Our aim is to <u>classify all closed surfaces</u>.

From now on surface will refer to a closed surface, unless specified otherwise.

The connected sum of surfaces

Let S be a surface, and Δ be the <u>closed</u> disc, $\Delta = \{ z \in \mathbb{C} : |z| \leq 1 \}$, B^2 the open disc, $B^2 \subset \Delta$.

Let $f : \Delta \to S$ be an embedding (i.e. f cts injection and $f(\Delta) \cong \Delta$).

Then $f(B^2)$ is homeomorphic to the open disc.

Now consider two surfaces S_1 and S_2, and embeddings $f_1 : \Delta \to S_1$, $f_2 : \Delta \to S_2$.

We remove the open discs $f_1(B^2)$, $f_2(B^2)$ from S_1, S_2 to obtain the top. spaces

$$S_1 \setminus f_1(B^2) \quad , \quad S_2 \setminus f_2(B^2) .$$

On the disjoint union $X = \left(S_1 \setminus f_1(B^2)\right) \sqcup \left(S_2 \setminus f_2(B^2)\right)$ we define the equivalence relation:

for $x \in S_1$ and $y \in S_2$, $x \sim y$ if $x = f_1(e^{2\pi i t})$, $y = f_2(e^{2\pi i t})$

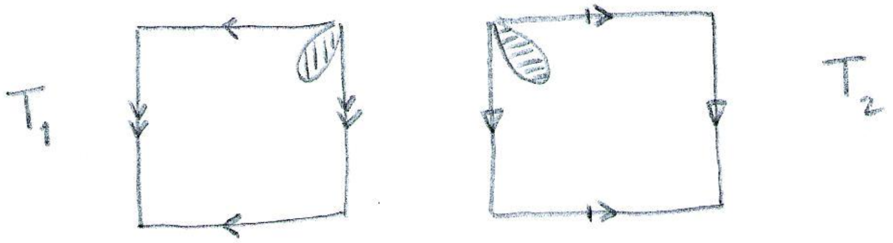
for $e^{2\pi i t} \in \Delta \setminus B^2 = S^1$.

The quotient space $X/_\sim$ is the <u>connected sum</u> of the surfaces S_1 and S_2, $S_1 \# S_2$.

<u>Theorem</u> The connected sum of surfaces S_1 and S_2 is a connected, compact 2-manifold, i.e. a closed surface. If we use different embeddings $f_1' : \Delta \to S_1$, $f_2' : \Delta \to S_2$, the resulting surface is homeomorphic to the one obtained using f_1, f_2.

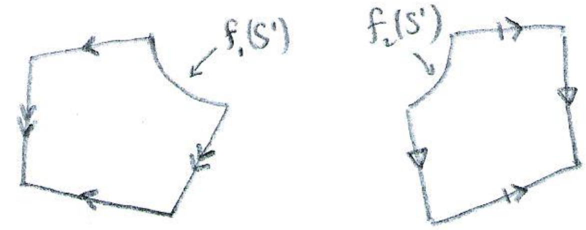
<u>Example</u> The connected sum of two tori is the surface with 2 holes, pretzel.

The torus is obtained from the square $I \times I$ by identification of opposite sides:

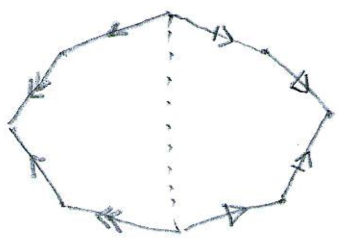


We remove a disc $f_1(B^2)$ and $f_2(B^2)$ from each torus.

$T_1 \setminus f_1(B^2)$ is the quotient of a pentagon with identifications of four of the sides, while the fifth side is $f_1(S^1)$.



On the disjoint union of the two pentagons we identify $f_1(e^{2\pi i t})$ with $f_2(e^{2\pi i t})$. We obtain an octagon, with identifications of the sides:



To describe the identifications of the sides of the octagon we choose one vertex as a start and an orientation. On each pair of identified sides we choose a direction. We denote each pair of identified sides by a letter. As we traverse the boundary of the octagon, starting from the chosen vertex, in the direction of the chosen orientation, we meet each side.
We write the corresponding letter with exponent $+1$ if the direction of the side agrees with the orientation of the boundary of the octagon, and with exponent -1 if the direction of the side is opposite to the orientation.

