < 0$ for $(s_1, s_2, \dots, s_{n-1}, 1) \in \operatorname{int} B_{n+1}$. □

Let now (X, ρ) be a metric space or even more generally let X be a topological space and $\rho : X \times X \to \mathbb{R}^+$ be a continuous function such that $\rho(x, y) = 0$ if and only if $x = y$. Let $c : [0, 1] \to X$ be an injective continuous curve, that is c is a topological embedding. Taking the continuous function $d : \Delta_2 \to \mathbb{R}^+$ defined by $d(s_1, s_2) = \rho(c(s_1), c(s_2))^2$, we see that $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$, since c is injective. In this case, $F_n^{-1}(0)$ is the set of solutions to the equipartition problem in $n + 1$ pieces for c. A direct application of Theorem 2.5 gives now the existence of solution to the general equipartition problem.

Theorem 2.6. *Let X be a topological space and $c:[0,1]\to X$ be a topological embedding. Then for every continuous function $\rho: X\times X\to\mathbb{R}^+$ with the property $\rho(x,y)=0$ if and only if $x=y$, and every positive integer n and real numbers $\lambda_0>0, \lambda_1>0,\dots,\lambda_n>0$, there exist $0<s_1<s_2<\cdots<s_n<1$ such that*

$$\lambda_0\rho\big(c(0),c(s_1)\big)=\lambda_1\rho\big(c(s_1),c(s_2)\big)=\cdots=\lambda_{n-1}\rho\big(c(s_{n-1}),c(s_n)\big)=\lambda_n\rho\big(c(s_n),c(1)\big).$$

3. Iso-level algorithm (ILA)

3.1. Description of the algorithm

In this section we propose a new approximate algorithm inspired from the level set approach to the EP of Section 2. To our knowledge, this kind of algorithm is used for the first time for the EP problem. It is named Iso-Level Algorithm (ILA), because an equipartition in $n+1$ pieces is determined by a finite sequence $(0,s_1),(s_1,s_2),\dots,(s_n,1)$ of points in Δ_2, which belong to the same level set of the function $d:\Delta_2\to\mathbb{R}^+$ of Theorem 2.5, if we take $\lambda_0=\lambda_1=\cdots=\lambda_n=1$, which we do in the sequel. In order to implement ILA we approximate d by a piecewise linear function $\hat{d}$ and solve precisely the problem for $\hat{d}$. The ILA computes all the solutions and is inductive. Thus, when it is executed for n, it also solves the problem for any positive integer number less than n.

We proceed now to the description of the ILA. Let $d:\Delta_2\to\mathbb{R}^+$ be a continuous function such that $d(s_1,s_2)=0$ if and only if $s_1=s_2$.

Let $\mathcal{P}=\{0=t_0<t_1<t_2<\cdots<t_{m-1}<t_m=1\}$ be a partition of $[0,1]$. On Δ_2 we consider the triangulation into triangles of the form

$$D^1_{ij}=\big\{(s_1,s_2)\in[t_i,t_{i+1}]\times[t_j,t_{j+1}]:\ s_2-t_j\leqslant s_1-t_i\big\},$$
$$D^2_{ij}=\big\{(s_1,s_2)\in[t_i,t_{i+1}]\times[t_j,t_{j+1}]:\ s_1-t_i\leqslant s_2-t_j\big\},$$

for $0\leqslant i<j<m$, or

$$D^2_{ii}=\big\{(s_1,s_2)\in[t_i,t_{i+1}]\times[t_i,t_{i+1}]:\ t_i\leqslant s_1\leqslant s_2\leqslant t_{i+1}\big\},$$

for $0\leqslant i<m$.

We take $\hat{d}:\Delta_2\to\mathbb{R}^+$ to be the simplicial function defined by the vertex map which sends each vertex (t_i,t_j) of this triangulation to $d(t_i,t_j)$, $0\leqslant i\leqslant j\leqslant m$. Then $\hat{d}$ is continuous and $\hat{d}(s_1,s_2)=0$ if and only if $s_1=s_2$. For each $1\leqslant j\leqslant n$ let $\hat{f}_j:\Delta_n\to\mathbb{R}$ be the function defined by

$$\hat{f}_j(s_1,s_2,\dots,s_n)=\hat{d}(s_j,s_{j+1})-\hat{d}(s_{j-1},s_j),$$

where $s_0=0$ and $s_{n+1}=1$. We put $\hat{f}=(\hat{f}_1,\hat{f}_2,\dots,\hat{f}_{n-1})$ and $\hat{F}_n=(\hat{f}_1,\hat{f}_2,\dots,\hat{f}_{n-1},\hat{f}_n)$. The points of $\hat{F}_n^{-1}(0)$ are approximate solutions to the EP for d.

At each iteration step k, $k=1,2,\dots,n$, the algorithm computes a (possibly non-connected) polygonal curve L_k from the corresponding polygonal curve L_{k-1} of the previous step. At the initial step we take $L_1=\{0\}\times[0,1]$ and at the k-th step we put

$$L_k=\big\{(u,z)\in\Delta_2:\ 0<u<z \text{ and } \hat{d}(u,z)=\hat{d}(v,u) \text{ for some } v<u \text{ with } (v,u)\in L_{k-1}\big\}.$$

Since L_{k-1} is a polygonal curve and $\hat{d}$ is a simplicial function, it is elementary to see that L_k is polygonal (possibly non-connected). Actually, a line segment of L_{k-1} in a triangle of the triangulation gives a set of line segments of L_k, each one of them is contained in a different triangle of the triangulation.

