Similarly we show h is cts at points $(e^{2\pi i s}, 1)$ and $(1,1)$. Hence h is cts and g is a homeomorphism.

Properties of topological spaces

Closure. C subset of top space X.

The closure of C is the smallest closed subset of X containing C,

$$\overline{C} = \bigcap \{F \subseteq X : F \text{ closed}, C \subseteq F\}.$$

Hausdorff. A top space X has the Hausdorff property if for any two points $x, y \in X$, there are open sets U, V in X with:

$$x \in U, \quad y \in V, \quad U \cap V = \phi.$$

- Sierpinski space is not Hdff: the only open set containing 1 is the whole space.
- A point in a Hdff space is closed: For every point $y \in X \setminus \{x\}$, there is open W_y with $y \in W_y$, $x \notin W_y$. Hence $X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} W_y$ is open.

Proposition. 1. If $f : X \to Y$ is a cts injection and Y is Hdff, then X is Hdff.

2. If X and Y are Hdff, then $X \times Y$ is Hdff.

Proof. 1. If $x \neq y$, then $f(x) \neq f(y)$. Since Y is Hdff, there are open U, V with $f(x) \in U$, $f(y) \in V$, $U \cap V = \phi$. Then $x \in f^{-1}(U)$, $y \in f^{-1}(V)$, $f^{-1}(U), f^{-1}(V)$ are open, and $f^{-1}(U) \cap f^{-1}(V) = \phi$.

2. If $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$.
Assume $x_1 \neq x_2$. Then there are open $U, V \subseteq X$, with $x_1 \in U$, $x_2 \in V$, $U \cap V = \phi$.
But then $U \times Y$ and $V \cap Y$ are disjoint open subsets of $X \times Y$. //

If X is Hdff, a quotient space $X/_\sim$ is not necessarily Hdff:
Sierpinski space is $I/_\sim$ for $x \sim 0$ if $x < 1$.

Compactness.

A family of open subsets A_j of a top space X is an open covering if $\bigcup_{j \in J} A_j = X$.

A top space X is compact if X is Hdff and every open covering of X contains a finite subfamily that is also a covering: there are j_i, $i = 1, \dots, n$ s.t. $\bigcup_{i=1}^{n} A_{j_i} = X$.

Ex. The interval $[0,1)$ is not compact: the open covering $[0, 1-\frac{1}{n})$ does not have a fin. subcovering.

The Heine-Borel theorem says that $[0,1]$ is compact.

Prop Every closed subset of a compact top space is compact.

Proof. Let A_i be an open covering of the subset $C \subseteq X$ with the relative topology: there are open sets $B_i \subseteq X$ s.t. $A_i = B_i \cap C$, and $\bigcup A_i = C$.
If C is closed, $X \setminus C$ is open, and $\{B_i\} \cup \{X \setminus C\}$ is an open covering of X. If X is compact there are finite $i_1, \dots, i_k$ s.t. B_{i_j} cover X, and hence the corresponding A_{i_j} cover C.
//

Thm If X, Y are compact, then $X \times Y$ is compact.

Proof Since every open subset of $X \times Y$ is the union of sets $A \times B$, for A open in X, B open in Y, it is enough to consider coverings by sets of this form.
Let A_j, B_j, $j \in J$, open sets in X and Y s.t.

$$X \times Y \subseteq \bigcup_{j \in J} A_j \times B_j .$$

For every $x \in X$, $\{x\} \times Y \cong Y$ is compact, so there are finite $j_{x,1}, \dots, j_{x,k(x)}$ s.t. $\{x\} \times Y \subseteq \bigcup_{i=1}^{k(x)} A_{j_{x,i}} \times B_{j_{x,i}}$.
Let $V_x = \bigcap_{i=1}^{k(x)} A_{j_{x,i}}$. This is open, so $\{V_x, x \in X\}$ is an open covering of X. Since X is compact, there are finite $x_1, \dots, x_m$ s.t. $X \subseteq \bigcup_{\ell=1}^{m} V_{x_\ell}$. Then

$$X \times Y = \bigcup_{\ell=1}^{m} V_{x_\ell} \times Y \subseteq \bigcup_{\ell=1}^{m} \left(\bigcup_{i=1}^{k(x_\ell)} A_{j_{x_\ell},i} \times B_{j_{x_\ell},i} \right).$$

//

Prop. Every compact subset of a Hdff space is closed.

