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Some aspects of the theory of asymptotic cycles

Konstantin Athanassopoulos

Abstract. This expository article presents the basic theory of asymptotic cycles of flows on compact spaces developed by S. Schwartzman in the late fifties and includes some explicit calculations in the case of suspension flows and 1-dimensional minimal sets.

1. Introduction

A flow on a compact metrizable space X is a continuous function $\phi : \mathbb{R}\times X \to X$ with the following properties :

(i) $\phi(0,x) = x$, for every $x \in X$.

(ii) $\phi(t+s,x) = \phi(t,\phi(s,x))$, for every $t, s \in \mathbb{R}$ and $x \in X$.

We shall use the convenient notation $\phi(t,x) = tx$ and $\phi(I\times A) = IA$ if $I \subset \mathbb{R}$ and $A \subset X$. The flow ϕ defines the continuous one-parameter group of homeomorphisms $\phi_t : X \to X$, $t \in \mathbb{R}$. The orbit of $x \in X$ is the set $C(x) = \{tx : t \in \mathbb{R}\}$.

A special kind of flow is the suspension of a homeomorphism. Let Y be a compact metrizable space and $h : Y \to Y$ a homeomorphism. On $[0,1] \times Y$ we consider the equivalence relation $(1,x) \sim (0,h(x))$, $x \in Y$. The quotient space $X = [0,1] \times Y/\sim$ is called the mapping torus of h and is compact and metrizable. We shall denote the class of $(s,x) \in [0,1] \times Y$ by $[s,x]$. The flow on X defined by

$$t[s,x] = [t+s-n, h^n(x)]$$

if $n \leq t+s < n+1$, $x \in Y$, is called the *suspension flow* of h. It is obvious that the identification $\pi : [0,1] \times Y \to X$ maps $[0,1) \times Y$ homeomorphically onto its image and every orbit meets $\pi(\{0\} \times Y)$. The first return of $[0,x]$ to $\pi(\{0\} \times Y)$ is $[0,h(x)]$. So the dynamics of the suspension flow are completely determined by the dynamics of h.

Let now $h : S^1 \to S^1$ be an orientation preserving homeomorphism. The mapping torus of h is the standard 2-torus $T^2 = S^1 \times S^1$. If $\bar{h} : \mathbb{R} \to \mathbb{R}$ is a lift of h, Poincare proved that the limit

$$\rho(h) = \lim_{n\to\infty} \frac{\bar{h}^n(t)}{n}$$

exists and is independent of $t \in \mathbb{R}$. $\rho(h)$ is called the *rotation number* of h and describes the average variation of the argument of the projection of an orbit on a meridian. The corresponding variation of the projection on an equator is 1, due to the special nature of the suspension flow.

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In the general case of an arbitrary flow on a compact metrizable space X a limit similar to that defining the rotation number corresponding to a projection of X to S^1 may not exist for every orbit. A theorem of Schwartzman asserts that for any invariant Borel probability measure μ the rotation number with respect to a continuous function $f : X \to S^1$ exists μ-almost everywhere and depends only on the homotopy class of f. This is proved in section 2 as a consequence of Birkhoff's Ergodic Theorem. The average rotation number on X defines a group homomorphism $A_\mu : \check{H}^1(X;\mathbb{Z}) \to \mathbb{R}$ called the *asymptotic cycle* of the flow with respect to the invariant measure μ, which describes how an μ-average orbit winds around X. The asymptotic cycles of smooth flows on compact manifolds are described in section 3. The asymptotic cycles of suspension flows are computed in section 4 and in section 6 we describe the set of asymptotic cycles of 1-dimensional minimal sets. Section 7 is devoted to the proof of a theorem of Schwartzman, which gives a necessary and sufficient condition for a flow to be topologically equivalent to a suspension in terms of asymptotic cycles.

In the sequel we shall make use of the following representation of the first Cech cohomology group with integer coefficients $\check{H}^1(X;\mathbb{Z})$ of a compact metrizable space X. Let $C(X, S^1)$ be the abelian group of all continuous functions of X with values in the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The set R of all $f \in C(X, S^1)$ which can be written in the form $f = \exp(2\pi i\psi)$ for some continuous function $\psi : X \to \mathbb{R}$ is a subgroup of $C(X, S^1)$. By a theorem of Bruschlinsky $\check{H}^1(X;\mathbb{Z}) \cong C(X, S^1)/R$.

2. Asymptotic cycles

Let X be a compact metrizable space carrying a flow ϕ. Let $C(X)$ be the set of real continuous functions of X. A *1-cocycle* for ϕ is a continuous function $g : \mathbb{R}\times X \to \mathbb{R}$ such that

$$g(t+s, x) = g(s, tx) + g(t, x)$$

for every t, $s \in \mathbb{R}$ and $x \in X$. The set $C^1(\phi)$ of all 1-cocycles of ϕ has a natural abelian group structure. Every $\psi \in C(X)$ defines a 1-cocycle g given by $g(t,x) = \psi(x) - \psi(tx)$, $t \in \mathbb{R}$, $x \in X$. The set Γ of all 1-cocycles defined from elements of $C(X)$ in this way is a subgroup of $C^1(\phi)$. Two 1-cocycles g, h are called *cohomologous* if $g - h \in \Gamma$. Let $H^1(\phi) = C^1(\phi)/\Gamma$.

2.1. Proposition. *For every continuous function $f : X \to S^1$ there exists a unique 1-cocycle g, called the 1-cocycle of f, such that $f(tx) = f(x)\exp(2\pi i g(t,x))$ for every $t \in \mathbb{R}$ and $x \in X$.*

Proof: The function $F : \mathbb{R}\times X \to S^1$ defined by $F(t,x) = f(tx)/f(x)$ is continuous and homotopic to a constant. So there is a continuous function $h : \mathbb{R}\times X \to \mathbb{R}$ such that $F = \exp(2\pi i h)$. Clearly, $h(0,x) \in \mathbb{Z}$ for every $x \in X$. Let $g : \mathbb{R}\times X \to \mathbb{R}$ be defined by $g(t,x) = h(t,x) - h(0,x)$. Then, $f(tx) = f(x)\exp(2\pi i g(t,x))$ and $g(0,x) = 0$ for every $t \in \mathbb{R}$ and $x \in X$. The connectedness of $\mathbb{R}$ implies that g is unique with respect to these properties. To prove now that g is a 1-cocycle note that for every t, $s \in \mathbb{R}$ and $x \in X$ we have

$$f(x)\exp(2\pi i g(t+s,x)) = f((t+s)x) = f(t(sx))$$
$$= f(sx)\exp(2\pi i g(t,sx)) = f(x)\exp(2\pi i(g(t,sx) + g(s,x))).$$

So $g(t+s) - g(t,sx) - g(s,x) \in \mathbb{Z}$ and by continuity and the connectedness of $\mathbb{R}\times\mathbb{R}$ necessarily $g(t+s,x) = g(t,sx) + g(s,x)$.