Note that if $(s_{n-1},s_n)\in L_n$, there exist $0<s_1<s_2<\cdots<s_{n-1}<s_n<1$, such that $(s_{n-2},s_{n-1})\in L_{n-1},\dots,(0,s_1)\in L_1$ and

$$\hat{d}(0,s_1)=\hat{d}(s_1,s_2)=\cdots=\hat{d}(s_{n-1},s_n).$$

Using the notations of Section 2, the level set

$$\hat{f}^{-1}(0)=\big\{(s_1,s_2,\dots,s_n)\in\Delta_n:\ (0,s_1)\in L_1,(s_1,s_2)\in L_2,\dots,(s_{n-1},s_n)\in L_n\big\}$$

approximates $f^{-1}(0)$. The points of $\hat{f}^{-1}(0)$ at which $\hat{f}_n$ vanishes are the solutions for the EP for $\hat{d}$ and approximate solutions for d in $n+1$ pieces.

We shall describe now a pseudo-code of the above procedure (see the end of this section). At each iteration step k, for each line segment $L_{k-1}(i)$ in L_{k-1}, $i=1,2,\dots,|L_{k-1}|$, where $|L_{k-1}|$ denotes the number of the line segments composing L_{k-1}, which belongs to a specific triangle $D^{\tau'}_{i'j}$, $\tau'=1,2$, $i'\leqslant j$, of the triangulation (we get it by using the getTriangle() function), we compute the line segment $LS(L_{k-1}(i),D^{\tau}_{jh})$. $LS(L_{k-1}(i),D^{\tau}_{jh})$ corresponds to $L_{k-1}(i)$, according to the definition of L_k, and is contained in the triangle D^{τ}_{jh}, $\tau=1,2$, $j\leqslant h$. An efficient way to store the line segments is by using their endpoints. In the initial step we break L_1 as follows:

$$L_1=\big[(0,0),(0,t_1)\big]\cup\big[(0,t_1),(0,t_2)\big]\cup\cdots\cup\big[(0,t_{m-1}),(0,1)\big].$$

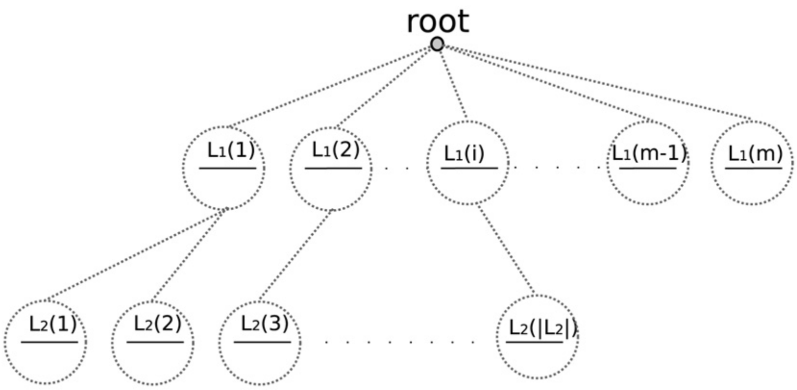

Fig. 2. An example of a tree data structure.

Tree.root $= [(0,0),(0,0)]$
for $i = 1$ **to** m **do**
  $L_1(i) = [(t_{i-1}, 0), (t_i, 0)]$
  Tree.addNode(Tree.root, $L_1(i)$)
end
for $k = 2$ **to** n **do**
  $count = 0$
  for $i = 1$ **to** $|L_{k-1}|$ **do**
    $D^{\tau'}_{i'j} = L_{k-1}(i)$.getTriangle()
    for $h = j$ **to** m **do**
      for $\tau = 1$ **to** 2 **do**
        if $LS(L_{k-1}(i), D^{\tau}_{jh}) \neq \varnothing$ **then**
          $count = count + 1$
          $L_k(count) = LS(L_{k-1}(i), D^{\tau}_{jh})$
          Tree.addNode($L_{k-1}(i)$, $LS(L_{k-1}(i), D^{\tau}_{jh})$)
        end
      end
    end
  end
  (L_k line segments simplification)
end
The solutions are computed inductively as the roots of $\hat{f}_n$ and by backtracking the tree data structure

Algorithm 1. Iso-level algorithm.

At each step we store the correspondence between $L_{k-1}(i)$ (parent node) and $LS(L_{k-1}(i), D^{\tau}_{jh})$ (child node) in a tree data structure. The root of tree data structure is the trivial segment $[(0,0),(0,0)]$. The function addNode($L_{k-1}(i)$, $LS(L_{k-1}(i), D^{\tau}_{jh})$) connects the new child node $LS(L_{k-1}(i), D^{\tau}_{jh})$ to the parent node $L_{k-1}(i)$. Finally, line segments of L_n are leaves of the Tree. Fig. 2 illustrates an example of tree data structure.

In the final n-th step we also compute the roots of $\hat{f}_n$ on L_n, which are points $(s_{n-1}, s_n) \in L_n$ and by moving backwards we compute the points $(s_{n-2}, s_{n-1}) \in L_{n-1}, \ldots, (0, s_1) \in L_1$, by backtracking (from leaves to root) the Tree. Fig. 3 illustrates a triangulation, the recursive computation of $\{s_1, s_2, s_3\}$, and the curves L_1, L_2 and L_3.