Proof X Hdff space, Y compact subspace. We show that $X \setminus Y$ is open.

Let $x \in X \setminus Y$. For every $y \in Y$ there are A_y, B_y open in X, s.t. $x \in A_y$, $y \in B_y$ and $A_y \cap B_y = \emptyset$.

Then $\{B_y, y \in Y\}$ is an open covering of the compact space Y, hence there are finite $y_1, \ldots, y_k \in Y$ s.t. $Y \subseteq \bigcup_{i=1}^{k} B_{y_i}$. But $A_{y_i} \subseteq X \setminus B_{y_i}$, and the set $A = \bigcap_{i=1}^{k} A_{y_i} \subseteq \bigcap (X \setminus B_{y_i}) = X \setminus \bigcup B_{y_i} \subseteq X \setminus Y$.

But A is open, and $x \in A$. Hence $X \setminus Y$ is open. //

Thm. X compact, Y Hdff. If $f : X \to Y$ is continuous then $f(X)$ is a compact subspace of Y, and f is closed.

Proof Let A_j, $j \in J$ be an open covering of $f(X)$. Then $f^{-1}(A_j)$, $j \in J$, is an open covering of X. There exist finite $j_1, \ldots, j_k$ s.t. $f^{-1}(A_{j_i})$, $i = 1, \ldots, k$ is an open covering of X. But then A_{j_i}, $i = 1, \ldots, k$ is an open covering of $f(X)$. $f(X)$ is Hdff, hence it is compact.

Let F be closed in X. Then F is compact, and hence $f(F)$ is compact. But since Y is Hdff, $f(F)$ is closed. Hence f is closed. //

Corollary. X compact, Y Hdff, $f : X \to Y$ continuous bijection. Then f is a homeomorphism.

Connectedness

A top. space X is connected if there is no cts surjection $f: X \longrightarrow \{0,1\}$.

Thm The only connected subspaces of $\mathbb{R}$ are the one-point sets, the intervals and the whole of $\mathbb{R}$.

$\mathbb{Q}$, with the relative topology as a subset of $\mathbb{R}$ is not connected. In fact, any subset of $\mathbb{Q}$ with more than one element is not connected.

Thm In a topological space X the following are equivalent:

1) X is connected.
2) There are no subset $A, B \subseteq X$ s.t. A and B are both non-empty, open and $A \cap B = \phi$, $A \cup B = X$.
3) The only open and closed subsets of X are ϕ, X.

Proof. $1 \Rightarrow 2$. Let X be connected. If there are A, B as in 2, define $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$. f is well defined since $A \cap B = \phi$ and $A \cup B = X$, f is cts since A, B are open. Since X is connected, f is not surjective, and one of A or B is empty. $\times$.

$2 \Rightarrow 3$. If there is an open and closed subset A of X then A and $B = X \setminus A$ satisfy the conditions of 2.

$3 \Rightarrow 1$. If there is a cts surjection $f: X \to \{0,1\}$, then $f^{-1}(0)$ is open and closed, different from ϕ, X.

//

Prop. If X is connected and $f: X \to Y$ is a cts surjection, then Y is connected.

Proof If Y is not connected, then there is $g: Y \to \{0,1\}$ cts surjection. Then $g \circ f$ is a cts surjection, and X is not connected.

Cor. The cts image of a connected top space is a connected space.

Prop. A_j, $j \in J$ family of connected subspaces of X.
If $\bigcap_{j \in J} A_j \neq \emptyset$, then $\bigcup A_j$ is connected.

Proof. Let $X = \bigcup A_i$, $f: X \to \{0,1\}$ cts surjection.
Assume $f^{-1}(0) \cap A_1$ non empty, $f^{-1}(1) \cap A_2$ non empty.
Then $A_1 \subset f^{-1}(0)$, $A_2 \subset f^{-1}(1)$, but $A_1 \cap A_2 \neq \emptyset$, contradiction. //

Prop If A is ctd subset of X, and B subset of X s.t. $A \subseteq B \subseteq \bar{A}$, then B is ctd.

