Classification of surfaces

A closed surface is a compact, connected topological 2-manifold.

Ex. The sphere, the torus, the projective plane are closed surfaces.

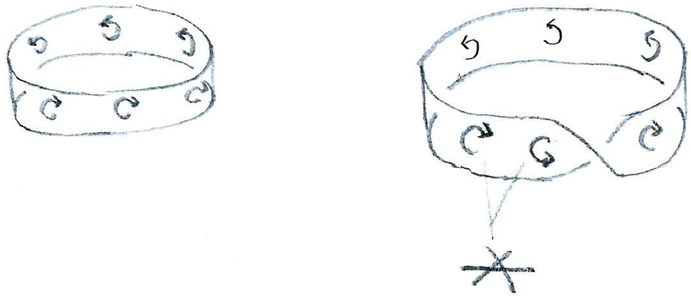
The plane $\mathbb{R}^2$ is not compact.

The cylinders $S^1 \times \mathbb{R}$, $S^1 \times (0,1)$ are not compact.

The cylinder $S^1 \times [0,1]$ is not a 2-manifold (it has boundary).

Surfaces can be orientable or non orientable.

Intuitively, a surface is orientable if we can choose a direction of rotation around each point, so that the directions are the same in nearby points.



To make this notion precise, we need more advanced tools. In this course we will define a surface to be non orientable if it contains a open subset homeomorphic to the Möbius strip. Otherwise it will be orientable.

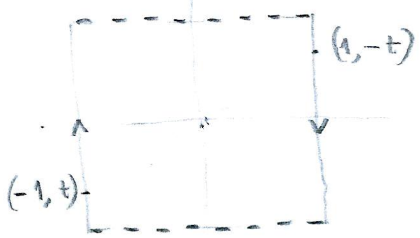
The Möbius strip is the quotient space obtained by identifying the opposite sides of a parallelogram, with a half twist:

Let $X = \{ (s,t) \in \mathbb{R}^2 : -1 \le s \le 1,\ -1 < t < 1 \}$.

We define the equivalence relation $\sim$ generated by

$$(-1,t) \sim (1,-t)$$

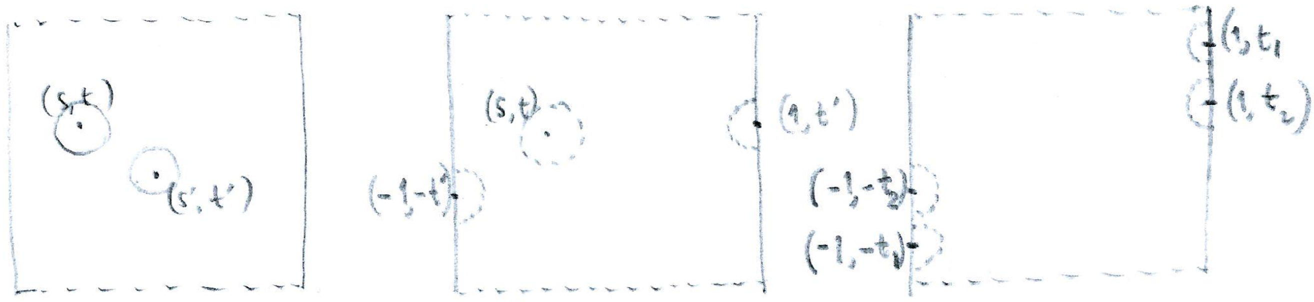
for every $t \in (-1,1)$.



We show that the Möbius strip $M = X/\sim$ is a Hdff space and a 2-manifold.

Let (s,t), (s',t') be two points with $s, s' \neq \pm 1$.
Then there are disjoint open discs around (s,t) and (s',t') in X, which are mapped injectively by the projection $q: X \to X/\sim$, to disjoint open nbds of the two points in $X/\sim$.

If (s,t) is a point with $|s| \neq 1$, and $[1,t']$ is the equiv. class of the point $(1,t') \in X$, then there exists $\delta > 0$ s.t.



$D((s,t),\delta)$, $D((1,t'),\delta) \cap X$ and $D((-1,-t'),\delta) \cap X$ are disjoint subsets of $\mathbb{R}^2$, which project to disjoint open nbds of (s,t) and $[1,t']$ in $X/\sim$.

Similarly for $[1,t_1]$, $[1,t_2]$, $t_1 \neq t_2$.

To show that $X/_\sim$ is a 2-manifold:

If $(s,t) \in X$, $|s| \neq 1$, there is $\varepsilon > 0$ with $\varepsilon < 1-|s|$ and $\varepsilon < 1-|t|$. Then $D((s,t),\varepsilon)$ is mapped injectively by q, and $q(D((s,t),\varepsilon))$ is homeomorphic to the open disc B^2.

For $(1,t) \in X$, there is $\varepsilon > 0$ s.t. $\varepsilon < 1-|t|$, and $\big(D((1,t),\varepsilon) \cup D((-1,-t),\varepsilon)\big) \cap X$ is mapped by q to a subset of $X/_\sim$ homeomorphic to the open disc B^2.

We show that the projective space $\mathbb{R}P^2$ is a non orientable closed surface.

We know that $\mathbb{R}P^2 = S^2/_{x\sim -x}$, is a 2-manifold.

Let $p: S^2 \to \mathbb{R}P^2$ be the projection. First we show that

$\mathbb{R}P^2$ is Hdff: let $p(x)$, $p(y)$ be different points in $\mathbb{R}P^2$. Then $x \neq \pm y$ and there is $\varepsilon > 0$ s.t.

$D(x,\varepsilon) \cap \big(D(y,\varepsilon) \cup D(-y,\varepsilon)\big) = \phi$.