There does not exist an upper bound of the number of line segments in L_k. As our experiments show, it may increase exponentially with k. In order to reduce this number and the computational cost, an optional line segments simplification procedure can be applied. If we want to bound the maximum number of line segments in L_k that belong to a triangle of the triangulation, we merge successive line segments in L_k in the same triangle that are almost collinear. We give more details about this optional procedure in the next section.

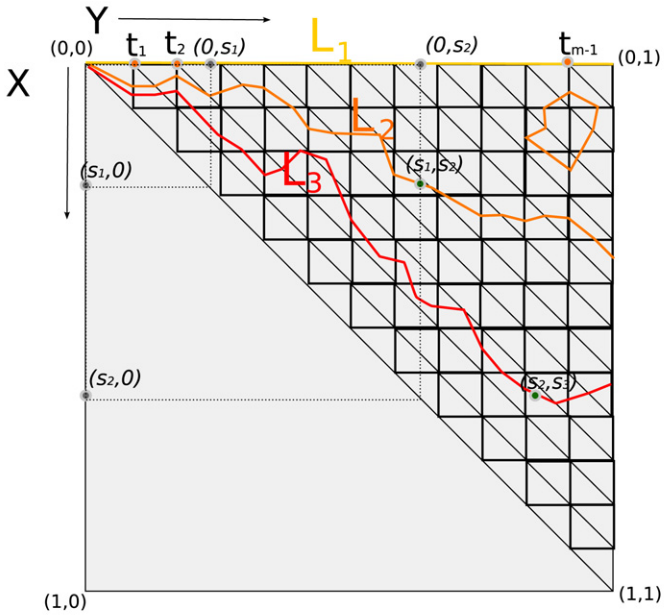

Fig. 3. An example of curve equipartition into 4 chords.

3.2. Computational complexity and error analysis

Concerning the method complexity, the computational cost for each line segment $L_{k-1}(i)$, $i = 1, 2, \dots, |L_{k-1}|$ of L_{k-1} is $O(m)$, since the search space for the LS computation has $O(m)$ triangles. The number $|L_k|$ of the line segments composing L_k is normally $O(m)$ and the total computational cost is $O(n \cdot m^2)$.

However, $|L_k|$ may increase exponentially. In these cases, the line segment simplification procedure is executed by keeping a limited number of line segments per triangle. Let $|L_k^{\tau,ij}|$ be the number of line segments of L_k that belong to the triangle D_{ij}^{τ}, $\tau = 1, 2$, $i \leqslant j$. According to the procedure, we merge the most collinear line segments of each triangle until $|L_k^{\tau,ij}| \leqslant T$. T is a threshold denoting the maximum allowed number of line segments that belong to the same triangle and is predefined by the user. Thus, in the worst case, $|L_k|$ is $O(T \cdot m^2)$, since there are $O(m^2)$ triangles on Δ_2. The total computational cost is $O(T \cdot n \cdot m^3)$.

We can estimate the normalized error (NE) of an approximated equipartition of length chords $r_1 = d(0, s_1), r_2 = d(s_1, s_2), \dots, r_n = d(s_{n-1}, s_n), r_{n+1} = d(s_n, 1)$ $(\hat{d}(0, s_1) = \hat{d}(s_1, s_2) = \dots = \hat{d}(s_{n-1}, s_n) = \hat{d}(s_n, 1))$ by getting the standard deviation of the $n+1$ estimated length chords of this equipartition σ divided by the mean length segment of this equipartition $(\bar{r})$, $\mathrm{NE} = \frac{\sigma}{\bar{r}}$.

It holds that NE decreases as m increases. Therefore, the mean error of the approximation $\hat{d}$ of d is

$$E\big(|\hat{d} - d|\big) = O\left(\frac{1}{m^2}\right).$$

Let $e(u, v) = \hat{d}(u, v) - d(u, v) = O(\frac{1}{m^2})$. Under the assumption that e has zero mean and e and d are independent, we can prove that NE decreases by the same factor $O(\frac{1}{m^2})$ (see Fig. 4(a)) when the line segment simplification procedure is not executed.

$$\mathrm{NE}^2 = \frac{\mathrm{var}(r)}{\bar{r}^2} = \frac{\frac{\sum_{i=1}^{n+1} d^2(s_{i-1}, s_i)}{(n+1)^2} - \bar{r}^2}{\bar{r}^2}$$

$$\Rightarrow \quad \mathrm{NE} = \frac{\sqrt{\sum_{i=1}^{n+1} (e(s_{i-1}, s_i))^2}}{(n+1)\bar{r}} \quad \text{(by hypothesis)}$$

$$\Rightarrow \quad \mathrm{NE} \leqslant \frac{\sum_{i=1}^{n+1} (|e(s_{i-1}, s_i)|)}{(n+1)\bar{r}} = O\left(\frac{1}{\bar{r}m^2}\right).$$

When the simplification procedure is executed, the error cannot be bounded.

