Constructing new topological spaces.

The ordering of subsets of $\mathcal{P}(X)$ induces an ordering of topologies on a set X.

A topology $\mathcal{T}$ is smaller than a topology $\mathcal{U}$ if $\mathcal{T} \subseteq \mathcal{U}$.

Ex. The indiscrete topology is smaller than any topology on X.

Any topology on X is smaller than the discrete topology.

Thm $X \neq \phi$, $\mathcal{C} \subseteq \mathcal{P}(X)$.

Then there exists a unique minimal topology $\mathcal{T}_{\mathcal{C}}$ on X such that $\mathcal{C} \subseteq \mathcal{T}_{\mathcal{C}}$.

$\mathcal{T}_{\mathcal{C}}$ consists of unions of finite intersections of sets in $\mathcal{C}$.

Proof Let $\mathcal{A}$ be the family of unions of fin. intersections of sets in $\mathcal{C}$: if $A \in \mathcal{A}$ then $A = \bigcup_{j \in J} A_j$,

and for each j there is $k_j \in \mathbb{N}$ and sets $A_{j,1}, \dots, A_{j,k_j} \in \mathcal{C}$, such that $A_j = \bigcap_{i=1}^{k_j} A_{j,i}$.

If $\mathcal{U}$ is a topology on X and $\mathcal{C} \subseteq \mathcal{U}$, then clearly $\mathcal{A} \subseteq \mathcal{U}$.

If we show that $\mathcal{A}$ is a topology, then $\mathcal{A}$ is the minimal topology containing $\mathcal{C}$.

Unions of sets in $\mathcal{A}$ clearly belong to $\mathcal{A}$.

We show that the intersection of two sets in $\mathcal{A}$ belongs to $\mathcal{A}$.

Then we can extend by induction to any finite intersection.

Let $A, B \in \mathcal{A}$. $A = \bigcup A_j$, $A_j = \bigcap_{i=1}^{k_j} A_{j,i}$

$B = \bigcup B_s$, $B_s = \bigcap_{t=1}^{m_s} B_{s,t}$.

$$A \cap B = \left(\bigcup_j A_j\right) \cap \left(\bigcup_s B_s\right)$$
$$= \bigcup_j \bigcup_s (A_j \cap B_s)$$
$$= \bigcup_j \bigcup_s \left(\left(\bigcap_{i=1}^{k_j} A_{ji}\right) \cap \left(\bigcap_{t=1}^{m_s} B_{st}\right)\right)$$

which is a union of finite intersections of sets in $\mathcal{C}$. //

We say that $\mathcal{C}$ is a <u>subbase</u> of the topology $\mathcal{T}_{\mathcal{C}}$.

A set $\mathcal{B}$ of subsets of X is a <u>base</u> of the topology $\mathcal{T}$ if every set in $\mathcal{T}$ is the union of sets in $\mathcal{B}$.

<u>Ex</u>. The open half lines from a subbase of the topology of $\mathbb{R}$.
The open intervals from a base of the topology of $\mathbb{R}$.

<u>Lemma</u> If $\mathcal{C}$ is a subbase of the topology of Y, $f : X \to Y$ is continuous if $f^{-1}(B)$ is open in X for every $B \in \mathcal{C}$.

Consider a function $f : X \to Y$ and a topology $\mathcal{U}$ of Y. The smallest topology on X with which f is continuous has as a subbase the set $\{ f^{-1}(B) : B \in \mathcal{U}\}$. We symbolize this topology by $\mathcal{U}^f$.

<u>Ex.</u> The relative topology on $V \subseteq Y$ is the smallest topology on V such that $\iota : V \hookrightarrow Y$ is continuous.

Topological product.

On the cartesian product of two sets we define the projections

$$\pi_1 : X \times Y \to X : (x, y) \mapsto x$$

$$\pi_2 : X \times Y \to Y : (x, y) \mapsto y$$

If X, Y are topological spaces, the product topology on $X \times Y$ is the smallest topology with which π_1 and π_2 are continuous.

If A is open in X, $\pi_1^{-1}(A) = A \times Y$, and
if B is open in Y, $\pi_2^{-1}(B) = X \times B$.

Hence a subbase of the product topology is

$$\{ A \times Y : A \text{ open in } X \} \cup \{ X \times B : B \text{ open in } Y \}.$$

Lemma Every open set in $X \times Y$ is the union of sets $A \times B$, where A is open in X and B is open in Y. //

The proof is an application of properties of union and intersection. See notes.

Ex. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The usual topology (basis: open discs) is the same as the product topology (basis: open rectangles): any open disc is the union of open rectangles, and any open rectangle is the union of open discs.

