<u>Theorem</u> (Classification of closed surfaces). (Poincaré, Dehn)

Every closed surface is homeomorphic to one of the following.

1) The sphere S^2 — surface of genus 0.

2) The connected sum of g tori, $g \geq 1$, — orientable surface of genus g

3) The connected sum of h proj. planes, $h \geq 1$. — nonorientable surface genus h

Any two of the above surfaces are not homeomorphic.

The proof of the Classification Theorem is based on triangulation of surfaces.

<u>Defn</u> A <u>triangulation</u> of a closed surface S is a subdivision of S into a finite number of subsets

$$\sigma_1, \sigma_2, \dots, \sigma_n \quad \text{s.t.}$$

1) $S = \bigcup_1^n \sigma_i$

2) Each σ_i is homeomorphic to a circle:

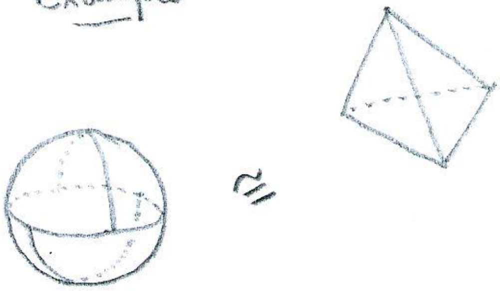
there are triangles in the plane $\tau_1, \dots, \tau_n$ and embeddings $f_i : \tau_i \longrightarrow S$ with $f_i(\tau_i) = \sigma_i$.

3) If two sets σ_i and σ_j, $i \neq j$ intersect then their intersection is exactly one of the following:

i) the image of a side of τ_i and a side of τ_j.

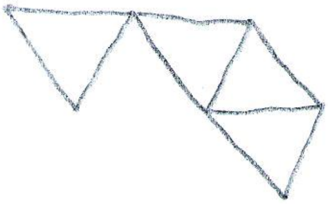
or ii) the image of a vertex of τ_i and a vertex of τ_j.

Example



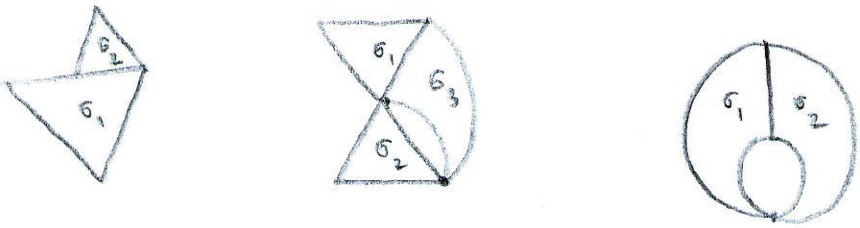
A tetrahedron is a triangulation of the sphere, with four triangles, 6 edges, 4 vertices.

Allowed intersections:



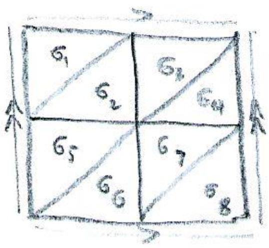
✓

Not allowed intersections:



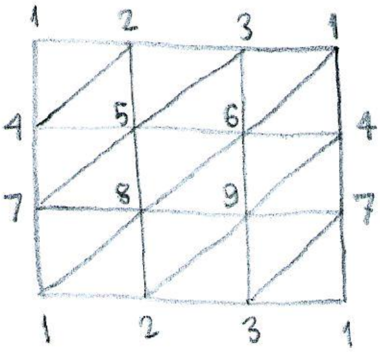
✗

Not a triangulation of the torus:



Why not?

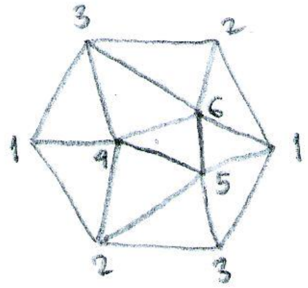
A triangulation of the torus: 18 triangles, 9 vertices, 27 edges.



Check that it is a triangulation!

There exists one with fewer triangles. Can you find it?

A triangulation of $\mathbb{RP}^2$.



Check that it is !

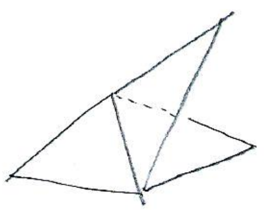
10 triangles: 126, 236, 346, 314,
124, 245, 235, 315,
165, 456.

6 vertices, 15 edges.

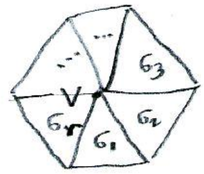
Three properties of triangulations

- Each edge of a triangulation belongs to exactly 2 triangles.

