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# The flow near non-trivial minimal sets on 2-manifolds

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## 1. *Introduction*

In this paper we give a description of the qualitative behaviour of the orbits near a non-trivial compact minimal set of a continuous flow on a 2-manifold. The first results in this direction were obtained in [1] and the present paper can be regarded as a continuation of that work. The main result can be stated as follows:

THEOREM 1·1. *Let $(\mathbb{R},M,f)$ be a continuous flow on a 2-manifold $M$ and $A \subset M$ a non-trivial compact minimal set. Then, there exists a connected, open, invariant neighbourhood $E$ of $A$ with the following properties:*

(*a*) *the restricted flow on $E\backslash A$ is completely unstable;*

(*b*) *if $x \in E$, then $L^+(x) \cup L^-(x) \subset A \cup \partial E$ and $L^+(x) = A$ or $L^-(x) = A$;*

(*c*) *every connected component of $E\backslash A$ contains at least one orbit $C(x)$ such that $L^+(x) = L^-(x) = A$;*

(*d*) *the boundary $\partial E$ of $E$ contains no non-trivial compact minimal set;*

(*e*) *if $M$ is closed and orientable, then $\partial E$ does not contain periodic orbits.*

The above theorem is proved in Section 2. In Section 3 we apply it to the case of continuous flows on closed, orientable 2-manifolds without saddle fixed points and prove a characterization of Denjoy flows. Finally, we show that the depth of the centre of continuous flows without saddle fixed points defined on closed, orientable 2-manifolds is 1.

## 2. *Non-trivial minimal sets*

Let $(\mathbb{R},M,f)$ denote a continuous flow on a metric space $M$. We shall use the notation $f(t,x) = tx$ and $IA = \{tx : t \in I,\ x \in A\}$, if $I \subset \mathbb{R}$ and $A \subset M$. The orbit of the point $x \in M$ will be denoted by $C(x)$, its positive semi-orbit by $C^+(x)$ and its negative semiorbit by $C^-(x)$.

We recall that

$$L^+(x) = \{y \in M : t_n x \to y \quad \text{for some } t_n \to +\infty\}$$

and

$$J^+(x) = \{y \in M : t_n x_n \to y \quad \text{for some } x_n \to x \text{ and } t_n \to +\infty\}$$

are the positive limit set and the positive prolongational limit set of the point $x \in M$ respectively, whose negative versions are defined analogously.

A set $A \subset M$ is called *minimal* if it is non-empty, closed, invariant and has no proper subset with these properties. A minimal set is called trivial if it consists of only one orbit or is homeomorphic to the torus $T^2$.

A set $A$ is called positively (resp. negatively) *stable* if every neighbourhood of $A$ contains a positively (resp. negatively) invariant neighbourhood of $A$. A positively stable set is called positively asymptotically stable if its region of positive attraction is an open neighbourhood of it. The region of positive attraction of a compact set $A$ is the set of points $x \in M$ such that $\emptyset \neq L^+(x) \subset A$ (see [3], p. 56).

A subset $A$ of $M$ is a *saddle set* if it has a neighbourhood $U$ such that every neighbourhood $V$ of $A$ contains a point $x$ with $C^+(x) \not\subset U$ and $C^-(x) \not\subset U$ ([2], definition 6·1).

We refer the reader to [3] and [2] for the undefined terms of this paper. The following result was proved in [1].

THEOREM 2·1. *A compact minimal set of a continuous flow on a 2-manifold is trivial whenever it is positively (or negatively) stable or a saddle set.*

In the rest of this section $A$ will always denote a non-trivial compact minimal set of a continuous flow defined on a 2-manifold $M$. Let $V$ be an open neighbourhood of $A$ such that its closure in $M$ is a connected, compact 2-manifold with boundary. Considered as a 2-manifold $V$ has finite genus. It follows from lemmas 5 and 6 in [8] that $A$ has a neighbourhood which contains no minimal set disjoint from $A$.