2.2. Lemma. *Let $f_1, f_2 : X \to S^1$ be two continuous functions with 1-cocycles g_1, g_2 respectively. If f_1 is homotopic to f_2, then g_1, g_2 are cohomologous.*

Proof: Since f_1 is homotopic to f_2, there is a continuous function $\theta : X \to \mathbb{R}$ such that $f_1 = f_2\exp(2\pi i\theta)$. So, for every $t \in \mathbb{R}$ and $x \in X$ we have

$$f_1(tx) = f_2(tx)\exp(2\pi i\theta(tx)) = f_2(x)\exp(2\pi i(\theta(tx) + g_2(t,x)))$$
$$= f_1(x)\exp(2\pi i(\theta(tx) + g_2(t,x) - \theta(x))).$$

Since g_1 is unique, we conclude that $g_1(t,x) = \theta(tx) + g_2(t,x) - \theta(x)$ for every $t \in \mathbb{R}$ and $x \in X$, i.e. g_1 is cohomologous to g_2.

The above show that there is a well defined homomorphism $\rho : \check{H}^1(X;\mathbb{Z}) \to H^1(\phi)$ which sends the homotopy class of f to the class of the corresponding 1-cocycle.

2.3. Lemma. *Let g, h be two cohomologous 1-cocycles and $x \in X$. Then, the limit $\lim_{t\to\infty} g(t,x)/t$ exists if and only if $\lim_{t\to\infty} h(t,x)/t$ exists and the two limits are equal.*

Proof: There is a continuous function $\theta : X \to \mathbb{R}$ such that $g(t,x) - h(t,x) = \theta(tx) - \theta(x)$ for every $t \in \mathbb{R}$ and $x \in X$. Hence

$$\left|\frac{g(t,x)}{t} - \frac{h(t,x)}{t}\right| \le \frac{2\|\theta\|}{|t|}$$

for every $t \neq 0$, and the conclusion follows.

2.4. Proposition. *Let μ be a ϕ-invariant Borel probability measure on X and g be a 1-cocycle. Then, the limit*

$$g^*(x) = \lim_{t\to\infty} \frac{g(t,x)}{t}$$

exists μ-almost for every $x \in X$. Moreover, $g^*(tx) = g^*(x)$, for every $t \in \mathbb{R}$ and $x \in X$ for which $g^*(x)$ exists and g^* is μ-integrable with

$$\int_X g^* d\mu = \int_X g(1,.)d\mu.$$

Proof: For every $t > 1$ we have $g(t,x) = g(t-[t],[t]x) + g([t],x)$, where $[t]$ denotes the integral part of t and inductively

$$g([t],x) = \sum_{k=0}^{[t]-1} g(1,kx).$$

Since X is compact, g is bounded on $[0,1]\times X$ and therefore

$$\lim_{t\to\infty}\frac{g(t-[t],x)}{t}=0.$$

By the Ergodic Theorem of Birkhoff

$$g^*(x)=\lim_{t\to\infty}\frac{g(t,x)}{t}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}g(1,kx)$$

exists μ-almost for every $x\in X$, g^* is ϕ-invariant, μ-integrable and its integral over X equals the integral of $g(1,.)$.

Note that if g, h are cohomologous 1-cocycles, then

$$\int_X g(1,.)d\mu=\int_X h(1,.)d\mu$$

for every ϕ-invariant Borel probability measure μ on X. So there is a well defined homomorphism $R_\mu : H^1(\phi)\to\mathbb{R}$ given by

$$R_\mu[g]=\int_X g^* d\mu=\int_X g(1,.)d\mu.$$

The homomorphism $A_\mu=R_\mu\circ\rho : \check{H}^1(X;\mathbb{Z})\to\mathbb{R}$ is called the *μ-asymptotic cycle* of the flow ϕ.

2.5. Definition: A continuous function $f : X\to\mathbb{C}$ is called differentiable with respect to the flow if there exists a continuous function $f' : X\to\mathbb{C}$ such that

$$\lim_{t\to 0}\frac{f(tx)-f(x)}{t}=f'(x)$$

uniformly for every $x\in X$.

2.6. Theorem. *Every continuous function $f : X\to\mathbb{C}$ is the uniform limit of a sequence of differentiable with respect to the flow functions.*

Proof: For every $n\in\mathbb{N}$ let $f_n : X\to\mathbb{C}$ be the continuous function defined by

$$f_n(x)=n\int_0^{1/n}f(tx)dt.$$

Let $\varepsilon>0$. Since $f\circ\phi$ is uniformly continuous on $[0,1]\times X$, there is a $\delta>0$ such that $|f(tx)-f(sy)|<\varepsilon$, whenever $d(x,y)<\delta$ and $|t-s|<\delta$, t, $s\in[0,1]$, where d is a compatible metric on X. Let $n_0\in\mathbb{N}$ be such that $1/n_0<\delta$. Then, for every $n\geq n_0$ and $0\leq t\leq 1/n$ we have $|f(tx)-f(x)|<\varepsilon$ for every $x\in X$. Hence

$$|f_n(x)-f(x)|\leq n\int_0^{1/n}\varepsilon dt=\varepsilon,$$

for every $x\in X$, $n\geq n_0$. This shows that $f_n\to f$ uniformly on X. It remains to show that every f_n is differentiable with respect to the flow. For every $x\in X$ and $0<|t|<1/n$ we have

$$\frac{f_n(tx)-f_n(x)}{t}=\frac{n}{t}\left[\int_{1/n}^{t+1/n}f(sx)ds-\int_0^t f(sx)ds\right].$$

Hence

$$\left|\frac{f_n(tx)-f_n(x)}{t}-n(f(\frac{1}{n}x)-f(x))\right|$$

$$\leq n\left|\frac{1}{t}\int_{1/n}^{t+1/n}f(sx)ds-f(\frac{1}{n}x)\right|+n\left|\frac{1}{t}\int_0^t f(sx)ds-f(x)\right|.$$