Moreover, NE is not straightforwardly affected as n increases (see Fig. 4(b)). However, $\mathrm{NE} = O(\frac{1}{\bar{r}m^2})$ increases when $\bar{r}$ decreases, that is true when n is getting very high. For example, in the case of the Euclidean distance it holds that

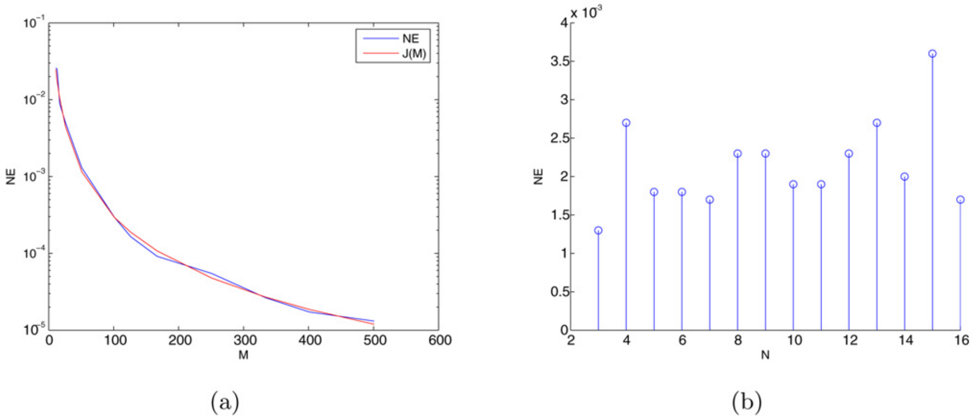

Fig. 4. The normalized error (NE) computed on a curve on which the EP has an infinite number of solutions for 4 segments. When we get more than one solutions, the NE was computed by the mean of normalized errors on these solutions. (a) The NE and its approximation function $J(m) = \frac{3}{m^2}$ computed for different values of M and for $n = 3$. (b) The NE computed for different values of N and for $m = 50$.

$\frac{d(1,0)}{n+1} < \bar{r} < \frac{\text{curveLength}}{n+1}$. This is the main reason why m should be greater than n, while our experiments show that it should be $m > 2n$. Moreover, m can be bounded by bounding NE or E. Therefore, m is an intrinsic input parameter of the algorithm and it is related to the error of the approximation.

3.3. Experimental results

The method has been implemented using C and Matlab. For our experiments, we used a Pentium 4 CPU at 2.8 GHz. A typical processing time, when $m = 100$ and $n = 10$, is about 4 seconds. Figs. 5 and 6 illustrate the results of the proposed algorithm for different 2-D or 3-D curves and values of n. The estimated solutions are projected via $\hat{d}$ with black circles and on input curve $c(t)$ (right) with the same color points belonging to the same equipartition.

The curves L_k, $k > 1$, are projected via $\hat{d}$, with gray colors, at both sides of the diagonal $x = y$, by mirroring L_k for odd values of k, for illustration reasons. L_1 is not projected, since it is always the trivial polygonal curve:

$$\big[(0,0),(0,t_1)\big] \cup \big[(0,t_1),(0,t_2)\big] \cup \cdots \cup \big[(0,t_{m-1}),(0,1)\big].$$

For each k, there exists a continuous curve from $(0,0)$ with endpoint on the axis $y = 1$. The second coordinate of L_k corresponds to a continuous path (sweep) along the curve $c(t)$ that starts at $c(0)$ ($(0,0) \in L_k$) and ends at $c(1)$ (intersection of L_k with the axis $y = 1$, see Fig. 3). The first coordinate of L_k gives the correspondence to L_{k-1} (isolevel). The complexity $|L_k|$ of L_k varies depending on the curve c. At least one solution approximately belongs to the connected component of $f^{-1}(0)$ which contains $(0, 0, \ldots, 0)$.

We shall give now a detailed description for how the L_k's and the tree data structure are developed using the curve of Fig. 5(f). In this example, we divide the curve into 12 pieces. Therefore, $L_1, L_2, \ldots, L_{11}$ are computed. Fig. 5(e) illustrates them apart from L_1. The curve L_2 is illustrated with black color on Fig. 5(e), consisting of two connected polygonal curves. It holds that some nodes of the first level (L_1 nodes) of the tree data structure are connected with at least two nodes of the next level (L_2 nodes), having at least two child nodes, e.g. the nodes that correspond to the second connected polygonal curve. In the case of two child nodes, it holds that there exists some pair of points of L_2: $(u_1, v), (u_2, v)$, $0 < v < u_1 < 1$, $0 < v < u_2 < 1$, so that $c(u_1)$ and $c(u_2)$ are equidistant from $c(v)$, that is $\hat{d}(v, u_1) = \hat{d}(v, u_2) = \hat{d}(0, v)$. Some of the nodes of first level (L_1 nodes) of the tree data structure are connected with a node of the next level (L_2 nodes), having one child node. According to the Fig. 5(e), about 15% of the nodes of the first level (L_1 nodes) are not connected with nodes of the next level (L_2 nodes), these nodes correspond to points close to the end of the curve. This is due to the fact that there exists v, $0 < v < 1$, so that there does not exist u, $v < u < 1$ with $\hat{d}(0, v) = \hat{d}(u, v)$. L_3 is illustrated with dark gray color on Fig. 5(e) consisting of two connected polygonal curves. L_k, $k > 3$, are illustrated with lighter gray colors as k increases on Fig. 5(e), consisting of one connected polygonal curve.

4. Applications

In this section, we present two EP based applications and the properties of the extracted solutions are discussed. First, an algorithm based on equal errors principle is proposed, which solves the General Polygonal Approximation problem (GPA) [18].