Proof. If B not connected, there is cts surjection $f: B \to \{0,1\}$.
Since A is ctd, $f|_A$ is not surjective. Assume $A \subseteq f^{-1}(0)$.
Then $f^{-1}(1)$ is open in B, and there is open G in X s.t. $G \cap B = f^{-1}(1) \neq \emptyset$.
But $A \subseteq f^{-1}(0) = B \setminus G \subseteq X \setminus G$. But $X \setminus G$ is closed in X,

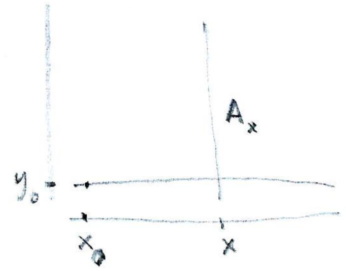
and $\overline{A} \subseteq X \setminus G$. But then $B \subseteq \overline{A} \subseteq X \setminus G$ and $G \cap B \neq \phi$. $\times$.

//

Thm The product of connected spaces is connected.

Proof (for finite products). Consider the product $X \times Y$, X, Y connected, and $x_0 \in X$, $y_0 \in Y$.

For $x \in X$ define $A_x = (X \times \{y_0\}) \cup (\{x\} \times Y)$



$X \times \{y_0\} \cong X$, $\{x\} \times Y \cong Y$ are connected,

and A_x is connected, since $X \times \{y_0\} \cap \{x\} \times Y \neq \phi$.

And $X \times Y = \bigcup_{x \in X} A_x$, which is connected,

since $\bigcap_{x \in X} A_x = \{(x_0, y_0)\} \neq \phi$. //

If $x \in X$, the <u>connected component</u> of x in X, $C(x)$, is the union of all connected subsets of X which contain x.

A cts mapping $\sigma : [0,1] \longrightarrow X$ is a <u>path</u> (δρόμος) in X, from $\sigma(0)$ to $\sigma(1)$.

A top. space X is <u>path connected</u> (κατα δρόμους συνεκτικός) if for every two points x, y in X, there is a path from x to y.

Lemma. A space X is path connected iff there is a point $x_0 \in X$ s.t. for every point $y \in X$ there is a path from x_0 to y.

Proof $\Rightarrow$ clear by dfn.

$\Leftarrow$ let y_1, y_2 be points in X, σ_1 path from x_0 to y_1, σ_2 from x_0 to y_2.

Define $\sigma(t) = \begin{cases} \sigma_1(1-2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \sigma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$ σ is well defined path from y_1 to y_2. //

<u>Thm</u> A path connected space is connected.

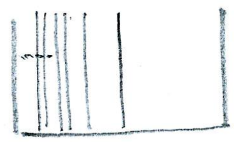
<u>Proof</u> Assume X path connected, and $f : X \to \{0,1\}$ cts surjection.

Let $x \in f^{-1}(0)$, $y \in f^{-1}(1)$, σ path from x to y.

Then $f \circ \sigma : I \to \{0,1\}$ is a cts surjection, but I is connected. $\cancel{\times}$ //.

A connected space is not necessarilly path connected.

<u>Example</u>. The topological comb:



The union of all path connected subsets of X which contain x is the <u>path component</u> of x in X, $K(x)$.

<u>Thm</u> An open subset A of $\mathbb{R}^n$ is connected iff it is path connected.

<u>Proof</u> We show that the path component of a point x in A is open and closed in A.

Let $y \in K(x)$. Since A is open, there is $\varepsilon > 0$ s.t. $D(y, \varepsilon) = \{z \in \mathbb{R}^n : |y - z| < \varepsilon\} \subseteq A$.

The segment from y to z also belongs to A.

Hence there is a path from x to z in A, and $D(y, \varepsilon) \subseteq K(x)$.

So $K(x)$ is open.

Let $y \in A \setminus K(x)$, and $\varepsilon > 0$ s.t. $D(y, \varepsilon) \subseteq A$.

Let $z \in D(y, \varepsilon)$. If $z \in K(x)$, then there would be a path from x to z and then to y. So $D(y, \varepsilon) \subseteq A \setminus K(x)$, and $A \setminus K(x)$ is open, hence $K(x)$ is closed.

//.