Let $\varepsilon>0$. There is $\delta>0$ such that $|f(s_1x)-f(s_2x)|<\varepsilon/2n$, for every $x\in X$ and s_1, $s_2\in[0,2]$ with $|s_1-s_2|<\delta$. So, for $0<t<\delta$ and $s\in[0,t]\cup[1/n,t+1/n]$ we have

$$\left|\frac{1}{t}\int_0^t f(sx)ds-f(x)\right|\leq\frac{1}{t}\int_0^t|f(sx)-f(x)|ds\leq\frac{\varepsilon}{2n},$$

$$\left|\frac{1}{t}\int_{1/n}^{t+1/n}f(sx)ds-f(\frac{1}{n}x)\right|\leq\frac{1}{t}\int_{1/n}^{t+1/n}|f(sx)-f(\frac{1}{n}x)|ds\leq\frac{\varepsilon}{2n}.$$

It follows now that

$$\left|\frac{f_n(tx)-f_n(x)}{t}-n(f(\frac{1}{n}x)-f(x))\right|\leq\varepsilon$$

for every $0<t<\delta$. Similarly for $-\delta<t<0$.

2.7. Corollary. *Every continuous function $f : X\to S^1$ is the uniform limit of a sequence of differentiable with respect to the flow functions $f_n : X\to S^1$, $n\in\mathbb{N}$.*

2.8. Corollary. *Every element of $\check{H}^1(X;\mathbb{Z})$ can be represented by a differentiable with respect to the flow continuous function with values in S^1.*

2.9. Theorem. *Let $[f]\in\check{H}^1(X;\mathbb{Z})$, where $f : X\to S^1$ is differentiable with respect to the flow. Then,*

$$A_\mu[f]=\int_X\frac{f'}{2\pi i f}d\mu.$$

for every ϕ-invariant Borel probability measure μ on X.

Proof: The 1-cocycle g which corresponds to f satisfies

$$g(t,x)=\int_0^t\frac{f'(sx)}{2\pi i f(sx)}ds$$

for every $t\in\mathbb{R}$ and $x\in X$. Hence, by Fubini's theorem,

$$A_\mu[f]=\frac{1}{2\pi i}\int_0^1\left(\int_X\frac{f'(sx)}{f(sx)}d\mu\right)ds=\frac{1}{2\pi i}\int_0^1\left(\int_X\frac{f'(x)}{f(x)}d\mu\right)ds=\frac{1}{2\pi i}\int_X\frac{f'}{f}d\mu.$$

3. Asymptotic cycles of smooth vector fields

Let ξ be a smooth vector field on a connected compact smooth n-manifold M with flow $\phi : \mathbb{R}\times M \to M$ and μ be a ϕ-invariant Borel probability measure on M. By de Rham's theorem and the Universal Coefficient theorem, $H^1_{DR}(M) \cong \check{H}^1(M;\mathbb{Z}) \otimes \mathbb{R}$. So, there is a basis of $H^1_{DR}(M)$ every element of which can be represented by a closed 1-form α on M for which there is a smooth function $f : M \to S^1$ and $r \in \mathbb{R}$ such that $[\alpha] = r[f^*(d\theta/2\pi)]$, where $[\alpha]$ is the de Rham cohomology class of α and $d\theta/2\pi$ is the representative of the natural generator of $H^1_{DR}(S^1) \cong \mathbb{R}$. We can extend A_μ on $H^1_{DR}(M)$ by setting $A_\mu[\alpha] = rA_\mu[f]$ and extending linearly.

It is easy to see that the 1-cocycle $g : \mathbb{R}\times M \to \mathbb{R}$ of a smooth function $f : M \to S^1$ is given by

$$g(t,x) = \int_0^t f^*(\frac{d\theta}{2\pi})(\xi(sx))ds.$$

So, for every $x \in X$ we have

$$f'(x) = 2\pi i f(x) g'(0,x) = 2\pi i f(x) f^*(\frac{d\theta}{2\pi})(\xi(x)).$$

It follows now from theorem 2.9 that

$$A_\mu[f] = \int_M f^*(\frac{d\theta}{2\pi})(\xi)d\mu.$$

Consequently, $A_\mu[\alpha] = \int_M (i_\xi\alpha)d\mu$ for every closed 1-form α on M, where $i_\xi\alpha$ denotes the interior product of α by ξ.

Suppose now that M is oriented by a volume element ω and the flow of ξ preserves volume. Then, $\phi_t^*\omega = \omega$ for every $t \in \mathbb{R}$ and $i_\xi\omega$ is a closed $(n-1)$-form. Since $\alpha\wedge\omega = 0$, for every closed 1-form α, we have

$$0 = i_\xi(\alpha\wedge\omega) = (i_\xi\alpha)\omega - \alpha\wedge(i_\xi\omega)$$

and therefore

$$A_\omega[\alpha] = \int_M (i_\xi\alpha)\omega = \int_M \alpha\wedge(i_\xi\omega).$$

This means that the asymptotic cycle A_ω is the Poincare dual of the de Rham cohomology class represented by $i_\xi\omega$.

4. The range of asymptotic cycles in suspension flows

Let Y be a compact metrizable space and $h : Y \to Y$ a homeomorphism. Let X be the natural mapping torus of h, as defined in the introduction, and $\pi : [0,1]\times Y \to X$ be the natural projection. If ν is an h-invariant Borel probability measure on Y and λ the Lebesgue measure on [0,1], then the product measure $\lambda\times\nu$ induces a Borel probability measure μ on X such that

$$\int_X \psi d\mu = \int_Y \Big(\int_0^1 \psi[t,x]dt\Big) d\nu$$

for every $\psi \in C(X)$. Using Fubini's theorem and the h-invariance of ν, it is easy to see that μ is invariant by the syspension flow.

Conversely, let μ be a Borel probability measure on X which is invariant by the suspension flow of h. Then, it is easy to see that $\mu(\pi(\{0\}\times A)) = 0$ and $\mu(\pi([0,t]\times A)) = t\mu(\pi([0,1]\times A))$ for every $0 \le t \le 1$ and every Borel set $A \subset Y$. Let $\nu(A) = \mu(\pi([0,1]\times A))$. Then, ν is an h-invariant Borel probability measure on Y and μ is the projection of $\lambda\times\nu$ on X as above.