Proposition $X \times Y$ top product, Z top space.
A mapping $f : Z \to X \times Y$ is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Proof $\Rightarrow$ obvious

$\Leftarrow$ Subbase of product topology: $A \times Y$, $X \times B$ for A open in X and B open in Y.

Since $f^{-1}(A \times Y) = f^{-1}(\pi_1^{-1}(A)) = (\pi_1 \circ f)^{-1}(A)$,

if $\pi_1 \circ f$ is cts, then $f^{-1}(A \times Y)$ is open.

Similarly, if $\pi_2 \circ f$ is cts, $f^{-1}(X \times B)$ is open. //

Note. The product topology can be defined on the cartesian product of any finite or infinite family of top. spaces. In each case a subbasis of the topology consists of the sets $\pi_i^{-1}(A_i)$ for A_i open in X_i. A basis of the topology consists of sets $\prod_{j \in J} A_j$, where A_j is open in X_j and $A_j = X_j$ for all but a finite set of j's.

Topological Quotient.

Let X be a topological space, and $p: X \to Y$ a surjective mapping onto a set Y.

Then the quotient topology on Y is the biggest topology with which p is continuous:

$B \subseteq Y$ is open iff $p^{-1}(B)$ is open in X.

We use this to define a topology on the set of equivalence classes of an equivalence relation: if $\sim$ is an equivalence relation on X, and $X/\sim$ is the set of equiv. classes, the natural mapping $p: X \longrightarrow X/\sim$ is a surjection

Ex. On the interval $I = [0,1]$, we define the equivalence relation $0 \sim 1$.

The equivalence classes are $\{t\}$, $0<t<1$, and $\{0,1\}$.

Intuitively, we glue together the two ends of the interval.

The quotient space $I/_\sim$ is homeomorphic to the circle S^1

We consider S^1 in $\mathbb{C}$: $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, and define

$$g : I/_\sim \longrightarrow S^1$$

by $g(\{t\}) = e^{2\pi i t}$, $g(\{0,1\}) = 1$.

g is bijective. (check!)

Let $h = g^{-1} : S^1 \to I/_\sim$. We show h is cts:

Let A be open in $I/_\sim$. Then $p^{-1}(A)$ is open in I.

If $\{t\} \in A$, then there exists $\varepsilon > 0$ s.t. $(t-\varepsilon, t+\varepsilon) \subseteq p^{-1}(A)$.

If $\{0,1\} \in A$, then there exists $\varepsilon > 0$ s.t. $[0,\varepsilon) \cup (1-\varepsilon, 1] \subseteq p^{-1}(A)$.

In any case, the arc $\{ e^{2\pi i s} : s \in (t-\varepsilon, t+\varepsilon) \}$

or the arc $\{ e^{2\pi i s} : s \in [0,\varepsilon) \cup (1-\varepsilon, 1] \}$ belongs to $h^{-1}(A)$.

Hence $h^{-1}(A)$ is open and g^{-1} is cts.

For the continuity of g we apply the following proposition.

Proposition X topological space, $\sim$ equiv. relation on X,
$p : X \to X/_\sim$ the projection onto the quotient space,
Z topological space.
A mapping $g : X/_\sim \to Z$ is cts iff $g \circ p : X \to Z$ is cts.

Proof. $\Rightarrow$ Since the projection p is cts, if g is cts, so is $g \circ p$.

$\Leftarrow$ If $g \circ p$ is cts, for every A open in Z, $(g \circ p)^{-1}(A)$ is open. But $(g \circ p)^{-1}(A) = p^{-1}(g^{-1}(A))$, hence $g^{-1}(A)$ is open in the quotient topology.

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A mapping $f : X \to Y$ is open, if for every open $A \subseteq X$, $f(A)$ is open in Y.

A mapping $g : X \to Y$ is closed, if for every closed $F \subseteq X$, $g(F)$ is closed in Y.

Ex. $f : [0,1] \to S^1 : t \mapsto e^{2\pi i t}$ is cts and closed, but not open: $[0, \frac{1}{2})$ is open, but $f[0, \frac{1}{2})$ is not open.

$g = \left(f|_{[0,1)}\right)^{-1} : S^1 \to [0,1)$ is open: any open arc in S^1 is mapped to an open subset of $[0,1)$. It is also closed: since it is bijective, $g(S^1 \setminus F) = [0,1) \setminus g(F)$, and if F is closed then $g(F)$ is closed. But g is not cts.

Proposition. X, Y top. spaces. If $f : X \to Y$ is a cts surjection and f is either open or closed, then Y has the quotient topology.

Proof. We must show that A is open in Y iff $f^{-1}(A)$ is open in X.

$\Rightarrow$ follows from continuity.

