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ASYMPTOTICALLY STABLE ONE-DIMENSIONAL COMPACT MINIMAL SETS

Konstantin Athanassopoulos

Abstract. It is proved that an asymptotically stable, 1-dimensional, compact minimal set A of a continuous flow on a locally compact, metric space X is a periodic orbit, if X is locally connected at every point of A. So, if the intrinsic topology of the region of attraction of an isolated, 1-dimensional, compact minimal set A of a continuous flow on a locally compact, metric space is locally connected at every point of A, then A is a periodic orbit.

1. Introduction

This note is concerned with Poincaré–Bendixson theory of 1-dimensional compact minimal sets in general locally compact, metric spaces. We are motivated by the question how the qualitative bahaviour of a continuous flow near a compact minimal set affects its structure. More precisely, we are interested in finding conditions refering to the flow near a 1-dimensional compact minimal set, which imply that it is a periodic orbit. Results in this direction, have been proved in [1] for almost periodic minimal sets. The almost periodicity is a rather restrictive internal property, which is equivalent to saying that the restricted flow on the compact minimal set is equicontinuous.

In [2] it was shown that on a locally compact ANR an asymptotically stable, 1-dimensional, invariant continuum without fixed points of a flow must be a periodic orbit. In particular this is true for asymptotically stable, 1-dimensional,

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compact minimal sets. The main tools used in the proof of this result were Čech cohomology and the Lefschetz Fixed Point Theorem. The purpose of this note is to weaken the assumption that the phase space is an ANR to local connectedness. More precisely, we prove that an asymptotically stable, 1-dimensional, compact minimal set A of a continuous flow on a locally compact, metric space X is a periodic orbit, if X is locally connected at every point of A (Theorem 4.2). This is based on a slight generalization of one of the two main results of [4], saying that an asymptotically stable, compact, invariant set A of a continuous map of a locally compact, metric space X has only a finite number of connected components, if X is locally connected at the points of A (Theorem 2.2). The main result follows then by constructing a suitable local section of the flow at some point of the minimal set, so that we have a well defined continuous Poincaré map and applying Theorem 2.2 to this map. The author makes no great claim to the originality of the methods, which do not exceed the level of pointset topology.

The reader may wonder about the necessity of the generalization from ANR to locally connected. One reason we need this, is the following. Recall that a compact invariant set is isolated if it is maximal in some of its compact neighbourhoods. This is a much more general property than asymptotic stability. An isolated invariant set A may not be asymptotically stable with respect to the restricted flow in its region of attraction $W^+(A)$. It is however possible to define on the set $W^+(A)$ a finer topology than the subspace topology such that the flow remains continuous and A becomes asymptotically stable (see [6], [3]). This is called the intrinsic topology. If $W^+_{\mathrm{i}}(A)$ denotes the set $W^+(A)$ equiped with the intrinsic topology, then $W^+_{\mathrm{i}}(A)$ is a locally compact, metric space. So, our main result implies that if A is an isolated, 1-dimensional, compact minimal set of a continuous flow on a locally compact, metric space and $W^+_{\mathrm{i}}(A)$ is locally connected at every point of A, then A is a periodic orbit (Corollary 4.3). Evidently, it is easier to check that $W^+_{\mathrm{i}}(A)$ is locally connected than an ANR.

Examples of nonperiodic, isolated, 1-dimensional, compact minimal sets are known. In [1, Section 4] examples of nonperiodic (but almost periodic) isolated 1-dimensional, compact minimal sets of flows on n-manifolds with $n \geq 4$ are constructed. Also, the C^1 counterexample to Seifert's conjecture constructed by P. A. Schweitzer in [7] is a nowhere vanishing C^1 vector field on the 3-sphere S^3 having two nonperiodic, isolated, 1-dimensional minimal sets, both copies of the Denjoy minimal set. It is worth to note that we do not know an example of a C^∞ vector field on a 3-manifold with a nonperiodic, isolated, 1-dimensional, compact minimal set. In any case, if such an example exists, the intrinsic topology of its region of attraction must be non-locally-connected, as it is in Schweitzer's C^1 example.