In order to describe the values of the asymptotic cycle A_μ in terms of Y, h and ν it will be useful to find the relation between $\check{H}^1(X;\mathbb{Z})$ and $\check{H}^1(Y;\mathbb{Z})$. Recall that the abelian group $C(Y,\mathbb{Z})$ of all integer valued continuous functions on Y can be identified with the zeroth Cech cohomology group of Y with integer coefficients.

Let $\gamma : C(Y,\mathbb{Z}) \to \check{H}^1(X;\mathbb{Z})$ be defined by $\gamma(\varphi) = [f]$, where $f : X \to S^1$ is the continuous function defined by

$$f[t,x] = \exp(2\pi i t\varphi(x))$$

for $t \in [0,1]$ and $x \in Y$. Let $\beta : \check{H}^1(X;\mathbb{Z}) \to \check{H}^1(Y;\mathbb{Z})$ be the homomorphism induced by inclusion. This means that for every continuous function $f : X \to S^1$ we set $\beta[f] = [f_Y]$ where f_Y is the restriction of f on $\pi(\{0\}\times Y)$.

4.1. Proposition. *The sequence*

$$C(Y,\mathbb{Z}) \xrightarrow{h^*-id} C(Y,\mathbb{Z}) \xrightarrow{\gamma} \check{H}^1(X;\mathbb{Z}) \xrightarrow{\beta} \check{H}^1(Y;\mathbb{Z}) \xrightarrow{h^*-id} \check{H}^1(Y;\mathbb{Z})$$

is exact.

Proof: In order to check the exactness at $C(Y,\mathbb{Z})$ let first $\varphi \in C(Y,\mathbb{Z})$ and $\gamma(h^*(\varphi)-\varphi) = [f]$. Then,

$$f[t,x] = \exp(2\pi i t(\varphi(h(x)) - \varphi(x)))$$

for every $(t,x) \in [0,1]\times Y$. So, we can define the function $\psi : X \to \mathbb{R}$ with

$$\psi[t,x] = t(\varphi(h(x)) - \varphi(x)) + \varphi(x)$$

which is continuous and $f = \exp(2\pi i\psi)$. Hence $[f] = 0$. This shows that $\gamma\circ(h^* - id) = 0$.

On the other hand, if $\gamma(\varphi) = 0$, there exists a continuous function $\theta : X \to \mathbb{R}$ such that $k[t,x] = \theta[t,x] - t\varphi(x) \in \mathbb{Z}$ for every $(t,x) \in [0,1]\times Y$. By continuity, $k[t,x] = k[0,x] = \theta[0,x]$ for every $(t,x) \in [0,1)\times Y$. Taking limits for $t \nearrow 1$ we get

$$k[0,x] = \theta[1,x] - \varphi(x) = k[0,h(x)] - \varphi(x).$$

This shows the exactness at $C(Y,\mathbb{Z})$.

To prove exactness at $\check{H}^1(X;\mathbb{Z})$ note first that for every $\varphi \in C(Y,\mathbb{Z})$, the Cech class $\beta(\gamma(\varphi))$ is represented by the constant function 1 and therefore it is zero. Conversely, let $f : X \to S^1$ be a continuous function with $\beta[f] = 0$, which means that there is a continuous function $\psi : Y \to \mathbb{R}$ such that $f[0,x] = \exp(2\pi i\psi(x))$ for every $x \in Y$. Let g be the 1-cocycle of f. Then, for every $(t,x) \in [0,1)\times Y$ we have

$$f[t,x] = f(t[0,x]) = \exp(2\pi i(\psi(x) + g(t,[0,x])))$$

and taking limits for $t \nearrow 1$,

$$\exp(2\pi i\psi(h(x))) = f[0,h(x)] = f[1,x] = \exp(2\pi i(\psi(x) + g(1,[0,x]))).$$

So, there is a function $\varphi : Y \to \mathbb{Z}$ such that

$$\psi(h(x)) + \varphi(x) = \psi(x) + g(1,[0,x])$$

for every $x \in Y$. Clearly φ is continuous and the Cech class $[f] - \gamma(\varphi)$ is represented by the function $\exp(2\pi i\theta)$, where $\theta : X \to \mathbb{R}$ is the continuous function defined by

$$\theta[t,x] = \psi(x) + g(t,[0,x]) - t\varphi(x)$$

for $(t,x) \in [0,1] \times Y$. Hence $[f] = \gamma(\varphi)$, which shows exactness at $\check{H}^1(X;\mathbb{Z})$.

Finally, to prove exactness at $\check{H}^1(Y;\mathbb{Z})$ we observe first that for every continuous function $f : X \to S^1$ we have

$$f[0,h(x)] = f[0,x]\exp(2\pi i g(1,[0,x]))$$

for every $x \in Y$, where g is the 1-cocycle of f. This implies that $h^*[f_Y] = [f_Y]$. Conversely, let $f : Y \to S^1$ be a continuous function such that $h^*[f] = [f]$. This means that there is a continuous function $\psi : Y \to \mathbb{R}$ such that $f \circ h = f\exp(2\pi i\psi)$. Thus, one can define the continuous function $\bar{f} : X \to S^1$ with

$$\bar{f}[t,x] = f(x)\exp(2\pi i t\psi(x))$$

for every $(t,x) \in [0,1] \times Y$. Obviously, $\beta[\bar{f}] = [f]$, which shows exactness at $\check{H}^1(Y;\mathbb{Z})$.

4.2. Theorem. *Let Y be a compact metrizable space, $h : Y \to Y$ a homeomorphism and ν an h-invariant Borel probability measure on Y. Let μ be the corresponding invariant Borel probability measure by the suspension flow of on the mapping torus X of h. Then,*

$$\mathrm{Im}A_\mu = \left\{\int_Y \varphi d\nu : \varphi \in C(Y) \text{ and } f \circ h = f\exp(2\pi i\varphi) \text{ for some continuous } f : Y \to S^1\right\}.$$