$\Leftarrow$ Assume $A \subseteq Y$ and $f^{-1}(A)$ open in X. If f is open, since f is a surjection, $A = f(f^{-1}(A))$ and is open.

If f is closed, the closed set $X \setminus f^{-1}(A)$ is mapped to $f(X \setminus f^{-1}(A))$ which is closed. But $f(X \setminus f^{-1}(A)) = Y \setminus A$, hence A is open.

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Ex. On the unit sphere, $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ define $x \sim y$ if $x = -y$. The quotient space is the projective space. The quotient topology is homeomorphic to the top. manifold topology.

Let $\mathbb{R}P^2_m$ be projective space with the manifold topology, and $\mathbb{R}P^2$ with the quotient topology.

A set B is open in $\mathbb{R}P^2_m$ if its intersections with V_u, V_f, V_r correspond to open subsets of U_u, U_f, U_r. But then $p^{-1}(B)$ is open in S^2. So $p : S^2 \to \mathbb{R}P^2_m$ is cts.

If A is open in S^2, then $p^{-1}(p(A)) = A \cup (-A)$, which is open. Hence the intersections of $p(A)$ with V_u, V_f, V_r correspond to open sets, and p is open.

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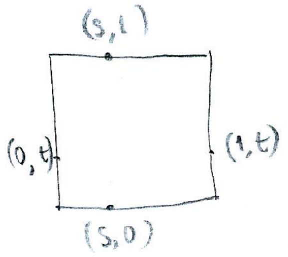
Ex.



$I \times I$ with equiv. relation:

for $s \in [0,1]$, $(s,0) \sim (s,1)$

for $t \in [0,1]$, $(0,t) \sim (1,t)$.

equiv. classes: $\{(s,t)\}$ if $0 < s < 1$, $0 < t < 1$.

$\{(s,0), (s,1)\}$ if $0 < s < 1$

$\{(0,t), (1,t)\}$ if $0 < t < 1$

$\{(0,0), (1,0), (0,1), (1,1)\}$.

$I \times I/_\sim \cong S^1 \times S^1$, the torus.

let $p : I \times I \to I \times I/_\sim$ be the projection.

$f : I \times I \to S^1 \times S^1 : (s,t) \longmapsto (e^{2\pi i s}, e^{2\pi i t})$.

$g : I \times I/_\sim \to S^1 \times S^1 : [s,t] \longmapsto (e^{2\pi i s}, e^{2\pi i t})$.

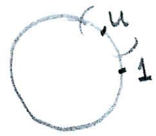
g well defined: if $e^{2\pi i s} = e^{2\pi i s'}$ then s, s' are 0 or 1.

For the same reason, g is a bijection.

g is cts, since $f = g \circ p$ is cts.

let $h = g^{-1} : S^1 \times S^1 \to I \times I/_\sim$. We'll show that h is cts.

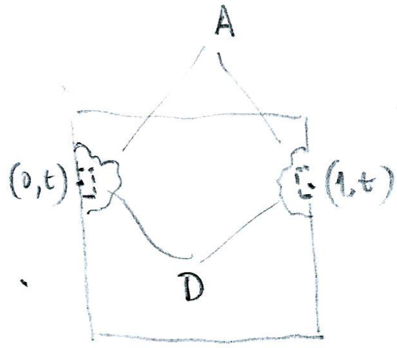
If $(u,v) \neq (1,1) \in S^1 \times S^1$,



then the principal argument $\operatorname{Arg} u$ is cts at u,

and $h(u,v) = \left(\frac{1}{2\pi}\operatorname{Arg} u, \frac{1}{2\pi}\operatorname{Arg} v\right)$ is cts at u, v.

If $(u,v) = (1, e^{2\pi i t})$ $0 < t < 1$, then



$h(u,v) = \{(0,t), (1,t)\}$

let V be a nbd of $\{(0,t),(1,t)\}$ in $I \times I/_\sim$.

Since $p : I \times I \to I \times I/_\sim$ is ct.,

$p^{-1}(V)$ contains an open subset $A \subseteq I \times I$ s.t. $\{(0,t),(1,t)\} \subseteq A$.

Then A containing a set

$$D = [0,\delta) \times (t-\varepsilon, t+\varepsilon) \cup (1-\delta, 1] \times (t-\varepsilon, t+\varepsilon)$$

such that $f(D)$ is open in $S^1 \times S^1$.

Then $f(D)$ is a nbd of $(1, e^{2\pi i t}) \in S^1 \times S^1$, and

$h(f(D)) = p(D) \subset V$. Hence h is cts at $(1, e^{2\pi i t})$.

Similarly we show h is cts at points $(e^{2\pi i s}, 1)$ and $(1,1)$. Hence h is cts and g is a homeomorphism.