2. Asymptotically stable sets

Let X be a locally compact, metric space and let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on X. A set $A \subset X$ is called invariant if $\phi_t(A) = A$ for every $t \in \mathbb{R}$. A set $A \subset X$ is called *minimal* if it is nonempty, closed, invariant and has no proper subset with these properties. The orbit of a point $x \in X$ is the set $C(x) = \{\phi_t(x) : t \in \mathbb{R}\}$. The point x is *fixed* if $C(x) = \{x\}$, and *periodic* if $C(x)$ is a simple closed curve.

The *positive limit set* of the orbit of the point $x \in X$ is the closed, invariant set

$$L^+(x) = \{y \in X : \phi_{t_n}(x) \to y \text{ for some } t_n \to \infty\}.$$

A compact invariant set A is minimal if and only if $A = L^+(x)$ for every $x \in A$. If $A \subset X$ is a compact invariant set, the invariant set

$$W^+(A) = \{x \in X : \emptyset \neq L^+(x) \subset A\}$$

is called the *region of attraction* of A. If $W^+(A)$ is an open set, then A is called *attractor*.

A compact invariant set $A \subset X$ is called (Lyapunov) *stable* if every open neighbourhood U of A contains a smaller open neighbourhood V of A such that $\phi_t(V) \subset U$ for every $t \geq 0$. An *asymptotically stable* set is a compact invariant set which is a stable attractor. If A is asymptotically stable, there exists an open neighbourhood V of A such that $\overline{\phi_t(V)} \subset \phi_s(V)$ for $t > s \geq 0$ and $A = \bigcap_{t\geq 0} \phi_t(V)$.

These notions can be defined also for continuous maps. Let $U \subset X$ be an open set and $f: U \to X$ be a continuous map. The *positive limit set of* $x \in U$ *with respect to* f is the set

$$L^+(x, f) = \{y \in X : f^{n_k}(x) \to y \text{ for some } n_k \to \infty\}.$$

We shall call a set $A \subset U$ f-invariant if $f(A) = A$. A compact f-invariant set $A \subset U$ is called (Lyapunov) *stable* with respect to f if every open neighbourhood $W \subset U$ of A contains a smaller open neighbourhood $V \subset U$ of A such that $f^n(V) \subset W$ for every integer $n \geq 0$. A compact f-invariant set $A \subset U$ is called *asymptotically stable* if it is stable and there is an open neighbourhood $V \subset U$ of A such that $\emptyset \neq L^+(x, f) \subset A$ for every $x \in V$.

LEMMA 2.1. *Let X be a locally compact, metric space, $U \subset X$ an open set and $f: U \to X$ a continuous map. Let $A \subset U$ be an asymptotically stable, compact, f-invariant set and W be an open neighbourhood of A with compact closure $\overline{W} \subset U$ such that $f^n(\overline{W}) \subset U$ for every $n \in \mathbb{N}$, and $\emptyset \neq L^+(x, f) \subset A$ for every $x \in \overline{W}$. Then, for every open neighbourhood $V \subset U$ of A there exists* *some* $n_0 \in \mathbb{N}$ *such that* $\overline{f^n(W)} \subset V$ *for all* $n \geq n_0$ *and*

$$A = \bigcap_{n \geq 0} f^{mn}(W) \quad \text{for every } m \in \mathbb{N}.$$

PROOF. Since X is locally compact, there is an open neighbourhood C of A such that $\overline{C}$ is a compact subset of V. By stability, there is an open neighbourhood N of A such that $f^n(N) \subset C$ for every integer $n \geq 0$. For every $x \in \overline{W}$ there is some $n(x)$ such that $f^{n(x)}(x) \in N$ and, by continuity, there is an open neighbourhood $W_x \subset U$ of x such that $f^{n(x)}(W_x) \subset N$. Since $\overline{W}$ is compact, there are $x_1, \ldots, x_k \in \overline{W}$ such that $\overline{W} \subset W_{x_1} \cup \ldots \cup W_{x_k}$. If now $n_0 = \max\{n(x_1), \ldots, n(x_k)\}$, then $f^n(W) \subset C \subset \overline{C} \subset V$ and so $\overline{f^n(W)} \subset V$ for all $n \geq n_0$.