- If v is a vertex of a triangulation, then the triangles which have v as a vertex can be ordered cyclically $\sigma_1, \sigma_2, \dots, \sigma_r$ so that σ_i and σ_{i+1} have a common edge if $1 \leq i < r$ and σ_r has a common edge with σ_1.



not a surface



- The edges of $\sigma_1, \dots, \sigma_r$ which do not have v as a vertex form a set homeomorphic to the circle, called the link of v, $C(v)$.

The set of triangles, with the identifications of the edges, gives us a model of the surface. More precisely:

<u>Proposition</u>

If $\sigma_1, \dots, \sigma_n$ is a triangulation of S, by $f_i : \tau_i \to \sigma_i \subseteq S$, then

$$S \cong \left(\bigsqcup_{i=1}^{n} \tau_i \right) \Big/ \sim$$

where $\sim$ is generated by $x \sim y$ for $x \in \tau_i$, $y \in \tau_j$ if $f_i(x) = f_j(y)$.

<u>Proof</u>. We define $f : \bigsqcup \tau_i \to S$ by $f|_{\tau_i} = f_i$.
Then f is cts, since its restrictions to the disjoint, closed sets τ_i are continuous.
It is surjective, since $S = \bigcup \sigma_i$.
But $\bigsqcup \tau_i$ is compact and S is Hdff. Hence f is a closed surjection and S has the quotient topology of f. //

The main thm on triangulations:

<u>Thm</u> Every closed surface S can be triangulated //.

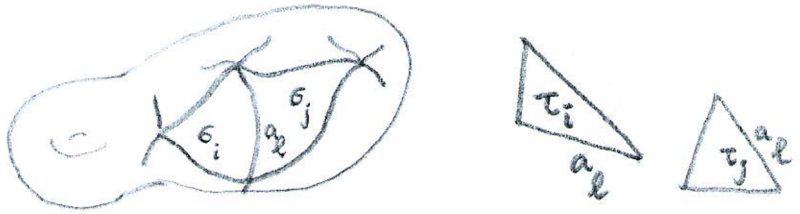
The proof can be found in ...

Next we show that every triangulated surface is homeomorphic with one of the surfaces of the Classification Thm.
First, from a triangulated surface S we construct a polygon P with identifications, so that $S \cong P/_\sim$.

Then we show that each such polygon can be transformed to one of the polygons of the Theorem.

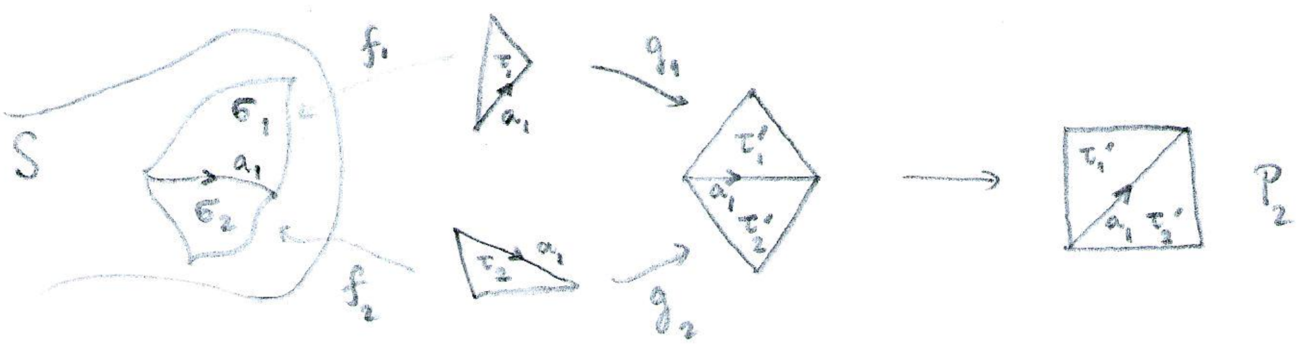
Step 1.

Let S be a surface with triangulation $\sigma_1, \dots, \sigma_n$.
The triangulation has $k = \frac{3n}{2}$ edges.
We number them $a_1, \dots, a_k$, and we pick a direction on each edge.
We use the embeddings f_i to transfer the numbering and the directions to the sides of the triangles τ_i. For each edge a_l of the triangulation, there are two triangles τ_i and τ_j with a side labeled a_l.