Let $E_s^+(A)$ (resp. $E_s^-(A)$) denote the region of positive (resp. negative) strong attraction of $A$. Since $A$ is compact and minimal $E_s^+(A)$ is the set of points $x \in M$ with the property $J^+(x) = A$ and similarly for $E_s^-(A)$ (see [3], p. 56). By Theorem 2·1 and [2], theorem 6·12, the set $E = A \cup E_s^+(A) \cup E_s^-(A)$ is an open invariant neighbourhood of $A$. If $x \in E \backslash A$, then $J^+(x) = A$ or $J^-(x) = A$ and hence $x \notin J^+(x)$. This means that the restricted flow on $E\backslash A$ is completely unstable. This also implies that the limit sets $L^+(x), L^-(x)$ do not intersect with $E\backslash A$ for every $x \in E$. Hence $L^+(x) \cup L^-(x) \subset A \cup \partial E$.

PROPOSITION 2·2. *Every connected component $K$ of $E\backslash A$ contains at least one orbit $C(x)$ such that $L^+(x) = L^-(x) = A$.*

*Proof.* We may assume without loss of generality that $M = E$. Suppose that the conclusion is false and let $K^\pm = \{x \in K : L^\pm(x) = A\}$. Then $K^+$, $K^-$ are disjoint and $K = K^+ \cup K^-$. Since $A$ is non-saddle, they are also open. Hence $K = K^+$ or $K = K^-$. We shall assume the former, the proof being similar if the latter holds. The closure $\bar{K} = A \cup K$ of $K$ is a connected, locally compact, invariant subspace of $M$ and $A$ is globally positively asymptotically stable with respect to the restricted flow on $\bar{K}$. It follows that the flow on $K$ is parallelizable with a compact global section $S$ which must be a simple closed curve (see [5], chapter VII, 1·6). Since $A$ is non-trivial, there is a closed transversal through $A$. More precisely, there exist $\epsilon > 0$ and a simple closed curve $C$ in $M$ such that $C \cap A$ is non-empty and the flow $f$ maps $[-\epsilon, \epsilon] \times C$ homeomorphically onto the closure of an open neighbourhood of $C$ ([4], lemma 2). We can choose $C$ so that it does not intersect with $S$. For every $x \in S$ let $g(x) = \inf\{t > 0 : tx \in C\}$. Then $g(x)x \in C$ and the function $g: S \to \mathbb{R}^+$ defined in this way is continuous. Let $h: S \to C$ be the map defined by $h(x) = g(x)x$. Since $S$ is a global section to the flow in $K$, $h$ is injective and therefore a topological embedding. Consequently $h(S) = C$, which contradicts the fact that $C \cap A$ is non-empty.

COROLLARY 2·3. *The boundary $\partial E$ of $E$ contains no non-trivial compact minimal set.*

*Proof.* Suppose that $\partial E$ contains a non-trivial compact minimal set $B$ and let $E' = B \cup E_s^+(B) \cup E_s^-(B)$. Then $E \cap E'$ is a non-empty, open, invariant set. Let $N$ be a connected component of $E \cap E'$ contained in some connected component $K$ of $E\backslash A$.

The set $N$ is open and closed in $K$ because $A$ is non-saddle. Therefore $N = K$, which is impossible according to Proposition 2·2.

COROLLARY 2·4. *Every orbit closure in a continuous flow on a 2-manifold contains at most one non-trivial compact minimal set.*

In other words two non-trivial compact minimal sets on a 2-manifold are not joined by an orbit.