To prove the last assertion, it suffices to show that if $y \in \bigcap_{n \geq 0} f^{mn}(W)$, then y belongs to every open neighbourhood $V \subset U$ of A. There exists a sequence $(x_n)_{n \in \mathbb{N}}$ in W such that $y = f^{mn}(x_n)$ for every $n \in \mathbb{N}$. Since $\overline{W}$ is compact, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ converging to some point $x \in \overline{W}$. Let N be an open neighbourhood of A such that $f^n(N) \subset V$ for every integer $n \geq 0$. Since $\emptyset \neq L^+(x, f) \subset A$, there is some $n_0 \in \mathbb{N}$ such that $f^{n_0}(x) \in N$ and so $f^{n_0}(W_x) \subset N$ for some open neighbourhood $W_x \subset U$ of x, by continuity. It follows that there exists $k_0 \in \mathbb{N}$ such that $f^{n_0}(x_{n_k}) \in N$ for every $k \geq k_0$ and therefore $f^n(x_{n_k}) \in V$ for every $n \geq n_0$ and $k \geq k_0$. Since $n_k \to \infty$, there is some $k \in \mathbb{N}$ such that $mn_k \geq n_0$ and therefore $y = f^{mn_k}(x_{n_k}) \in V$. □

Recall that a space X is locally connected at a point $x \in X$ if x has a neighbourhood basis in X which consists of connected, open sets.

THEOREM 2.2. *Let X be a locally compact, metric space, $U \subset X$ an open set and $f: U \to X$ a continuous map. Let $A \subset U$ be a nonempty, asymptotically stable, compact, f-invariant set. If X is locally connected at every point of A, then A has a finite number of connected components and they are permuted by f.*

PROOF. Since X is locally compact and A is asymptotically stable, there exists an open neighbourhood V of A with compact closure $\overline{V} \subset U$ such that $f^n(\overline{V}) \subset U$ for every $n \in \mathbb{N}$ and $\emptyset \neq L^+(x, f) \subset A$ for every $x \in \overline{V}$. By assumption, every point $x \in A$ has a connected, open neighbourhood $W_x \subset V$. There are $x_1, \ldots, x_l \in A$ such that $A \subset W_{x_1} \cup \ldots \cup W_{x_l}$. If now $W = W_{x_1} \cup \ldots \cup W_{x_l}$, then by Lemma 2.1 there is some $m \in \mathbb{N}$ such that $\overline{f^m(W)} \subset W$ and

$$A = \bigcap_{n \geq 0} f^{mn}(W) = \bigcap_{n \geq 0} \overline{f^{mn}(W)}.$$

It is evident that W has a finite number of connected components $C_1, \ldots, C_k$, where $k \leq l$, and $A \cap C_i \neq \emptyset$ for every $1 \leq i \leq k$. Each $\overline{f^m(C_i)}$ is contained in a connected component C_j. Since $f^m(W)$ has at most k connected components and each C_j contains at least one connected component of $f^m(W)$, because A is f^m-invariant, it follows that $f^m(W)$ has exactly k connected components. Each one of them is contained in a unique connected component of W. Inductively, $f^{mn}(W)$ has exactly k connected components and each one is contained in a unique connected component of $f^{m(n-1)}(W)$. It follows that A has exactly k connected components, because A is nonempty and compact, and they are permuted by f. $\square$

The preceeding theorem generalizes the following result, which was proved in [4].

