The torus.

The torus as a set is the cartesian product of two circles,

$$T^2 = S^1 \times S^1 .$$

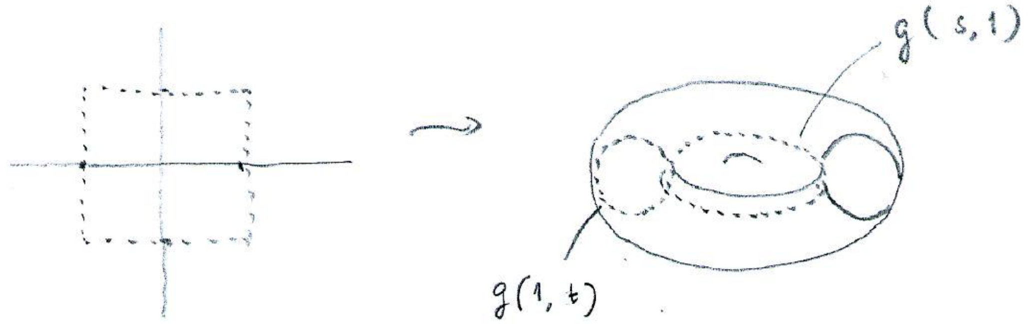
A point on the torus is $\big((x_1,y_1), (x_2,y_2) \big)$, with $x_1^2 + y_1^2 = 1$ and $x_2^2 + y_2^2 = 1$.

We define a mapping $g : \mathbb{R}^2 \longrightarrow T^2$ by

$$(s,t) \longmapsto \big((\cos s\pi, \sin s\pi), (\cos t\pi, \sin t\pi) \big)$$

g "wraps" the plane around the torus, covering each point infinitely many times. We want to choose suitable subsets of the torus, where g has a right inverse, to define the charts.

Let $\mathcal{U}_1 = g\big((-1,1) \times (-1,1) \big)$.



Then $\mathcal{U}_1$ covers the torus except two circles.

On $\mathcal{U}_1$ we can define $h_1 : \mathcal{U}_1 \longrightarrow (-1,1) \times (-1,1) \subset \mathbb{R}^2$, by

$$h_1 : \big((\cos s\pi, \sin s\pi), (\cos t\pi, \sin t\pi) \big) \longmapsto (s,t).$$

To obtain a chart, we only have to map the square $(-1,1) \times (-1,1)$ onto the disc B^2. One way to do this is $f : (-1,1) \times (-1,1) \longrightarrow B^2$, defined by

$$f(s,t) = \begin{cases} \dfrac{|t|}{\sqrt{s^2+t^2}}\,(s,t) & |t| \geq |s| \\[2ex] \dfrac{|s|}{\sqrt{s^2+t^2}}\,(s,t) & |t| < |s| \end{cases}$$

Check that f does map the square onto the disc.

Now we can define the chart $\varphi_1 : U_1 \longrightarrow B^2$ by

$\varphi_1 = f \circ h_1$, so that $\varphi_1\big(g(s,t)\big) = f(s,t)$.

We can use the image of different squares to cover the rest of the torus, for example $U_2 = g\big((0,2)\times(-1,1)\big)$, $U_3 = g\big((-1,1)\times(0,2)\big)$ and $U_4 = g\big((0,2)\times(0,2)\big)$.

and the corresponding right inverses for g ,

$h_2 : U_2 \rightarrow (0,2)\times(-1,1)$, $h_3 : U_3 \rightarrow (-1,1)\times(0,1)$,

$h_4 : U_4 \rightarrow (0,2)\times(0,2)$.

Alternatively, we note that any open disc of radius ≤ 1 in $\mathbb{R}^2$ maps injectively into T^2. Hence its image can be used as a chart.

Continuous mappings between top-manifolds.