Then $p(D(x,\varepsilon) \cap S^2)$ and $p(D(y,\varepsilon) \cap S^2)$ are disjoint open nbds of $p(x)$ and $p(y)$ in $\mathbb{R}P^2$.

Since $\mathbb{R}P^2$ is Hdff and S^2 is compact, the image of the cts mapping p is compact.

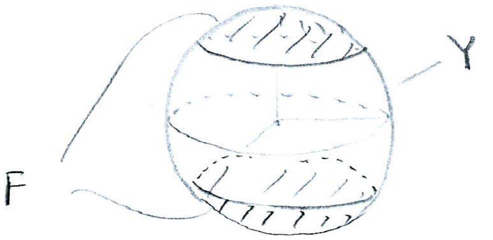
Since S^2 is connected, $p(S^2) = \mathbb{R}P^2$ is connected.

Hence $\mathbb{R}P^2$ is a closed surface.

Now we want to show that $\mathbb{R}P^2$ contains a Möbius strip.

Consider the closed subset of S^2, $F = \{ (x,y,z) \in S^2 : |z| \geq \frac{1}{2} \}$.

The projection p is a closed mapping (since it maps compact to Hdff) hence $p(F)$ is a closed subset of $\mathbb{R}P^2$. Let $Y = S^2 \setminus F$.



We claim $p(Y)$ is homeomorphic to the Möbius strip.

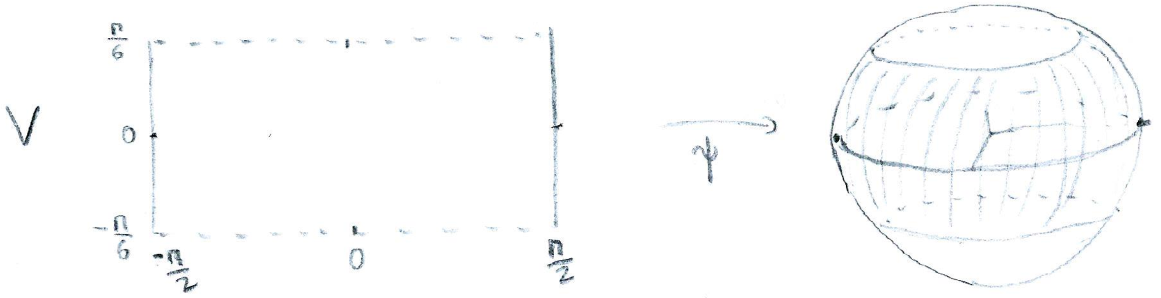
Consider polar coordinates on Y:

The point $(x,y,z) \in Y$ corresponds to (ϑ, φ) with $-\pi < \vartheta \leq \pi$ and $-\frac{\pi}{6} < \varphi < \frac{\pi}{6}$:

$$\psi : (\vartheta, \varphi) \mapsto (x,y,z) = (\cos\varphi \cos\vartheta,\ \cos\varphi \sin\vartheta,\ \sin\varphi)$$

The set $V = \{ (\vartheta, \varphi) \in \mathbb{R}^2 : -\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2},\ -\frac{\pi}{6} < \varphi < \frac{\pi}{6} \}$

is mapped by $p \circ \psi$ onto $p(Y) \subset \mathbb{R}P^2$, because for any point $(x,y,z) \in Y$, either itself or its opposite belongs to $\psi(V)$.



For $X = [-1,1] \times (-1,1)$, we let f be the linear homeomorphism

$$f : X \longrightarrow V \; : \; (s,t) \longmapsto \left(\tfrac{\pi}{2} s, \tfrac{\pi}{6} t \right).$$

We define a map $\bar{f}$ from the Möbius strip $M = q(X)$ to the set $p(Y) \subset \mathbb{R}P^2$:

$$\bar{f} : q(s,t) \longmapsto p \circ \psi \circ f(s,t).$$

This mapping is well defined: $(-1,t) \sim (1,-t)$ in X,

and
$$p \circ \psi \left(-\tfrac{\pi}{2}, \tfrac{\pi}{6} t \right) = \left(\cos \tfrac{\pi t}{6} \cos\left(-\tfrac{\pi}{2}\right), \cos \tfrac{\pi t}{6} \sin\left(-\tfrac{\pi}{2}\right), \sin \tfrac{\pi t}{6} \right)$$
$$= \left(0, -\cos \tfrac{\pi t}{6}, \sin \tfrac{\pi t}{6} \right).$$

$$p \circ \psi \left(\tfrac{\pi}{2}, -\tfrac{\pi}{6} t \right) = \left(\cos\left(-\tfrac{\pi t}{6}\right) \cos \tfrac{\pi}{2}, \cos\left(-\tfrac{\pi t}{6}\right) \sin \tfrac{\pi}{2}, \sin\left(-\tfrac{\pi t}{6}\right) \right)$$
$$= \left(0, \cos \tfrac{\pi t}{6}, -\sin \tfrac{\pi t}{6} \right).$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q"'] & V \arrow[d, "p \circ \psi"] \\
M \arrow[r, "\bar{f}"'] & p(Y)
\end{tikzcd}$$

Since f is bijective, and $q(s,t) = q(s',t')$ iff $p \circ \psi \circ f(s,t) = p \circ \psi \circ f(s',t')$, the mapping $\bar{f}$ is also bijective.

Since $p \circ \psi \circ f$ is cts, $\bar{f}$ is cts.

Since $q \circ f^{-1}$ is cts, $\bar{f}^{-1}$ is cts.

So $\bar{f}$ maps the Möbius strip homeomorphically onto the open subset $p(Y)$ of $\mathbb{R}P^2$, and $\mathbb{R}P^2$ is non orientable. //