Corollary 2.3. *Let X be a locally connected, locally compact, metric space, $U \subset X$ an open set and $f\colon U \to X$ a continuous map. If $A \subset U$ is an asymptotically stable, compact f-invariant set, then A has a finite number of connected components and they are permuted by f.*

3. Local sections of continuous flows

Let X be a locally compact, metric space and let $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on X. A set $S \subset X$ is called a *local section* of extent $\varepsilon > 0$ if the flow maps $(-\varepsilon, \varepsilon) \times S$ homeomorphically onto an open subset of X. Local sections can be constructed from continuous functions that are monotone along the pieces of orbits in an open set.

Proposition 3.1. *Let $V \subset X$ be an open set and $g\colon V \to \mathbb{R}$ be a continuous function such that $g(\phi_t(x)) > g(x)$ whenever $t > 0$ is such that $\phi_t(x) \neq x$ and $\phi_s(x) \in V$ for every $0 \leq s \leq t$. Let $W \subset X$ be an open set such that $\overline{W}$ is compact and contained in V. If $c \in \mathbb{R}$ is such that $S = W \cap g^{-1}(c)$ is nonempty and $\overline{S}$ contains no fixed point of the flow, then S is a local section.*

Proof. Since $\overline{W}$ is compact, so is $\overline{S}$ and there exists $\varepsilon > 0$ such that $\phi_t(x) \in V$ for every $|t| \leq 2\varepsilon$ and $x \in \overline{S}$. We shall prove that S is a local section of extent ε. First we shall show that the flow maps $[-\varepsilon, \varepsilon] \times \overline{S}$ homeomorphically onto a compact subset of X. For this, it suffices to show that if t, $s \in [-\varepsilon, \varepsilon]$ and x, $y \in \overline{S}$ are such that $\phi_t(x) = \phi_s(y)$, then $t = s$ and $x = y$, because $[-\varepsilon, \varepsilon] \times \overline{S}$ is compact. Indeed, suppose that $t > s$. If $\phi_{t-s}(x) \neq x$, then

$$c = g(y) = g(\phi_{t-s}(x)) > g(x) = c,$$

because $0 \leq t - s \leq 2\varepsilon$, contradiction. So, $\phi_{t-s}(x) = x$, which means that the orbit of x is periodic, since S contains no fixed points by assumption. Let $0 < \tau < t - s$ be such that $\phi_\tau(x) \neq x$. Then,

$$g(x) < g(\phi_\tau(x)) < g(\phi_{t-s}(x)) = g(x),$$

contradiction again.

It remains to prove that the set $U = \{\phi_t(x) : |t| < \varepsilon,\ x \in S\}$ is open. We proceed by contradiction. Suppose that U is not open, and so there exists some $|t| < \varepsilon$ and $x \in S$ for which there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $X \setminus U$ converging to $\phi_t(x)$. Since V is an open neighbourhood of $\phi_t(x)$, we may assume that $x_n \in V$ for all $n \in \mathbb{N}$.

Suppose first that $t = 0$. Then $x_n \to x$ and we may moreover assume that $x_n \in W$ for all $n \in \mathbb{N}$. There exists some $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $|s_n| < \varepsilon$ with $g(\phi_{s_n}(x_n)) = c$. Indeed, if this is not true, there exists a sequence $n_k \to \infty$ such that $g(\phi_s(x_{n_k})) < c$ for every $|s| < \varepsilon$ or $g(\phi_s(x_{n_k})) > c$ for every $|s| < \varepsilon$. In the first case

$$c = g(x) < g(\phi_s(x)) = \lim_{k\to\infty} g(\phi_s(x_{n_k})) \leq c$$

for $0 < s < \varepsilon$ and in the second case

$$c \leq \lim_{k\to\infty} g(\phi_s(x_{n_k})) = g(\phi_s(x)) < g(x) = c$$

for $-\varepsilon < s < 0$, contradiction. Let now $|s| \leq \varepsilon$ be a limit point of the sequence $(s_n)_{n\geq n_0}$. Then, $g(\phi_s(x)) = c$ and so $s = 0$, by monotonicity. This implies that $s_n \to 0$ and $\phi_{s_n}(x_n) \to x$. Thus, eventually $\phi_{s_n}(x_n) \in W \cap g^{-1}(c) = S$, and therefore $x_n \in U$, contradiction.