The <u>symbol</u> for the surface $T_1 \# T_2$ is

$$\alpha \beta \alpha^{-1} \beta^{-1} \gamma \delta \gamma^{-1} \delta^{-1}.$$

If we start at a different vertex, the letters in the symbol are permuted cyclically. If we change the direction, the exponents on the symbol are interchanged. So the above symbol describes the same surface as the symbol $\gamma^{-1} \beta \alpha \beta^{-1} \alpha^{-1} \delta \gamma \delta^{-1}$.

<u>The Klein surface.</u> is the quotient of the square $X = [-1,1] \times [-1,1]$, under the equivalence relation generated by

$$(s,-1) \sim (s,1)$$

$$(-1,t) \sim (1,-t).$$

$K = X/_\sim$ cannot be embedded in $\mathbb{R}^3$.

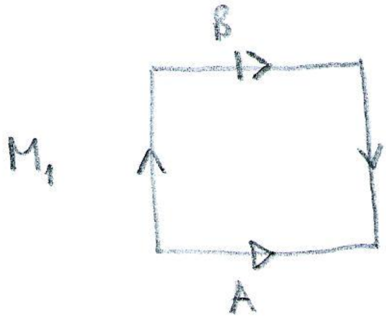
K has symbol $\alpha\beta\alpha\beta^{-1}$.

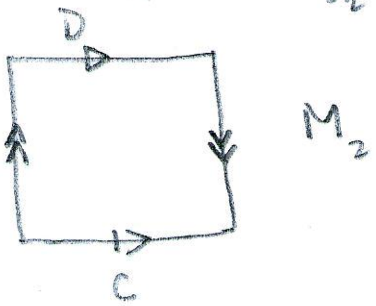
We'll show that K is homeomorphic to the connected sum of two projective planes.

Let X_1, X_2 be two copies of the proj. plane, and $f_1 : \Delta \to X_1$, $f_2 : \Delta \to X_2$ two embeddings of the closed disc to X_1 and X_2.

We know that $X_1 \setminus f_1(B^2)$, $X_2 \setminus f_2(B^2)$ are homeomorphic to copies of the Möbius band with boundary, M_1, M_2.

$$\mathbb{R}P^2 \# \mathbb{R}P^2 = M_1 \sqcup M_2 \Big/ f_1(e^{i\vartheta}) \sim f_2(e^{i\vartheta})$$