The EP can be also applied to the key frames selection problem [15] yielding a key frames selection algorithm based on Iso-Content Distance, Distortion principles. In both cases, the equality principle provides selected key frames with the property that they are equivalent in terms of content video summarization. Moreover, the EP has been successfully applied

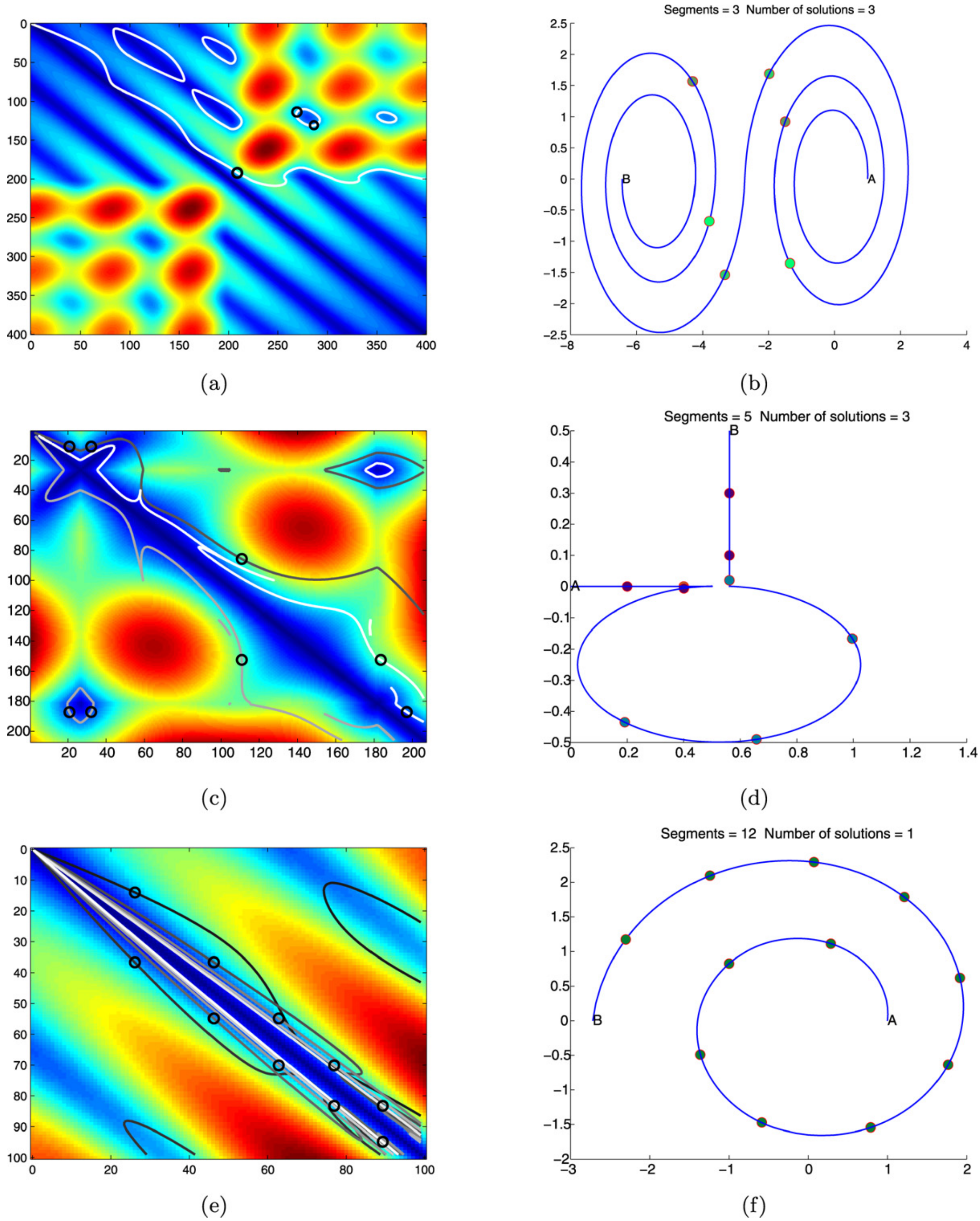

Fig. 5. Results of the of ILA for 2-D input curves.

to snake motion analysis [17] yielding equally spaced skeleton points, so that the time correspondence between the tracked skeleton points is done automatically.

The polygonal approximation [18] is an important topic in the area of pattern recognition, computer graphics and computer vision, because the polygonal approximation process saves memory space, reduces the rendering time on graphics applications and gives a more compact representation.