PROPOSITION 2·5. *If $M$ is closed and orientable, then $\partial E$ does not contain periodic orbits.*

*Proof.* Suppose that $\partial E$ contains a periodic orbit $C(z)$. By Corollary 2·3 and the compactness of $M$, $C(z)$ is isolated from minimal sets with respect to the flow on $\bar{E}$. So we may assume that there is a point $y \in \bar{E}$ such that $L^+(y) = C(z)$ by [**2**], theorem 6·12 and corollary 6·11, that is, the positive semiorbit $C^+(y)$ spirals around $C(z)$. There is an open neighbourhood $U$ of $C(z)$ homeomorphic $(-1, 1) \times S^1$ so that $C(z)$ separates $U$ in two annular regions $U_1$, $U_2$ such that $U_1$ is positively invariant, contains $y$ and every positive semiorbit in $U_1$ spirals around $C(z)$. This follows from the orientability of $M$ and [**6**], chapter VII, 5·1. The set $E \cap U_1$ is non-empty, open and closed in $U_1$, since $A$ is non-saddle. Hence $U_1$ is contained in some connected component $K$ of $E \backslash A$. Note that $C(z)$ is globally positively asymptotically stable with respect to the flow on the locally compact subspace $\mathbb{R}U_1 \cup C(z)$ of $M$. Therefore the flow on $\mathbb{R}U_1$ is parallelizable and admits a compact global section $S$, which is a simple closed curve bounding an annulus with $C(z)$. It follows from this that $K = \mathbb{R}U_1$ and hence $L^+(x) = C(z)$ for every $x \in K$. This contradicts Proposition 2·2.

The proof of Theorem 1·1 is now complete.

## 3. *Flows without saddle fixed points*

In this section we shall apply Theorem 1·1 to the study of the qualitative behaviour of continuous flows on closed, orientable 2-manifolds without saddle fixed points. By the term saddle fixed point we mean a fixed point of the flow which is a saddle set as defined in Section 2. Thus the notion of saddle fixed point used here is more general than the usual one. For instance an isolated saddle fixed point in our sense may have index zero.

The main result of the present section gives a characterization of Denjoy flows. A continuous flow on the torus $T^2$ is called a Denjoy flow if it is topologically equivalent to the suspension of an orientation-preserving homeomorphism of $S^1$ onto itself with a unique Cantor minimal set.

Let $(\mathbb{R}, M, f)$ be a continuous flow without saddle fixed points on a closed, orientable 2-manifold $M$. We allow the flow to have infinitely many (hence non-isolated) fixed points. For every $x \in M$, the positive limit set $L^+(x)$ contains a minimal set $X$. If $X$ is a periodic orbit or non-trivial then $X = L^+(x)$. If $X$ is a singleton, then again we have $X = L^+(x)$ because otherwise $X$ would consist of one saddle fixed point. These show that in such a flow every positive limit set is minimal and the same is true for negative limit sets. Therefore an orbit is positively (or negatively) Poisson stable and not periodic or singular if and only if it is contained in a non-trivial minimal set, unless the flow is topologically equivalent to some irrational flow on the torus.

THEOREM 3·1. *A continuous flow on a closed, orientable 2-manifold is a Denjoy flow if and only if it satisfies the following two conditions:*

(*a*) *there are no saddle fixed points;*

(*b*) *there exists at least one non-trivial minimal set.*

*Proof.* Let $A \subset M$ be a non-trivial minimal set and $E$ the open invariant neighbourhood of $A$ given by Theorem 1·1. We shall show that $\partial E$ must be empty by proving that the opposite assertion contradicts condition (*a*). If $\partial E$ is non-empty, then it is a saddle set with respect to the restricted flow on $\bar{E}$, by Theorem 1·1 and [2], theorem 6·12. This means that there are a sequence $\{x_n : n \in \mathbb{N}\}$ of points of $E \backslash A$ converging to a point $x \in \partial E$ and an open neighbourhood $U$ of $\partial E$ such that $C^+(x_n) \not\subset U$ and $C^-(x_n) \not\subset U$. By (*d*) and (*e*) of Theorem 1·1, $L^+(x)$ must be a fixed point. Let $V$ be an open neighbourhood of $L^+(x)$ contained in $U$ and $t > 0$ such that $tx \in V$. There is a neighbourhood $W$ of $x$ such that $tW \subset V$ and $[0,t]\,W \subset U$. Thus eventually $tx_n \in W$ and $[0,t]\,x_n \subset U$. This implies that $C^+(tx_n) \not\subset U$ and $C^-(tx_n) \not\subset U$, i.e. $L^+(x)$ is a saddle fixed point, which is contrary to condition (*a*). Hence $M = E$ and there are no fixed points or periodic orbits. It follows that $M$ must be a torus and the flow a Denjoy flow (see [7], chapter I, 4·3·3).