Suppose now that $t > 0$. Since $g(\phi_t(x)) > g(x) = c$, we may assume that $g(x_n) > c$ for all $n \in \mathbb{N}$. As before we see that there is some $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $-\varepsilon < s_n < 0$ such that $g(\phi_{t+s_n}(x_n)) = c$. Indeed, otherwise there exists a sequence $n_k \to \infty$ such that $g(\phi_{t+s}(x_{n_k})) > c$ for every $-\varepsilon < s < 0$ and by continuity

$$c \leq g(\phi_s(\phi_t(x))) < g(x) = c$$

for $-\varepsilon < s < -t$, contradiction. If $-\varepsilon \leq s \leq 0$ is a limit point of the sequence $(s_n)_{n\geq n_0}$, then $g(\phi_{t+s}(x)) = c$ and so $s = -t$, because $|t + s| < \varepsilon$. Therefore, $s_n \to -t$ and eventually $\phi_{t+s_n}(x_n) \in W \cap g^{-1}(c) = S$. Thus, eventually $x_n \in U$, contradiction. The case $t < 0$ is treated similarly. □

REMARK 3.2. Note that the local section constructed in Proposition 3.1 has the additional property that $\overline{S} \setminus S$ is a closed subset of X. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{S} \subset \overline{W} \cap g^{-1}(c)$ converging to $x \in S$, then eventually $x_n \in W$ and so $x_n \in S = W \cap g^{-1}(c)$. This property will be useful later.

THEOREM 3.3. *Let X be a locally compact, metric space and $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on X. If $x_0 \in X$ is not a fixed point of the flow, there exists a locally compact local section S through x_0 such that $\overline{S} \setminus S$ is a closed subset of X. If X is moreover locally connected at x_0, then S is in addition locally connected at x_0.*

Proof. Since x_0 is not fixed, there exists $T > 0$ such that $\phi_T(x_0) \neq x_0$. Let d be a compatible distance on X and $g: X \to \mathbb{R}$ be the continuous function

$$g(x) = \int_0^T d(x_0, \phi_s(x))\, ds.$$

If $G: \mathbb{R} \times X \to \mathbb{R}$ is defined by $G(t,x) = g(\phi_t(x))$, then

$$\frac{\partial G}{\partial t}(t,x) = d(x_0, \phi_{t+T}(x)) - d(x_0, \phi_t(x)).$$

Since $(\partial G/\partial t)(0, x_0) = d(x_0, \phi_T(x_0)) > 0$ and $\partial G/\partial t: \mathbb{R} \times X \to \mathbb{R}$ is continuous, there exists an open neighbourhood V of x_0 and some $\delta > 0$ such that $\partial G/\partial t > 0$ on $(-\delta, \delta) \times V$. Let now $x \in V$ and $t > 0$ be such that $\phi_t(x) \neq x$ and $\phi_s(x) \in V$ for every $0 \leq s \leq t$. Let $0 = t_0 < \ldots < t_k = t$ be a partition of $[0, t]$ with $t_{i+1} - t_i < \delta$ for $i = 0, \ldots, k-1$. Then,

$$g(x) < g(\phi_{t_1}(x)) < \ldots < g(\phi_{t_{k-1}}(x)) < g(\phi_t(x)).$$

Let $c = g(x_0)$. Since X is locally compact, there exists an open neighbourhood W of x_0 such that $\overline{W}$ is a compact subset of V. By Proposition 3.1, the set $S = W \cap g^{-1}(c)$ is a local section through x_0. It is also locally compact, because it is the intersection of an open subset and a closed subset of the locally compact space X.