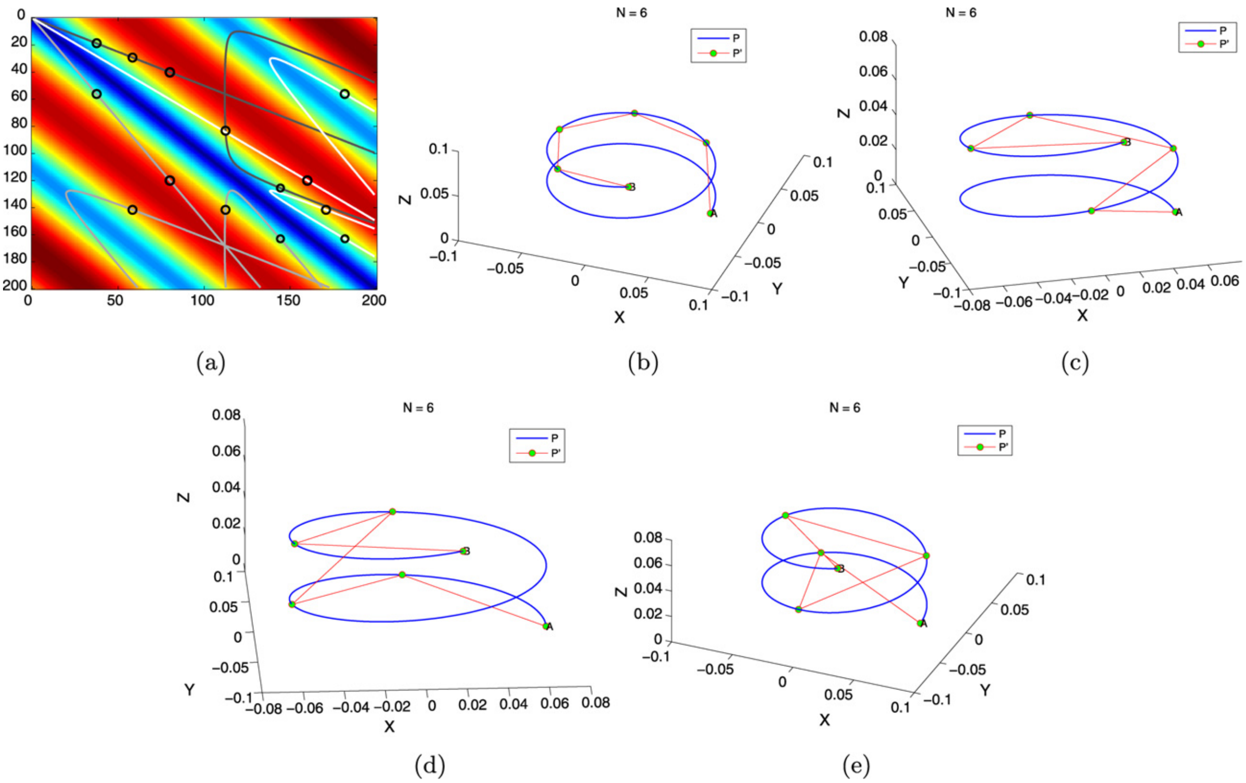

Fig. 6. Results of ILA for a helix [13] for $N = 6$. (a) The four solutions are projected via $\hat{g}$ with black circles and (b), (c), (d), (e) on $c(t)$ (blue curve) with the green color points connected with red line segments. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

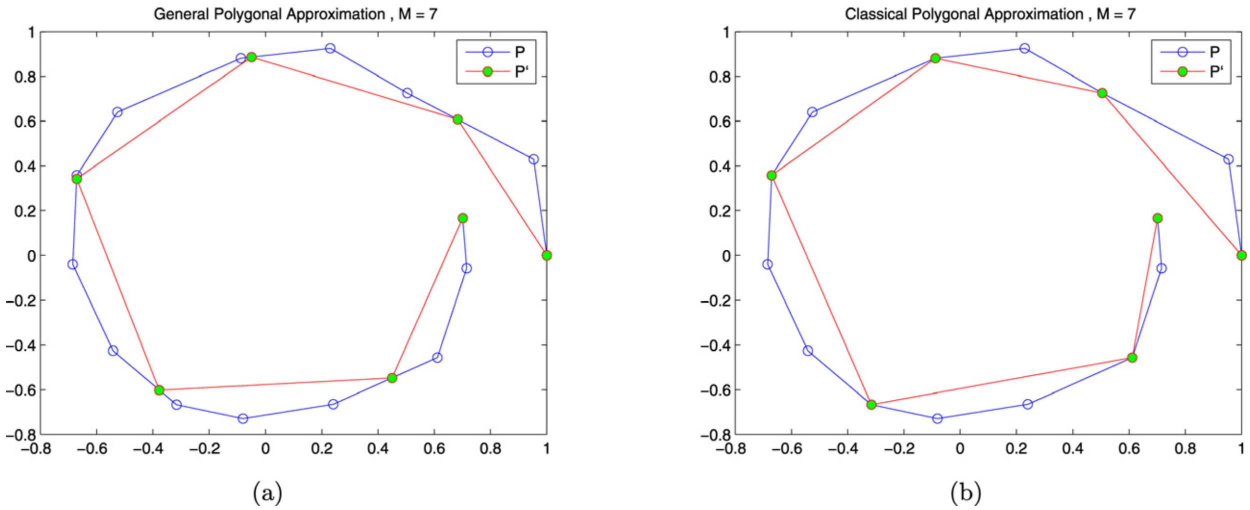

Fig. 7. Polygonal approximations (red polygon) with six segments of the same given curve (blue polygon). (a) A general polygonal approximation and (b) a "classical" polygonal approximation of the given curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Given a polygonal curve c in $\mathbb{R}^m$ with N vertices, a curve approximation of c is another polygonal curve c' with, say, M vertices that approximates the original curve c, according to a predefined error criterion. Let $P = \{p_1, p_2, \ldots, p_N\}$ and $P' = \{p'_1, p'_2, \ldots, p'_M\}$ be the set of the vertices of the given polygonal curve c and its approximation c', respectively. According to the general polygonal approximation problem (GPA), the vertices of c' are an ordered finite sequence of points on the trace of c, which need not be vertices of c as the polygonal approximation problem (PA) demands (see Fig. 7). Therefore, under this constraint relaxation, the solutions of the GPA problem give approximations of the polygonal curve c with possibly smaller error than the error of the solutions of PA problem.

Different error criteria have been proposed for polygonal approximation problems.

- A frequently used error criterion is the tolerance zone criterion [3,8]. Let $\overline{p'_k p'_{k+1}}$ be a segment of c', for some $1 \leqslant k < M$, and $S = [p'_k, p_m, p_{m+1}, \dots, p_{m+s}, p'_{k+1}]$ be the corresponding part of c. Under this criterion, the error between the segment $\overline{p'_k p'_{k+1}}$ and S is defined to be the maximum distance between $\overline{p'_k p'_{k+1}}$ and each point on S, with respect to one of the distance functions L_1, L_2 or L_∞.
- Another frequently used error criterion is the local integral square error (LISE) [4]. Under this criterion, the error between the segment $\overline{p'_k p'_{k+1}}$ and S is defined to be the sum of squared Euclidean distances of $\overline{p'_k p'_{k+1}}$ from each vertex of S.
- Finally, according to these error criteria the approximation error between c' and c is defined to be the maximum error between the segments of c' and their corresponding parts of c like S.