*Remark* 3·2. A fixed point $s$ of a continuous flow on a closed, orientable 2-manifold $M$ is called simple if the restriction of the flow to some open neighbourhood of $s$ is topologically equivalent to the restriction of the flow of a linear vector field on $\mathbb{R}^2$ with non-zero determinant to an open neighbourhood of the origin. If $M$ is not the sphere $S^2$ or the torus $T^2$ and the flow has only simple fixed points, then it must have at least one saddle fixed point. On the other hand, the sphere $S^2$ does not carry flows with non-trivial minimal sets. Thus, within the class of flows having only simple fixed points, Theorem 3·1 takes the following form:

COROLLARY 3·3. *Let* $(\mathbb{R}, T^2, f)$ *be a continuous flow on the torus having only simple fixed points and no saddle fixed points. If there exists a non-trivial minimal set, then the flow is topologically equivalent to a Denjoy flow.*

Finally, we shall prove that the non-wandering set of a continuous flow without saddle fixed points on a closed, orientable 2-manifold consists of fixed and periodic points, unless the flow is a Denjoy or irrational flow. We shall need the following general proposition.

PROPOSITION 3·4. *A periodic orbit in a continuous flow on a closed, orientable 2-manifold $M$ is a non-saddle set.*

*Proof.* Let $C(z)$ be a periodic orbit and suppose first that it is isolated from other periodic orbits. Then it is also isolated from minimal sets. If there exist points $x, y$ such that $L^+(x) = L^-(y) = C(z)$, then $C(z)$ has an open neighbourhood $V$ homeomorphic to $(-1,1) \times S^1$ separated by $C(z)$ in two annular regions $V_1$ and $V_2$ such that the orbit of every point in $V_1$ (resp. $V_2$) spirals around $C(z)$ in positive (resp. negative) time. Clearly then $C(z)$ is non-saddle. If there do not exist points $x, y$ with the above property, then $C(z)$ is non-saddle by [2], corollary 6·11. So the conclusion is true in case $C(z)$ is isolated from other periodic orbits. Let now $U$ be an annular open neighbourhood of $C(z)$ separated by $C(z)$ in two annular regions $U_1$ and $U_2$ and a sequence $\{C(z_n) : n \in \mathbb{N}\}$ of disjoint periodic orbits converging to $C(z)$. If the convergence is two-sided, then obviously $C(z)$ is stable and hence non-saddle. Suppose that the convergence is one-sided. Let $n \in \mathbb{N}$ be large enough so that $C(z_n)$ and $C(z)$ bound an invariant closed annulus $G$. The connected component $Y$ of the invariant set $M \backslash \operatorname{int}(G)$ which contains $C(z)$ is a connected, compact 2-dimensional submanifold with boundary of $M$, one of whose boundary components is $C(z)$. Clearly $C(z)$ is isolated from minimal sets with respect to the restricted flow on $Y$. If $C(z)$ is a saddle set in $M$, then it is also a saddle set in $Y$ and there exist $x, y \in Y$ such that $L^+(x) = L^-(y) = C(z)$. This is impossible by [**6**], chapter VII, 5·1, and hence $C(z)$ must be non-saddle.

COROLLARY 3·5. *Let $(\mathbb{R}, M, f)$ be a continuous flow without saddle fixed points on a closed orientable 2-manifold, which is not a Denjoy or irrational flow on the torus $T^2$. Then the non-wandering set consists of fixed and periodic points.*

*Proof.* Let $x \in M$ be non-wandering, that is $x \in J^+(x)$. The positive limit set $L^+(x)$ is either a singleton or a periodic orbit, by Theorem 3·1. If $x \notin L^+(x)$, then $L^+(x)$ is a saddle set because $x$ is non-wandering. This contradicts Proposition 3·4 and our assumptions.

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