If X is in addition locally connected at x_0, and $\varepsilon > 0$ is the extent of S, the open set $U = \{\phi_t(x) : |t| < \varepsilon,\ x \in S\}$ is locally connected at x_0. Since U is homeomorphic to $(-\varepsilon, \varepsilon) \times S$, it follows that S is locally connected at x_0. $\square$

Remark 3.4. If X is an open subset of $\mathbb{R}^n$, $n \geq 1$, and the flow is generated by a locally Lipschitz vector field ξ on X, then through every point $x_0 \in X$ such that $\xi(x_0) \neq 0$ there exists a local section which is a $(n-1)$-dimensional open ball with center x_0. This follows from Proposition 3.1 taking the continuous function $g: X \to \mathbb{R}$ defined by

$$g(x) = \langle x, \xi(x_0) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product, and proceeding as in the proof of Theorem 3.3. Here $c = g(x_0) = \|\xi(x_0)\|^2$ and $g^{-1}(c) = X \cap H$, where H is the hyperplane through x_0 which is orthogonal to $\xi(x_0)$. Also W can be chosen to be an open n-ball with center x_0. Thus, $S = W \cap g^{-1}(c)$ is an $(n-1)$-dimensional open ball with center x_0.

4. One-dimensional compact minimal sets

Let X be a locally compact, metric space and $(\phi_t)_{t \in \mathbb{R}}$ be a continuous flow on X. Let $S \subset X$ be a local section of extent $\varepsilon > 0$. For every $x \in S$ let

$$r(x) = \inf\{t > 0 : \phi_t(x) \in S\}.$$

The continuity of the flow implies that the set $\{x \in S : r(x) < \infty\}$ is open in S, but maybe empty. If there exists a point $x \in S$ such that $x \in L^+(x)$, then it is not empty. If moreover, the orbit of x does not pass through $\overline{S} \setminus S$, there exists an open neighbourhood U of x in S such that the function $r\colon U \to (\varepsilon, \infty)$ is continuous.

Let now $A \subset X$ be a compact minimal set and suppose that there exists a local section S of extent $\varepsilon > 0$ through some point $x \in A$ with the property $A \cap (\overline{S} \setminus S) = \emptyset$. Since $x \in L^+(x)$ for every $x \in A$, there exists an open neighbourhood U of the compact set $A \cap S$ in S such that $r\colon U \to (\varepsilon, \infty)$ is continuous. The map $f\colon U \to S$ defined by $f(x) = \phi_{r(x)}(x)$, for $x \in U$, is a topological embedding of U onto an open subset of S, and is called the *Poincaré map*.

Lemma 4.1. *If $A \subset X$ is a 1-dimensional compact minimal set, then through any point of A there exists a locally compact local section S such that $\overline{S} \setminus S$ is a closed set and $A \cap (\overline{S} \setminus S) = \emptyset$. If X is locally connected at the points of A, then S is in addition locally connected at every point of $A \cap S$.*

Proof. Through any point of A there exists a locally compact local section S_0 of some extent $\varepsilon > 0$ such that $\overline{S_0} \setminus S_0$ is a closed set, by Theorem 3.3 and Remark 3.2. Moreover, the flow maps $[-\varepsilon, \varepsilon] \times \overline{S_0}$ homeomorphically onto a compact subset of X, as the proof of Proposition 3.1 shows. Since A has dimension 1, the set $A \cap \overline{S_0}$ is a 0-dimensional, compact, metric space (see [5, Remark 2, p. 302] and the reference therein). If now $x \in A \cap S_0$, there exists an open neighbourhood $B \subset S_0$ of x in $\overline{S_0}$ such that $B \cap A$ is an open-compact neighbourhood of x in $A \cap \overline{S_0}$ [5, Theorem 1, p. 277]. Let S be an open neighbourhood of $B \cap A$ in S_0 such that $\overline{S} \subset B$. Then S is a local section through x of extent $\varepsilon > 0$ such that $\overline{S} \setminus S$ is a closed set and $\overline{S} \cap A \subset B \cap A \subset S \cap A$, which implies that $A \cap (\overline{S} \setminus S) = \emptyset$. The rest follows as in the proof of Theorem 3.3. □