The polygonal approximation problem can be formulated in two ways: The problem of minimum error (*min*-ε) and the problem of minimum number of segments (*min*-#) [8,9]:

- The problem of minimum error (*min*-ε), where the approximation error is minimized given the number of segments.
- The problem of minimum number of segments (*min*-#), where the approximation error is bounded and the goal is to find the minimum number of segments that gives error lower than the given error.

Concerning the 2-D *min-# problem* and the *min-ϵ problem* under the tolerance zone criterion [8], the lowest computation cost method [2] has cost $O(M^2)$ and $O(M^2 \log M)$, respectively, M being the initial number of segments. The 3-D and 4-D polygonal approximation problems require near-quadratic time and sub-cubic time, respectively [1].

A near optimal solution of the GPA problem is achieved when the approximation errors per line segment $D(p'_1, p'_2), \dots, D(p'_{M-1}, p'_M)$ are equal, as the error is shared between all the segments and the total (maximum) error ϵ (e.g. under tolerance zone criterion or LISE) is minimized [18], that is

$$\epsilon = D(p'_1, p'_2) = D(p'_2, p'_3) = \cdots = D(p'_{M-1}, p'_M). \tag{1}$$

The solution under the Equal Error (EE) criterion can be computed approximately using the EquiPartition method (EP). We have seen that for a specific M the EP algorithm computes M vertices of c' under the EE criterion and a predefined error distance function. The input of the EP algorithm is the number M and the symmetric matrix $D(t_k, t_l)$, $k, l = 1, 2, \dots, N$ solving the *min*-ϵ problem directly. The *min*-# problem can be solved by the EP method under the same time-space requirements as the *min*-ϵ problem [18].

We have approximated various 2-D, 3-D and higher dimensional polygonal curves under the tolerance zone or the LISE criterion using the distance function L_2. We compare our method with the optimal PA methods for *min*-# [8] and *min*-ε [19]. Fig. 8 illustrates results of Perez–Vidal [19] and proposed GPA methods under LISE or tolerance zone criterion for different values of M.

Consequently, the proposed GPA algorithm approximates in a lot of cases the given curve with lower error than the optimal PA algorithm. When,

1. c', derived by the EE criterion, is the optimal solution of the GPA problem,
2. the EP method accuracy is high
3. and the error of the optimal GPA approximation is significantly lower than the error of optimal PA approximation,

the proposed solution will be, with a great probability, better than the optimal solution of the PA problem. It holds that when the given polygonal curve is smooth, the proposed algorithm yields, with a great probability, lower error results than the optimal PA method. Otherwise, the result of which algorithm gives better solution is unpredictably changing with M. A detailed analysis on optimality of the proposed solutions and more comparisons are given in [18].

5. Conclusions and discussion

In this paper, we have discussed the curve equipartition problem (EP). We have given a new geometric proof that it has at least one solution for every injective continuous curve and for any number of chords. Our approach is based on the analysis of the connected components of the zero level set of a certain function using methods of Combinatorial and Algebraic Topology. Inspired by this proof, we have implemented a new approximate algorithm (ILA) for solving the equipartition problem, which is the first EP algorithm.

The EP is a generic problem and can be applied to many applications. Energy minimizations problems that MINimize the MAXimum "per segment" (frame) distortion (MINMAX) [10] can be solved (almost optimally [18]) under equal error criterion. Applying the EP to polygonal approximation under LISE or tolerance zone criterion, we get equal errors per segment,