Proof: Let $f : X \to S^1$ be a continuous function with cocycle g. For every $t \in [0,1)$ and $x \in Y$ we have

$$g(1,[t,x]) = g(t,[0,h(x)]) + g(1,[0,x]) - g(t,[0,x]).$$

By Fubini's theorem now,

$$A_\mu[f] = \int_0^1 \left(\int_Y g(1,[t,x])d\nu\right)dt$$

$$= \int_0^1 \left(\int_Y g(1,[0,x])d\nu\right)dt + \int_0^1 \left(\int_Y (g(t,[0,h(x)]) - g(t,[0,x]))d\nu\right)dt$$

$$= \int_Y g(1,[0,x])d\nu.$$

Conversely, for every $\varphi \in C(Y)$ for which there is a continuous function $f : Y \to S^1$ such that $f \circ h = f\exp(2\pi i\varphi)$, we can define the continuous function $\bar{f} : X \to S^1$ by

$$\bar{f}[t,x] = f[0,x]\exp(2\pi i t\varphi(x))$$

for every $(t,x) \in [0,1] \times Y$. Let $\bar{g}$ be the 1-cocycle of $\bar{f}$. Then for every $(t,x) \in [0,1] \times Y$ there is an integer $k[t,x]$ such that

$$t\varphi(x) = \bar{g}(t,[0,x]) + k[t,x].$$

Since $k[0,x] = 0$, the continuity implies that $k[t,x] = 0$ for every $(t,x) \in [0,1] \times Y$. Hence $\bar{g}(1,[0,x]) = \varphi(x)$ and from what we have already proved

$$A_\mu[\bar{f}] = \int_Y \bar{g}(1,[0,x])d\nu = \int_Y \varphi d\nu.$$

One case of special interest is when $\check{H}^1(Y;\mathbb{Z})=0$. Then, the exact sequence of proposition 4.1 implies that

$$\check{H}^1(X;\mathbb{Z}) \cong C(Y,\mathbb{Z})/\{\varphi \circ h - \varphi : \varphi \in C(Y,\mathbb{Z})\}.$$

This means that every element of $\check{H}^1(X;\mathbb{Z})$ can be repercesented by a continuous function $f : X \to S^1$ for which there is a $\varphi \in C(Y,\mathbb{Z})$ such that

$$f[t,x] = \exp(2\pi i t\varphi(x))$$

for every $(t,x) \in [0,1] \times Y$.

For instance, one case where all these are true is when Y is 0-dimensional.

4.3. Theorem. *Let Y be a compact metrizable space with $\check{H}^1(Y;\mathbb{Z})=0$, $h : Y \to Y$ a homeomorphism and ν an h-invariant Borel probability measure on Y. If μ is the invariant by the suspension flow Borel probability measure on the mapping torus X of h induced by ν, then*

$$\mathrm{Im}A_\mu = \left\{\int_Y \varphi d\nu : \varphi \in C(Y,\mathbb{Z})\right\}.$$

Proof: From the above remarks, every element of $\check{H}^1(X;\mathbb{Z})$ is represented by a continuous function $f : Y \to S^1$ for which there is a $\varphi \in C(Y,\mathbb{Z})$ such that $f[t,x] = \exp(2\pi i t\varphi(x))$ for every $(t,x) \in [0,1] \times Y$. The proof now of theorem 4.2 shows that

$$A_\mu[f] = \int_Y \varphi d\nu.$$

5. Global sections and change of velocity

Let X be a compact metrizable space carrying a flow ϕ. A closed set $S \subset X$ is called *section* to the flow if there exists $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon)S$ is an open subset of X and ϕ maps $(-\varepsilon,\varepsilon) \times S$ homeomorphically onto $(-\varepsilon,\varepsilon)S$. If moreover $\mathbb{R}S = X$, then S is called *global section*. In this section we shall see how a flow admitting a global section can be obtained from a suspension flow by a measurable change of velocity.

We assume in the rest of the section that the flow has a global section S. For every $x \in X$ let $T(x) = \inf\{t > 0 : tx \in S\}$. The first return map $h : S \to S$ defined by

$h(x) = T(x)x$ is a homeomorphism. Let $\psi : [0,1] \times S \to X$ be the continuous, onto map defined by $\psi(t,x) = (tT(x))x$. Then, $\psi(t,x) = \psi(s,y)$ if and only if either $t = s$ and $x = y$ or $t = 1$, $s = 0$ and $y = h(x)$. Hence ψ induces a homeomorphism, which we denote also by ψ, of the mapping torus $\tilde{X}$ of h onto X which sends orbits of the suspension flow of h onto orbits in X preserving their time orientation. In other words, ψ is a topological equivalence. In fact, if $F : \mathbb{R}\times\tilde{X} \to \mathbb{R}$ is the continuous function defined by

$$F(s,[t,x]) = (t+s-n)T(h^n(x)) - tT(x) + \sum_{k=0}^{n-1} T(h^k(x)),$$

if $n \leq t+s < n+1$ and $x \in S$, then clearly

$$\psi(s[t,x]) = F(s,[t,x])\psi[t,x]$$

for every $s \in \mathbb{R}$ and $(t,x) \in [0,1] \times S$. Let $l : X \to [\epsilon, +\infty)$ be the function defined by $l(\psi[t,x]) = T(x)$, for every $0 \leq t < 1$ and $x \in S$. Then, l is measurable, continuous at the points of $X \setminus S$ and

$$\int_0^{F(s,[t,x])} \frac{1}{l(r\psi[t,x])} dr = s$$

for every $s \in \mathbb{R}$ and $(t,x) \in [0,1] \times S$.

5.1. Proposition. *Let μ be a ϕ-invariant Borel probability measure on X. There is a Borel probability measure ν on $\tilde{X}$ which is invariant under the suspension flow of h such that*

$$\int_{\tilde{X}} \varphi d\nu = \frac{1}{\int_X \frac{1}{l} d\mu} \int_X \frac{\varphi \circ \psi^{-1}}{l} d\mu$$

for every $\varphi \in C(\tilde{X})$.

Proof: Let $G : \mathbb{R}\times X \to \mathbb{R}$ be the continuous function defined by

$$G(s,y) = \int_0^s \frac{1}{l(ty)} dt.$$

Then, for every $\varphi \in C(\tilde{X})$ we have

$$\int_0^{G(s,y)} \varphi(t\psi^{-1}(y))dt = \int_0^{G(s,y)} \varphi(\psi^{-1}(F(t,\psi^{-1}(y))y))dt = \int_0^s \frac{\varphi(\psi^{-1}(ty))}{l(ty)} dt$$

and by the Ergodic Theorem of Birkhoff

$$\int_X \frac{\varphi \circ \psi^{-1}}{l} d\mu = \int_X \left(\lim_{s\to+\infty} \frac{1}{s} \int_0^{G(s,y)} \varphi(t,\psi^{-1}(y))dt \right) d\mu.$$

Since for every τ, $s > 0$ we have

$$\frac{1}{s} \left| \int_0^{G(s,y)} (\varphi((t+\tau)\psi^{-1}(y)) - \varphi(t\psi^{-1}(y)))dt \right| \leq \frac{2\tau\|\varphi\|}{s},$$

it follows that the positive linear functional $\nu : C(\tilde{X}) \to \mathbb{R}$ defined by

$$\nu(\varphi) = \int_X \frac{\varphi \circ \psi^{-1}}{l} d\mu$$

is invariant under the suspension flow of h. So, its normalization represents an invariant Borel probability measure on $\tilde{X}$.