Let M be an m-manifold, with charts $\varphi_i : U_i \to B^m$, $i \in \mathcal{A}$

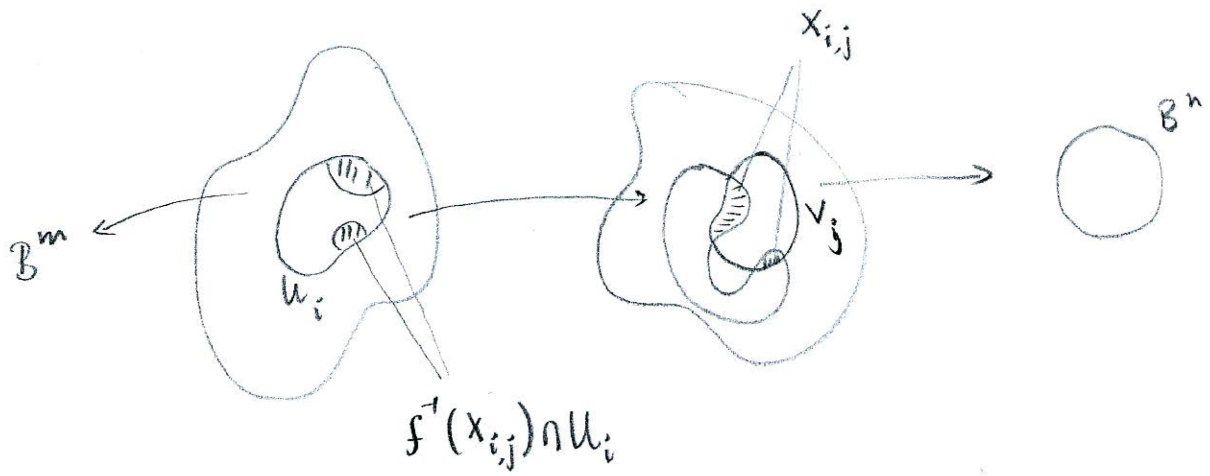
and N an n-manifold, with charts $\psi_j : V_j \to B^n$, $j \in \mathcal{B}$

A mapping $f : M \to N$ is continuous if

for every $i \in \mathcal{A}$, $j \in \mathcal{B}$ and $X_{i,j} = f(U_i) \cap V_j$,

the map $\psi_j \circ f \circ \varphi_i^{-1}$, which is defined on

$\varphi_i\left(f^{-1}(X_{ij}) \cap U_i \right)$, is continuous. $\left(\begin{array}{c}\text{note that} \\ f^{-1}(X_{ij}) \cap U_i = f^{-1}(V_j) \cap U_i\end{array}\right)$



In this way the continuity of a mapping between manifolds is reduced to the continuity of mappings between subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$.

We recall that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in \mathbb{R}^n$ if $\lim_{x \to x_0} f(x) = f(x_0)$, that is if for every $\varepsilon > 0$, there is $\delta > 0$ s.t. for every $x \in \mathbb{R}^n$ if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

If we define $U_\delta = \{ x \in \mathbb{R}^n : |x - x_0| < \delta \}$

and $V_\varepsilon = \{ y \in \mathbb{R}^m : |y - f(x_0)| < \varepsilon \}$, we have:

f is cts at $x_0 \in \mathbb{R}^n$ if for every $\varepsilon > 0$

there is $\delta > 0$ s.t. if $x \in U_\delta$ then $f(x) \in V_\varepsilon$.

which is the same as:

f is cts at $x_0 \in \mathbb{R}^n$ iff for every $\varepsilon > 0$

there is $\delta > 0$ s.t. $U_\delta \subseteq f^{-1}(V_\varepsilon)$.

A set is called <u>open</u> in $\mathbb{R}^n$ if for every $x \in A$ there is $\varepsilon > 0$ s.t. $D(x, \varepsilon) = \{ u \in \mathbb{R}^n : |u - x| < \varepsilon \} \subseteq A$.

<u>Theorem</u> A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous (at every point) <u>iff</u> for every open subset A of $\mathbb{R}^m$, the inverse image $f^{-1}(A)$ is open in $\mathbb{R}^n$.

Proof.

$\Rightarrow$ Assume f cts, A open in $\mathbb{R}^m$ and $x_0 \in f^{-1}(A)$.

We must find δ s.t. $D(x_0, \delta) \subseteq f^{-1}(A)$.

Let $y_0 = f(x_0)$. Since A is open, there is $\varepsilon > 0$ s.t.

$D(y_0, \varepsilon) \subseteq A$. Since f cts, there is $\delta > 0$ s.t.

$D(x_0, \delta) \subseteq f^{-1}(D(y_0, \varepsilon)) \subseteq f^{-1}(A)$.