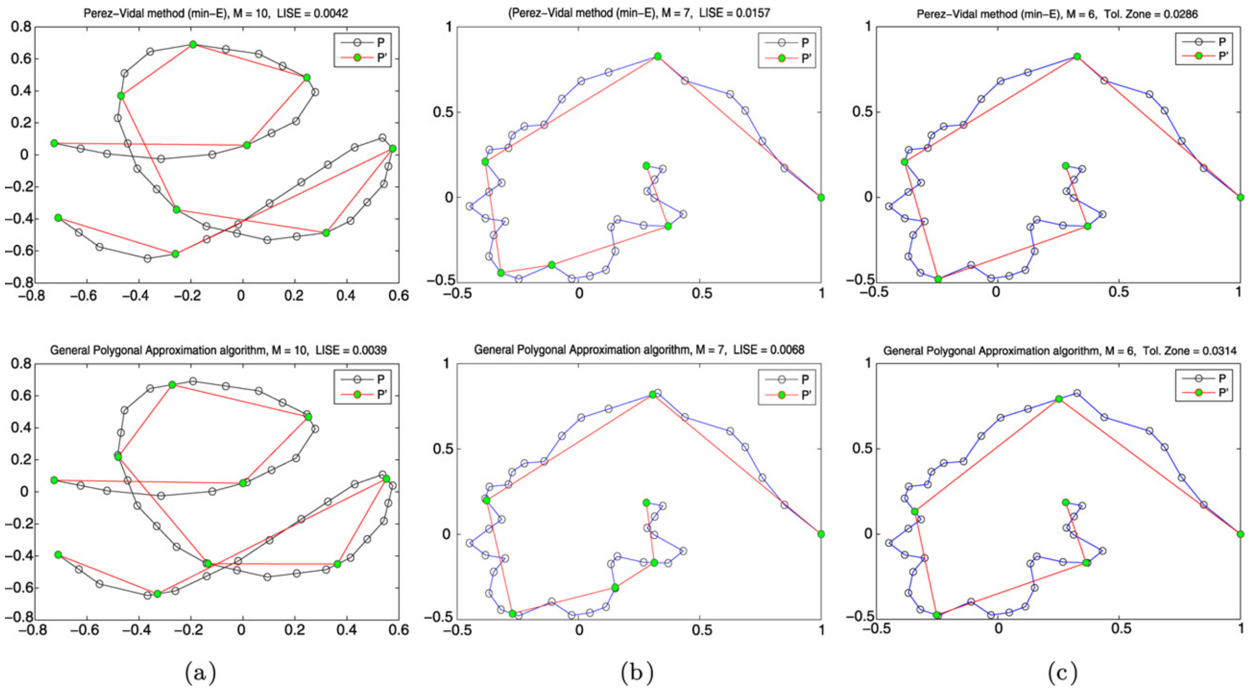

Fig. 8. *Min-*ϵ results of the Perez–Vidal and proposed GPA methods under LISE or tolerance zone criterion for different values of M. The curves P, P' are projected with blue and red colors. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

yielding low error values, since the global error will be shared between all the segments. The derived approximations are sufficient to provide a lower error than the optimal polygonal simplification methods with about the same computational cost. This result comes from the relaxation of constraint that approximate polygon vertices are a subset of the initial polygon vertices.

Another challenging application is the key-frames extraction out of a video sequence. The equipartition applied to video summarization provides selected key frames with the property of equivalent content. We can use any distance function relevant to the video content. Moreover, undulatory locomotion analysis can be based on the proposed method. Until now, we have applied the methodology to snake motion analysis [17]. The time correspondence between the tracked points is done automatically, since they are equally spaced. The EP can be probably applied to approximate curves [6] and to compute minimal energy curves [13].

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References

[1] G. Barequet, D. Chen, O. Daescu, M. Goodrich, J. Snoeyink, Efficiently approximating polygonal paths in three and higher dimensions, Algorithmica 33 (2002) 2539–2548.
[2] W. Chan, F. Chin, On approximation of polygonal curves with minimum number of line segments or minimum error, Internat. J. Comput. Geom. Appl. 6 (1996) 59–77.
[3] D. Chen, O. Daescu, Space-efficient algorithms for approximating polygonal curves in two-dimensional space, in: Proc. 4th Ann. Inter. Conf. on Computing and Combinatorics, in: Lecture Notes in Computer Science, vol. 1449, Springer, Berlin, 1998, pp. 45–54.
[4] K.-L. Chung, W.-M. Yan, W.-Y. Chen, Efficient algorithms for 3-d polygonal approximation based on LISE criterion, Pattern Recognition 35 (2002) 2539–2548.
[5] A. Dold, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1972.
[6] M.S. Floater, High order approximation of rational curves by polynomial curves, Comput. Aided Geom. Des. 23 (8) (2006) 621–628.
[7] H.B. Griffiths, The topology of square pegs in round holes, Proc. London Math. Soc. 62 (1991) 647–672.
[8] H. Imai, M. Iri, Polygonal approximations of a curve (formulation and algorithms), in: G.T. Tousaint (Ed.), Computational Morphology, North-Holland, Amsterdam, 1988, pp. 71–86.
[9] Y. Kurozumi, W. Davis, Polygonal approximation by the minimax method, Computer Vis. Graph. Image Process. (1982) 248–264.
[10] Z. Li, G. Schuster, A. Katsaggelos, Minimax optimal video summarization, IEEE Trans. Circuits Syst. Video Techn. 15 (10) (2005) 1245–1256.
[11] M.A. Lopez, S. Reisner, A note on curves equipartition, preprint, arXiv:0707.4298v1 [cs.CG], 29 Jul 2007.
[12] M. Marji, P. Siy, Polygonal representation of digital planar curves through dominant point detection nonparametric algorithm, Pattern Recognition 37 (2004) 2113–2130.

[13] M. Moll, L.E. Kavraki, Path planning for deformable linear objects, IEEE Trans. Robotics 22 (4) (2006) 625–636.
[14] M.H.A. Newman, Elements of the Topology of Plane Sets of Points, Cambridge University Press, Cambridge, 1951.
[15] C. Panagiotakis, A. Doulamis, G. Tziritas, Equivalent key frames selection based on iso-content principles, IEEE Trans. Circuits Syst. Video Techn. 2008, in press.
[16] C. Panagiotakis, G. Georgakopoulos, G. Tziritas, The curve equipartition problem: a brief exposition of basic issues, in: 22nd European Workshop on Computational Geometry, Delphi, 2006, pp. 33–36.
[17] C. Panagiotakis, G. Tziritas, Snake terrestrial locomotion synthesis in 3d virtual environments, Visual Computer 22 (8) (2006) 562–576.
[18] C. Panagiotakis, G. Tziritas, Any dimension polygonal approximation based on equal errors principle, Pattern Recognition Lett. 28 (5) (2007) 582–591.
[19] J. Perez, E. Vidal, Optimum polygonal approximation of digitized curves, Pattern Recognition Lett. 15 (1994) 743–750.
[20] K. Urbanik, Sur un probléme de J.F. Pal sur les courbes continues, Bull. Acad. Polon. Sci. 2 (1954) 205–207.