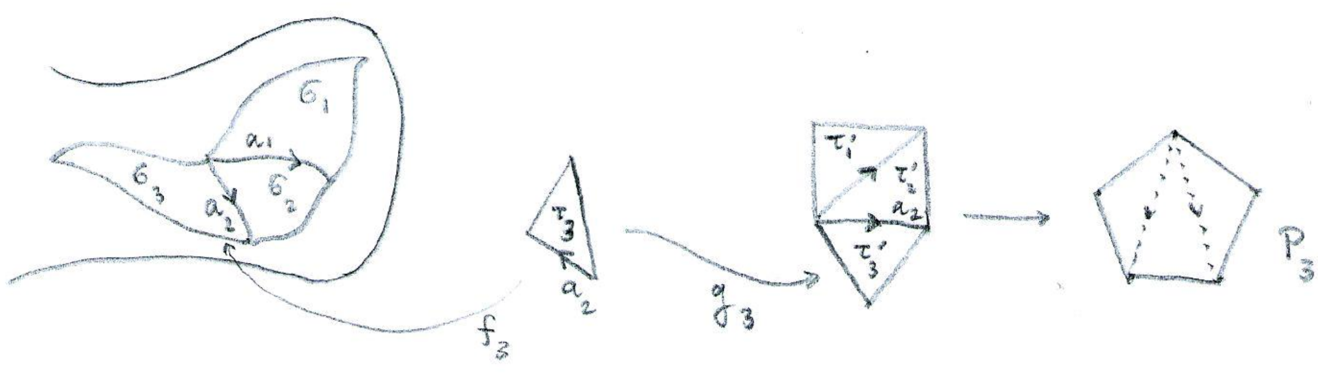
We map the triangle τ_1 homeomorphically to an equilateral triangle τ_1', and we choose a side of τ_1, let it be a_1.
Let τ_2 be the other triangle with a side labeled a_1. We map τ_2 homeomorphically to an equilateral triangle τ_2' so that the sides labeled a_1 are identified, and the directions match:
we use homeomorphisms $g_i: \tau_i \to \tau_i'$ such that

$$\left(f_2|_{a_1}\right)^{-1} \circ f_1|_{a_1} = \left(g_2|_{a_1}\right)^{-1} \circ g_1|_{a_1}$$



Then we map the rhombus $\tau_1' \cup \tau_2'$ homeomorphically to a square P_2.

We choose a side of the square, which corresponds to an edge, let it be a_2. Since σ_1 and σ_2 intersect only along side a_1, there is exactly one other triangle, let it be τ_3, with a side labeled a_2.



We map τ_3 homeomorphically to an equilateral triangle τ_3' so that side a_2 is identified with side a_2 of P_2, and the directions match. Then we map the pentagon $\tau_1' \cup \tau_2' \cup \tau_3'$ to a canonical pentagon P_3.

We continue in this way. When we have used triangles $\tau_1, \dots, \tau_m$, for $m < n$, we have constructed a canonical $(m+2)$-gon, P_m, with labels and chosen directions on its sides.

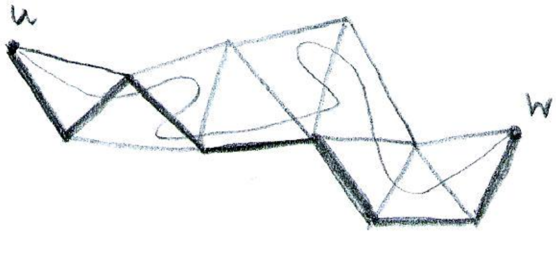
Some of the sides of the polygon may have the same label, which means that they correspond to the same edge on S.

We show that there is at least one side of the polygon whose label is different from all the other sides.

Assume that all sides of P_m form pairs with the same label. Consider $K = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_m$, $L = \sigma_{m+1} \cup \dots \cup \sigma_n$ in S. They are both non empty.

Choose vertices $u \in K$, $w \in L$.

If $w \notin K$, we connect u to w with a polygonal path consisting of edges of the triangulation.



a path in S can be deformed to a path along edges.

Along this path there is a last vertex belonging to K, let it be a. The next vertex, b, does not belong to K.

Now we look at the link $C(a)$: since a belongs to a triangle in K, there is a vertex c in $C(a)$ belonging to K. The vertex b, which does not belong to K, also belongs to $C(a)$.

If $w \in K \cap L$, in the link $C(w)$ there is a vertex c belonging to K and a vertex b which does not belong to K.

In both cases, since the link of a vertex is a circle, there is a polygonal path on $C(a)$ or $C(w)$ joining the vertex c to the vertex b. On this path there is a last vertex d belonging to K, with the next vertex e not belonging to K. Since $e \notin K$, the triangle ade belongs to L. But the edge ad belongs to K, and we assumed that each edge in K belongs to two different triangles in K. Hence edge ad belongs to 3 different triangles, a contradiction. $\times$

So we conclude that it is possible to continue the process until we have used all the triangles of the triangulation, to form an $(n+2)$-gon, P_n, whose sides form pairs with the same label, and embeddings $g_i : \tau_i \longrightarrow P_n$.

This completes the first part of the proof.