We shall find now the relation between the asymptotic cycles of μ and ν.

5.2. Lemma. *Let $f : X \to S^1$ be a continuous function with 1-cocycle g and suppose that the time derivative $g'(0,x)$ exists for every $x \in X \setminus S$ and is continuous and bounded on $X \setminus S$. Then,*

$$A_\mu[f] = \int_{X\setminus S} g'(0,x)d\mu$$

for every ϕ-invariant Borel probability measure μ on X.

Proof: We first observe that $g'(t,x)$ exists for every $t \in \mathbb{R}$ and $x \in X$ such that $tx \in X \setminus S$ and $g'(t,x) = g'(0,tx)$. Let $f'(x) = 2\pi i f(x)g'(0,x)$, if $x \in X \setminus S$ and $f'(x) = 0$, if $x \in S$. Then, $f' : X \to \mathbb{C}$ is a measurable function and $f'(tx) = 2\pi i f(tx)g'(t,x)$ for every $t \in \mathbb{R}$ and $x \in X$ such that $tx \in X\setminus S$. Since S is a global section, $\mu(S) = 0$ and for every $x \in X$ the set of times $t \in \mathbb{R}$ such that $tx \in S$ is discrete countable. Hence

$$g(t,x) = \int_0^t \frac{f'(sx)}{2\pi i f(sx)} ds$$

for every $t \in \mathbb{R}$ and $x \in X$ and as in the proof of theorem 2.9 we have

$$A_\mu[f] = \int_X \frac{f'}{2\pi i f} d\mu = \int_{X\setminus S} g'(0,x)d\mu.$$

5.3. Theorem. *Let μ be a ϕ-invariant Borel probability measure on X and ν the corresponding measure on $\tilde{X}$ given by proposition 5.1. Then*

$$A_\mu[f] = \left(\int_X \frac{1}{l} d\mu \right) A_\nu[f \circ \psi]$$

for every $[f] \in \check{H}^1(X;\mathbb{Z})$.

Proof: Let $\tilde{f} : \tilde{X} \to S^1$ be a differentiable with respect to the suspension flow function with 1-cocycle $\tilde{g}$ such that $\tilde{f} = f \circ \psi$. The 1-cocycle g of f is given by

$$g(t,x) = \tilde{g}\left(\int_0^t \frac{1}{l(sx)} ds, \psi^{-1}(x) \right)$$

for every $t \in \mathbb{R}$ and $x \in X$ and therefore satisfies the assumptions of Lemma 5.2. Moreover, $l(x)g'(0,x) = \tilde{g}'(0,\psi^{-1}(x))$ for every $x \in X \setminus S$. Hence

$$A_\mu[f] = \int_X \frac{\tilde{g}'(0,\psi^{-1}(x))}{l(x)} d\mu = \int_X \frac{\tilde{f}' \circ \psi^{-1}}{2\pi i l(\tilde{f} \circ \psi^{-1})} d\mu = \left(\int_X \frac{1}{l} d\mu \right) A_\nu[\tilde{f}].$$

6. Asymptotic cycles of 1-dimensional minimal sets

In this section we shall apply the content of the preceding sections to the study of the asymptotic cycles of 1-dimensional minimal sets. Recall that an *ordered group* is a pair (G,G^+), where G is an abelian group and G^+ is a subset of G, the positive cone, such that (i) $G^+ + G^+ \subset G^+$, (ii) $G = G^+ - G^+$ and (iii) $G^+ \cap (-G^+) = \{0\}$. One can define a partial ordering in G by setting $g_1 \geq g_2$ if $g_1 - g_2 \in G^+$. A *unit* is an element $u \in G^+$ such that for every $g \in G^+$ there is some $n \in \mathbb{N}$ such that $nu - g \in G^+$. The triple (G,G^+,u) is called a *unital ordered group*.

If Y is a compact space, then the group $C(Y,\mathbb{Z})$ becomes a unital ordered group in the obvious way. The constant function 1 is the distinguished unit.

6.1. Proposition. *Let Y be a compact metrizable space. If $h: Y \to Y$ is a minimal homeomorphism, that is every orbit of h is dense in Y, then the quotient group*

$$C(Y,\mathbb{Z})/\{\varphi \circ h - \varphi : \varphi \in C(Y,\mathbb{Z})\}$$

has an ordered group structure with unity induced from $C(Y,\mathbb{Z})$.

Proof: Properties (i) and (ii) are obvious. We prove (iii). Let $\varphi \geq 0$ be such that there are $\psi \geq 0$ and $\gamma \in C(Y,\mathbb{Z})$ with $\varphi + \psi = \gamma \circ h - \gamma$. Since Y is compact, γ takes a maximum value at some point $x_0 \in Y$. The set $A = \gamma^{-1}(\gamma(x_0))$ is nonempty, open and closed in Y. If $x \in A$, then $0 \geq \gamma(h(x)) - \gamma(x) = \varphi(x) + \psi(x) \geq 0$. This shows that $h(A) \subset A$ and the minimality implies that $A = Y$. This means that γ is constant and therefore $\varphi = \psi = 0$.

6.2. Corollary. *Let Y be a compact metrizable space with $\check{H}^1(Y;\mathbb{Z})=0$. If $h: Y \to Y$ is a minimal homeomorphism with mapping torus X, then $\check{H}^1(X;\mathbb{Z})$ is a unital ordered group.*

For the remainder of the section let X be a 1-dimensional compact metrizable space carrying a minimal flow ϕ, that is $\overline{C(x)} = X$ for every $x \in X$. Then, the flow has a 0-dimensional global section $S \subset X$ and is therefore obtained from the suspension of a minimal homeomorphism $h: S \to S$ by a measurable change of velocity l, according to section 5. More precisely, if $T(x) = \inf\{t > 0 : tx \in S\}$, then $l(tx) = T(x)$ for every $x \in S$ and $0 \leq t \leq T(x)$. Since S is 0-dimensional, $\check{H}^1(S;\mathbb{Z})=0$ and hence

$$\check{H}^1(X;\mathbb{Z}) \cong C(S,\mathbb{Z})/\{\varphi \circ h - \varphi : \varphi \in C(S,\mathbb{Z})\}$$

has an ordered group structure with unity, by Corollary 6.2.