$\Leftarrow$ Assume that for every open $A \subseteq \mathbb{R}^m$, $f^{-1}(A)$ is open in $\mathbb{R}^n$.

Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Then $D(f(x_0), \varepsilon)$ is open in $\mathbb{R}^m$. Hence $f^{-1}(D(f(x_0), \varepsilon))$ is open in $\mathbb{R}^n$. Since $x_0 \in f^{-1}(D(f(x_0), \varepsilon))$, there is $\delta > 0$ s.t. $D(x_0, \delta) \subseteq f^{-1}(D(f(x_0), \varepsilon))$.

The theorem shows that continuity can be defined once we know the open sets of a space.

What are the characteristic properties of open sets of $\mathbb{R}^n$?

1) The union of any family of opens sets is open.

2) The intersection of any finite family of open sets is open.

Recall that the intersection of an infinite family of open sets need not be open. E.g. $\bigcap (-1/n, 1/n) = \{0\}$.

<u>Proof</u> 1. If A_i is a family of open sets, and $x \in \bigcup A_i$, then $x \in A_j$ for some j. So there is ε s.t. $D(x,\varepsilon) \subseteq A_j \subseteq \bigcup A_i$.

2. If $x \in \bigcap A_i$, for every i there is ε_i s.t. $D(x,\varepsilon_i) \subseteq A_i$. Let $\varepsilon = \min\{\varepsilon_i\}$. Then $D(x,\varepsilon) \subseteq A_i$ for all i, hence $\subseteq \bigcap A_i$. //

Definition of a topology.

A <u>topology</u> on a set X is a family $\mathcal{T}$ of subsets of X s.t.

1) $\emptyset$ and X belong to $\mathcal{T}$.

2) The union of any family of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.

3) The intersection of any finite family of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.

$(X, \mathcal{T})$ is a <u>topological space</u>. The elements of $\mathcal{T}$ are the <u>open sets</u> of the topology $\mathcal{T}$.

Def. Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be top spaces. A mapping $f: X \longrightarrow Y$ is continuous if for every open set B of Y, $B \in \mathcal{U}$, the set $f^{-1}(B) \subseteq X$ is open, $f^{-1}(B) \in \mathcal{T}$.

Examples

1) $\mathbb{R}^n$, open subsets of $\mathbb{R}^n$ form a topology.

2) Discrete topology on a set X, $\mathcal{T} = \mathcal{P}(X)$.

3) Trivial topology on a set X, $\mathcal{T} = \{\emptyset, X\}$.

4) M a top manifold. We define open sets in M and show that they form a topology:

M top. manifold. A subset $A \subseteq M$ is <u>open</u>

if for every chart $\varphi_i : U_i \longrightarrow B^n$,

$\varphi_i (A \cap U_i)$ is an open subset of B^n.

We show that the open sets of M form a topology

on M.

It is clear that M and $\emptyset$ are open.

If A and B are open, consider $A \cap B$ and a U_i.

$(A \cap B) \cap U_i = (A \cap U_i) \cap (B \cap U_i)$, and since φ_i is injective,

$\varphi_i((A \cap B) \cap U_i) = \varphi_i(A \cap U_i) \cap \varphi_i(B \cap U_i)$

which is open.

If A_α is a family of open sets in M,

$\left(\bigcup A_\alpha\right) \cap U_i = \bigcup (A_\alpha \cap U_i)$ and

$\varphi_i\left(\left(\bigcup A_\alpha\right) \cap U_i\right) = \bigcup \varphi_i(A_\alpha \cup U_i)$ which is open.

<u>Examples of continuous maps</u>

1) We have shown that a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous in the ε-δ sense, <u>iff</u> the inverse image of each open set is open.

2) If X has the discrete topology, then any mapping $f : X \to Y$ is cts.

3) If X has the trivial topology, then any mapping $f : Y \to X$ is cts.

4) $f : M \to N$ mapping between top. manifolds. We want to show that the two notions of continuity coincide.

α) f is cts if for any open subset $B \subseteq N$, $f^{-1}(B)$ is open.

β) f is cts if for any charts $\varphi_i : U_i \to B^m$, $\psi_j : V_j \to B^n$ the map $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i\left(f^{-1}(V_j) \cap U_i\right) \longrightarrow B^n$ is continuous.