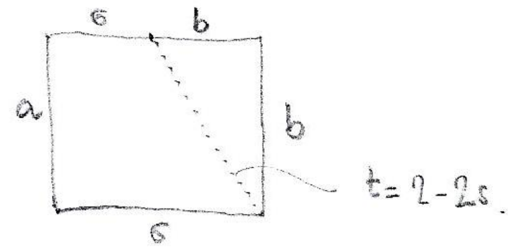
11.3. Let σ be a path from a to b. We want to show $\sigma \sim \sigma \cdot b$. Define

$$G(s,t) = \begin{cases} \sigma\left(\dfrac{s}{1-\frac{t}{2}}\right) & t \leq 2-2s \\ b & t \geq 2-2s \end{cases}$$



Then G is continuous, since $\dfrac{s}{1-t/2} = 1$ when $t = 2-2s$.

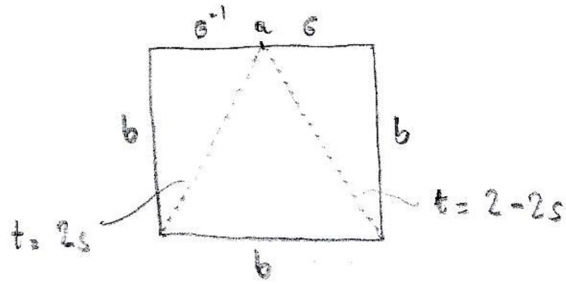
$G(s,0) = \sigma(s)$, $G(s,1) = \sigma(2s)$ for $s \leq \frac{1}{2}$ and

$G(s,1) = b$ for $s \geq \frac{1}{2}$.

11.4. Let σ be a path from a to b.

We want to show $b \sim \sigma^{-1} \cdot \sigma$.

Define



$$G(s,t) = \begin{cases} \sigma(1-2s) & 2s \leq t \\ \sigma(1-t) & t \leq 2s \leq 2-t \\ \sigma(2s-1) & 2s \geq 2-t \end{cases}$$

Then G is cts, $G(s,0) = \sigma(1)$ and

$G(s,1) = \sigma^{-1}(2s)$ for $s \leq \frac{1}{2}$ and

$G(s,1) = \sigma(2s-1)$ for $s \geq \frac{1}{2}$.

11.6. Recall that $\sin\vartheta = \sin(\pi - \vartheta)$. Hence $\sin(\pi t) = \sin(\pi(1-t))$.

Define the path $\tau(t) = \sin(\pi t)$. Intuitively, the path τ goes from 0 to 1 and then back from 1 to 0. Hence the path $\sigma(t) = e^{2\pi i\, \tau(t)}$ goes round the circle, and then back again

To express precisely this intuitive idea we define a path

$\rho: I \to I$, $\rho(t) = \sin\left(\frac{\pi}{2} t\right)$, and show that $\tau = \rho \cdot \rho^{-1}$.

We know $\rho^{-1}(t) = \sin\left(\frac{\pi}{2}(1-t)\right)$, and

$$\rho \cdot \rho^{-1}(t) = \begin{cases} \rho(2t) = \sin \frac{\pi}{2}(2t) = \sin \pi t & \text{for } 0 \le t \le \frac{1}{2} \\ \rho^{-1}(2t-1) = \sin\left(\frac{\pi}{2}\left(1-(2t-1)\right)\right) & \\ \quad = \sin\left(\frac{\pi}{2}(2-2t)\right) = \sin\left(\pi(1-t)\right) = \sin \pi t & \\ & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Hence $\rho \cdot \rho^{-1}(t) = \tau(t)$ for all $t \in [0,1]$.

We know that there is a homotopy H from $\rho \cdot \rho^{-1}$ to the constant path 0. Then $G(t,s) = e^{2\pi i H(t,s)}$ is a homotopy from σ to the constant path at $e^{2\pi i 0} = 1$.