If μ is a ϕ-invariant Borel probability measure on X, by Theorems 4.3 and 5.3 there is an h-invariant Borel probability measure ν on S such that

$$A_\mu[f] = \left(\int_X \frac{1}{l} d\mu\right) \int_S \varphi d\nu$$

where $[f] \in \check{H}^1(X;\mathbb{Z})$ is represented by $\varphi \in C(S,\mathbb{Z})$. Thus, $A_\mu[f] \geq 0$, if $[f] \geq 0$.

A *state* of a unital ordered group (G,G^+,u) is a group homomorphism $p: G \to \mathbb{R}$ such that $p(g) \geq 0$ for every $g \in G^+$ and $p(u) = 1$. If G is countable, the set of all states $S(G)$ is a compact, convex subset of the product space $\mathbb{R}^G$. Since X is compact metrizable, this holds in particular for $S(\check{H}^1(X;\mathbb{Z}))$.

Let $M_\phi(X)$ denote the set of ϕ-invariant Borel probability measures on X endowed with the weak* topology, which makes it a compact metrizable space.

6.3. Theorem. *The map $A: M_\phi(X) \to S(\check{H}^1(X;\mathbb{Z}))$ defined by*

$$A(\mu) = \frac{1}{\left(\int_X \frac{1}{l} d\mu\right)} A_\mu$$

maps line segments onto line segments and is a homeomorphism.

Proof: The continuity of A is obvious. We shall first prove that A is onto. Let p be a state of $\check{H}^1(X;\mathbb{Z})$. If B is an open-closed subset of S, then $\chi_B \in C(S,\mathbb{Z})$ and represents a unique Cech class $[f_B]$. We define $\nu(B) = p[f_B]$. Since S is 0-dimensional, the open-closed subsets of S form a basis for its topology and thus ν extends to a unique Borel probability measure on S. Moreover, for every open-closed set $B \subset S$ we have

$$\nu(h^{-1}(B)) = p[f_{h^{-1}(B)}] = p[f_B \circ h] = p[f_B] = \nu(B)$$

and hence ν is h-invariant. There is a ϕ-invariant Borel probability measure μ on X such that

$$\int_X \psi d\mu = \frac{1}{\int_S (l|S) d\nu} \int_S \left(\int_0^{T(x)} \psi(tx) dt\right) d\nu$$

for every $\psi \in C(X)$. Clearly, $\left(\int_X (1/l) d\mu\right)\left(\int_S (l|S) d\nu\right) = 1$ and

$$\left(\int_S (l|S) d\nu\right) A_\mu[f] = \int_S \varphi d\nu$$

by Theorems 4.3 and 5.3, where $\varphi \in C(S,\mathbb{Z})$ represents $[f]$. It is now obvious from the definitions that $p = A(\mu)$. This shows that A is onto and it remains to prove that it is one-to-one. Let μ, $\mu' \in M_\phi(X)$ be such that $A(\mu) = A(\mu')$. There are h-invariant Borel probability measures ν, ν' on S such that

$$\int_S \varphi d\nu = A(\mu)[f] = A(\mu')[f] = \int_S \varphi d\nu'$$

where $[f] \in \check{H}^1(X;\mathbb{Z})$ is represented by $\varphi \in C(S,\mathbb{Z})$. Since S is 0-dimensional, $C(S,\mathbb{Z})$ generates a dense subspace of $C(S)$. It follows that $\nu = \nu'$ and hence $\mu = \mu'$ also. Since $M_\phi(X)$ is compact metrizable, A is a homeomorphism.

7. Asymptotic cycles and existence of global sections

Let X be a compact metrizable space carrying a flow ϕ and let S be a section to the flow of some extent $2\varepsilon > 0$. The continuous function $f_S: X \to S^1$ defined by

$$f_S(x) = \begin{cases} 1, & \text{for } x \in X \setminus [0,\varepsilon]S \\ \exp(2\pi i t/\varepsilon), & \text{for } x \in tS,\ 0 \leq t \leq \varepsilon. \end{cases}$$

is called the *cosection map* of S. It is clear that the homotopy class of f_S does not depend on ε but only on S. The class $[f_S] \in \check{H}^1(X;\mathbb{Z})$ is called the *flow class* of S.

7.1. Proposition. *Let $f : X \to S^1$ be a continuous function with 1-cocycle g. Suppose that $g(.,x) : \mathbb{R} \to \mathbb{R}$ is strictly increasing for every $x \in X$. Then $S = f^{-1}(1)$ is a global section to the flow and f represents its flow class.*

Proof: We shall prove first that $\mathbb{R}S = X$. For this it suffices to show that $g(\mathbb{R},x) = \mathbb{R}$ for every $x \in X$. Suppose that the set $g(\mathbb{R},x)$ is bounded above. Then the limit $\lim_{t\to+\infty} g(t,x)$ exists in $\mathbb{R}$. Since X is compact, there is a sequence $t_n \to +\infty$ and a point $y \in X$ such that $t_n x \to y$. So,

$$f(ty) = \lim_{n\to+\infty} f((t+t_n)x_n) = \lim_{n\to+\infty} f(t_n x) = f(y)$$

for every $t \in \mathbb{R}$. This means that $g(t,y) = 0$ for every $t \in \mathbb{R}$, contrary to the assumption.