We can prove now the main result of this note.

Theorem 4.2. *Let X be a locally compact, metric space and $(\phi_t)_{t \in \mathbb{R}}$ be a continuous flow on X. Let $A \subset X$ be an asymptotically stable, 1-dimensional, compact minimal set. If $W^+(A)$ is locally connected at every point of A, then A is a periodic orbit.*

Proof. From Lemma 4.1 and since $W^+(A)$ is an open neighbourhood of A, through any point of A there exists a locally compact local section $S \subset W^+(A)$ of some extent $\varepsilon > 0$ such that $\overline{S} \setminus S$ is a closed set and $A \cap (\overline{S} \setminus S) = \emptyset$. So, we have a well defined continuous Poincaré map $f\colon U \to S$ on some open neighbourhood U of $K = A \cap S$ in S. Moreover, K is a 0-dimensional, compact, f-invariant (actually minimal) set, as the proof of Lemma 4.1 shows, and S is locally connected at every point of K. Also, K is stable with respect to f. Indeed, let G be an open neighbourhood of K in U. Let $V = \{\phi_t(x) : 0 \leq t \leq r(x),\ x \in G\}$. Since A is contained in the interior of V and it is stable, there exists an open neighbourhood W of A such that $\phi_t(W) \subset \operatorname{int} V$ for every $t \geq 0$. The set $H = G \cap W$ is an open neighbourhood of K in U and $f^n(H) \subset G$ for every integer $n \geq 0$, because if $x \in H$ is such that $f^n(x) \notin G$ for some $n > 0$ such that $f^k(x) \in G$ for $0 \leq k < n$, then $\phi_t(f^n(x)) \notin V$ for $t = r(f^n(x)) + \frac{\varepsilon}{2}$, contradiction. Moreover, K is asymptotically stable with respect to f, since $L^+(x) = A$ for every $x \in V$, and in particular for $x \in G$. It follows now from Theorem 2.2 that K has a finite number of connected components and f permutes them. This means that K is a periodic orbit of f and so A is a periodic orbit of the flow. □

The above theorem gives directly a periodicity criterion for isolated 1-dimensional minimal sets. Recall that a compact invariant set $A \subset X$ is call *isolated* if it has a compact neighbourhood V such that A is the maximal invariant set in V. It is not true in general that a compact invariant set A is asymptotically stable with respect to the restricted flow in $W^+(A)$, even if the latter is locally compact. If however A is isolated, there is a finer topology on the set $W^+(A)$ than the subspace topology, called the *intrinsic topology*, with respect to which the flow remains continuous and A becomes asymptotically stable (see [6], [3]). Let $W^+_{\mathrm{i}}(A)$ denote the set $W^+(A)$ equiped with the intrinsic topology. The space $W^+_{\mathrm{i}}(A)$ is locally compact and metrizable [3]. From Theorem 4.2 we have immediately the following.

Corollary 4.3. *Let X be a locally compact, metric space and $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on X. Let $A \subset X$ be an isolated,* 1*-dimensional, compact minimal set. If $W^+_{\mathrm{i}}(A)$ is locally connected at every point of A, then A is a periodic orbit.*

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KONSTANTIN ATHANASSOPOULOS
Department of Mathematics
University of Crete
GR-71409 Iraklion, GREECE

E-mail address: athanako@math.ouc.gr