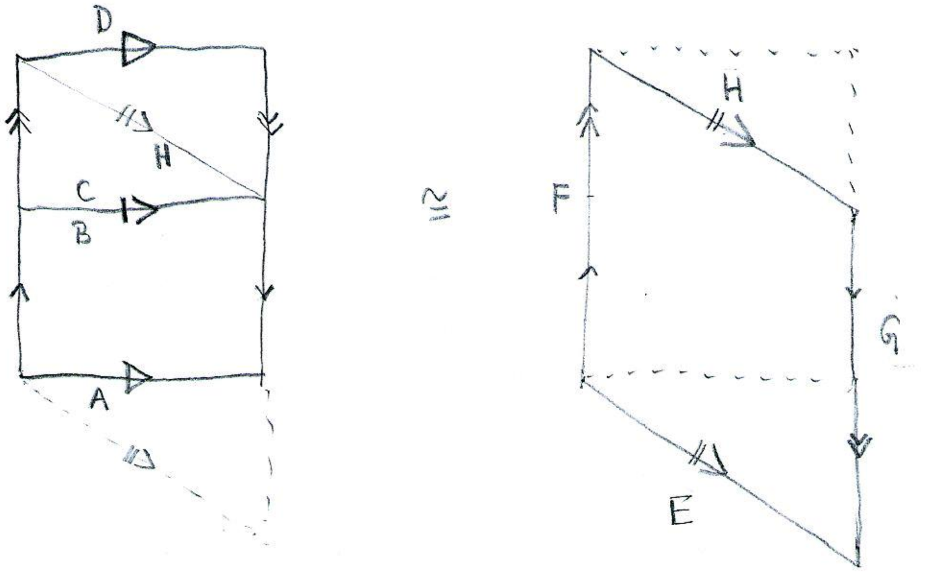


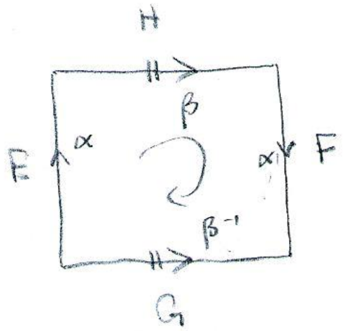


The boundary of M_1 is made up of A and B.

The boundary of M_2 is made up of C and D.

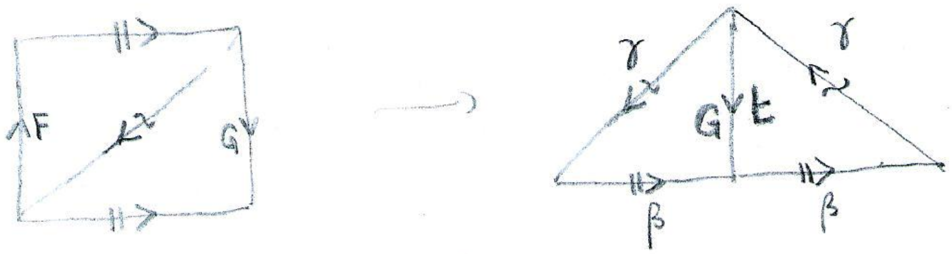
We identify the side B of M_1 with the side C of M_2, to obtain a parallelogram, with sides identified as in the diagram.





$\cong K.$

We cut along H, and paste D to A:
we have a par/gram with identifications of
H to E and G to F, which make a Klein surface,
with symbol $\alpha\beta\alpha\beta^{-1}$.
We'll do a further change, which will give the
symbol in a simpler form:



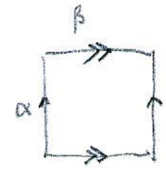
In this form the symbol is

$$\beta\beta\gamma\gamma.$$

We show inductively that the connected sum of n tori is the surface obtained from a $4n$-gon with identifications given by the symbol

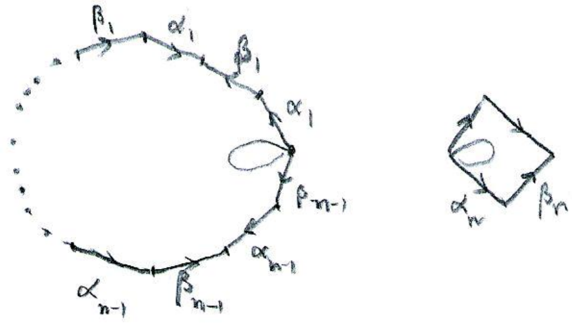
$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}.$$

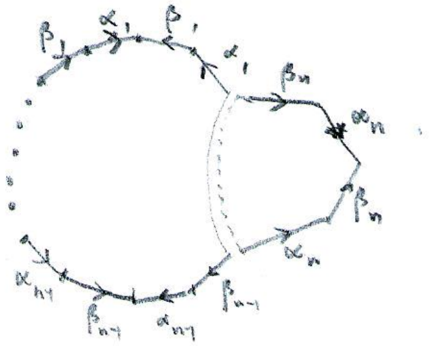
$\underline{n=1}$



$\alpha\beta\alpha^{-1}\beta^{-1}$

Assume $\#_{1}^{n-1} T$:





Similarly we show that the connected sum of n projective planes is the surface obtained from a $2n$-gon with identifications given by the symbol

$$\alpha_1\alpha_1\alpha_2\alpha_2\cdots\alpha_n\alpha_n.$$

$n=2$ Klein surface, $\cong \alpha_1\alpha_1\alpha_2\alpha_2$, assume $\#_{1}^{n-1}\mathbb{RP}^2$:

$\#\,\mathbb{RP}^2$