There is an open neighbourhood V of S and a continuous function $\theta : V \to \mathbb{R}$ such that $S = \theta^{-1}(0)$ and $f|V = \exp(2\pi i\theta)$. Thus, $g(t,x) = \theta(tx) - \theta(x)$ for every $x \in V$ and $t > 0$ such that $[0,t]x \subset V$. Consequently, θ is strictly increasing along the pieces of orbits in V. By the continuity of the flow, there is $\varepsilon > 0$ such that $[-2\varepsilon, 2\varepsilon]S \subset V$. The restriction of the flow $\phi|[-2\varepsilon,2\varepsilon] \times S$ is a homeomorphism onto $[-2\varepsilon,2\varepsilon]S$. Indeed, let $t, s \in [-2\varepsilon, 2\varepsilon]$ and $x, y \in S$ be such that $tx = sy$. Then, $\theta((t-s)x) = \theta(y) = 0$ and therefore $t - s = 0$ and $x = y$. We shall show now that $(-2\varepsilon,2\varepsilon)S$ is open in X. Suppose that $0 < t < 2\varepsilon$, $x \in S$ and $x_n \in X \setminus (-2\varepsilon,2\varepsilon)S$, $n \in \mathbb{N}$, are such that $x_n \to tx$. Then, eventually $\theta(x_n) > 0$ and therefore $\theta(sx_n) > 0$ for every $|s| < 2\varepsilon$. It follows that $\theta(sx) \geq 0$ for every $-2\varepsilon + t < s < 2\varepsilon + t$, contradiction. Similarly we arrive at a contradiction if $t < 0$ or $t = 0$.

The above show that S is a global section to the flow and it remains to prove that f is homotopic to the cosection map f_S. For every $x \in X$ let $T(x) = \inf\{t > 0 : tx \in S\}$, $\tau(x) = \sup\{t \leq 0 : tx \in S\}$ and $h : S \to S$ be the first return map defined by $h(x) = T(x)x$ for $x \in S$. Then, h is a homeomorphism and T, τ are continuous at the points of $X \setminus S$ but only "right continuous" at the points of S. Let $u : X \to \mathbb{R}$ be the function defined by $u(x) = (-\tau(x)/\varepsilon) + g(\tau(x),x)$, if $-\tau(x) \leq \varepsilon$ and $u(x) = 1 + g(\tau(x),x)$, if $-\tau(x) \geq \varepsilon$. Then, $f_S = f\exp(2\pi i u)$ and so it suffices to show that u is "left continuous" at the points of S. Let $x \in S$ and $x_n \to x$ be such that $x_n \in X \setminus [0,\varepsilon]S$ for every $n \in \mathbb{N}$. Then, $\lim_{n\to+\infty} \tau(x_n) = -T(h^{-1}(x))$ and so

$$\lim_{n\to+\infty} u(x_n) = 1 + g(-T(h^{-1}(x)),x) = 1 - g(T(h^{-1}(x)), h^{-1}(x)).$$

However, $\exp(2\pi i g(T(h^{-1}(x)), h^{-1}(x))) = f(x) = 1$ and since $T(h^{-1}(x))$ is the first positive time at which the orbit of x returns to S necessarily $g(T(h^{-1}(x)), h^{-1}(x)) = 1$. Hence, $\lim_{n\to+\infty} u(x_n) = 0$ and the proof is complete.

7.2. Corollary. *Let $f : X \to S^1$ be a differentiable with respect to the flow continuous function. If*

$$\frac{f'(x)}{2\pi i f(x)} > 0$$

for every $x \in X$, then the set $f^{-1}(1)$ is a global section to the flow and f represents its flow class.

7.3. Theorem. *Let X be a compact metrizable space carrying a flow ϕ. The following are equivalent :*

(a) The flow has a global section.

(b) There exists a continuous function $f : X \to S^1$ such that $A_\mu[f] > 0$ for every ϕ-invariant Borel probability measure μ on X.

Proof: Suppose first that the flow has a global section S of extent $2\varepsilon > 0$. There is a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi^{-1}(0) = (-\infty, 0]$, $\varphi^{-1}(1) = [\varepsilon, +\infty)$ and $\varphi'(t) > 0$ for every $0 < t < \varepsilon$. The function $f : X \to S^1$ defined by $f(x) = 1$, if $x \in X \setminus [0,\varepsilon]S$ and $f(x) = \exp(2\pi i\varphi(t))$, if $x \in tS$, $0 \leq t \leq \varepsilon$ represents the flow class of S and is differentiable with respect to the flow. If μ is any ϕ-invariant Borel probability measure on X, then $\mu([0,\varepsilon]S) > 0$, because S is a global section and X is compact. Thus, $A_\mu[f] > 0$.

Conversely, suppose that $f : X \to S^1$ is a continuous function satisfying (b). According to Corollary 2.8 we may assume that f is differentiable with respect to the flow. Let $D = \{g : g = u' \text{ for some } u \in C(X)\}$ and C be the positive cone in $C(X)$. Let also $F = D \oplus < f'/2\pi i f >$. If $F \cap C = \emptyset$, then by the geometric Hahn-Banach theorem there exists a normalized positive linear functional $\mu_0 : C(X) \to \mathbb{R}$ such that $F \subset \mathrm{Ker}\mu_0$. The normalized positive linear functionals $\mu_n : C(X) \to \mathbb{R}$, $n \in \mathbb{N}$, defined by

$$\mu_n(g) = \frac{1}{n}\int_0^n \mu_0(g \circ \phi_t)dt,$$

satisfy $F \subset \mathrm{Ker}\mu_n$ for every $n \in \mathbb{N}$ and have a weak* limit point μ. Clearly, $F \subset \mathrm{Ker}\mu$ and μ represents a ϕ-invariant Borel probability measure which we shall denote with the same symbol. So, $A_\mu[f] = 0$ contrary to our assumption. This shows that $F \cap C \neq \emptyset$, that is, there is a differentiable with respect to the flow $u \in C(X)$ such that

$$u'(x) + \frac{f'(x)}{2\pi i f(x)} > 0$$

for every $x \in X$. Let $\tilde{f} = f\exp(2\pi i u)$. Then $\tilde{f}$ is differentiable with respect to the flow and

$$\frac{\tilde{f}'(x)}{2\pi i \tilde{f}(x)} = \frac{f'}{2\pi i f(x)} + u'(x) > 0$$

for every $x \in X$. By Corollary 7.2 the set $\tilde{f}^{-1}(1)$ is a global section to the flow and f represents its flow class.

8. Remarks

The exposition of the basic theory given in sections 2 and 3 is based on [4], [7] and [8]. The range of asymptotic cycles in suspension flows described in section 4 has been related in [2] and [5] to K-theoretic invariants of C^* algebras associated to flows. The criterion for the existence of a global section to a flow given in section 7 is due to Schwartzman [8].

The theory of asymptotic cycles has been generalized in [9] to foliations. A homotopy version of the notion has been recently defined in [1] for smooth flows on manifolds (see also [6]). Finally, the theory of asymptotic cycles has found interesting applications to the study of the structure of the group of measure preserving homeomorphisms [3].