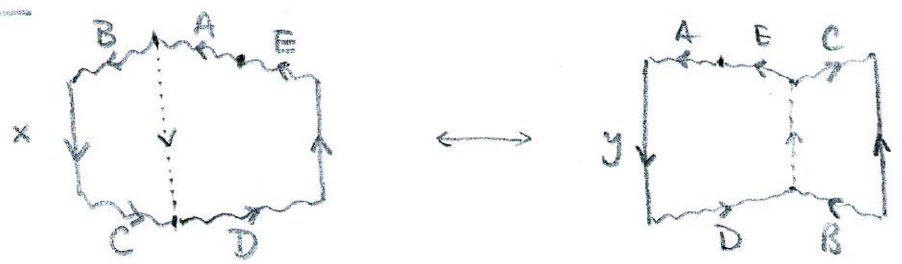
Step 2.

We define 3 types of transformations of a polygon, by cutting and pasting, which do not change the homeomorphism type of the surface obtained by the identification of the sides of the polygon.

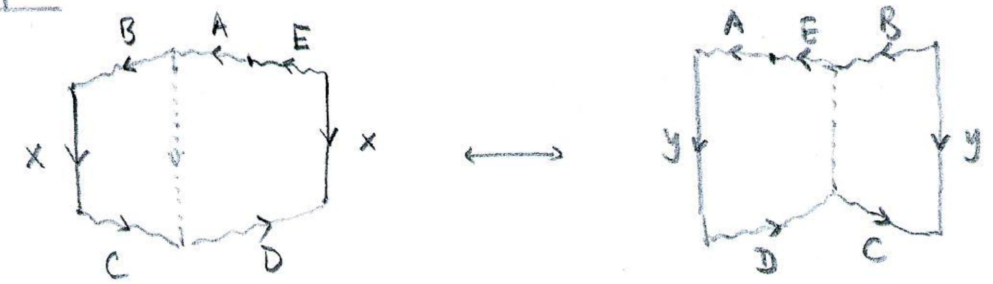
We use small letters to represent single sides, and capitals to represent sets of consecutive sides of the polygon, which may be empty.

Type 1



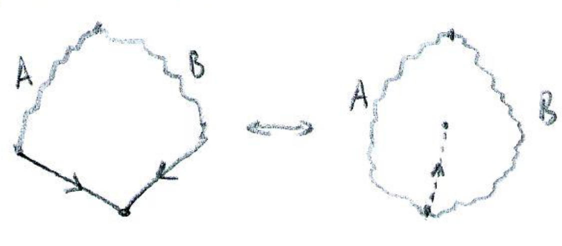
$$A B x C D x E \longleftrightarrow A y D B^{-1} y C^{-1} E$$

Type 2.



$$A B x C D x^{-1} E \longleftrightarrow A y D C y^{-1} B E$$

Type 3. If AB contain at least 4 sides, we define the "zip up". $A x x^{-1} B \longleftrightarrow A B$.



Theorem Every polygon with at least four sides and identifications of the sides in pairs, can be brought by transformations of types 1, 2, 3 to one of the following:

1. M_0 with symbol $x x^{-1} y y^{-1}$ or $x y y^{-1} x^{-1}$.

2. M_g with symbol $x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1}$, $g \geq 1$.

3. N_1 with symbol $x x y y^{-1}$

4. N_h with symbol $x_1 x_1 x_2 x_2 \cdots x_h x_h$, $h \geq 2$.

Proof
We consider a polygon with at least 4 sides.
Each side has a label, and each label appears on exactly two different sides.
On each side we choose a direction, so that the directions are matched by the identifications.
We choose a starting vertex and an orientation of the polygon, and write the symbol of the polygon.

$$x_{l_1}^{\varepsilon_1} x_{l_2}^{\varepsilon_2} \cdots x_{l_n}^{\varepsilon_n},$$

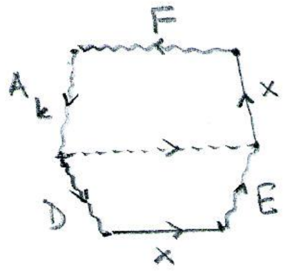
where l_i is the label of the i^{th} side, and $\varepsilon_i = +1$ if the direction of the side x_{l_i} is the same as the orientation of the polygon, otherwise it is -1.

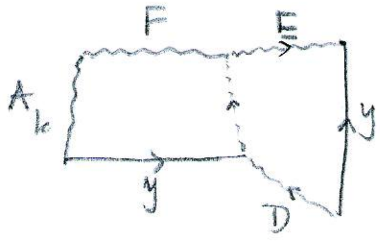
A pair of sides with the same label, $l_i = l_j$, form a <u>similar pair</u> if $\varepsilon_i = \varepsilon_j$, and a <u>reversed pair</u> if $\varepsilon_i = -\varepsilon_j$.

Two pairs of reversed sides are <u>interlocked</u> if they appear on the symbol in the order $\dots x \dots y \dots x^{-1} \dots y^{-1} \dots$, and <u>non interlocked</u> if $\dots x \dots x^{-1} \dots y \dots y^{-1} \dots$ or $x \dots y \dots y^{-1} \dots x^{-1} \dots$.

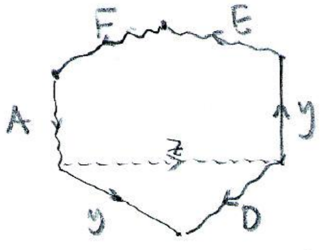
<u>Step 2.a</u> We collect all similar pairs at the begining of the symbol, to obtain a symbol AB, where A is of the form $x_1 x_1 x_2 x_2 \cdots x_r x_r$, and B does not contain any similar pairs. We use induction on k:

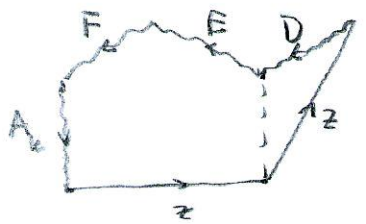
If $A_k = x_1 x_1 \cdots x_k x_k$, and the polygon has symbol of the form $A_k D x E x F$, we apply type 1, by cutting from the end of A_k to the end of E,





and then we apply type 1, by cutting from the end of A_k to the beginning of D (i.e. the end of D^{-1})





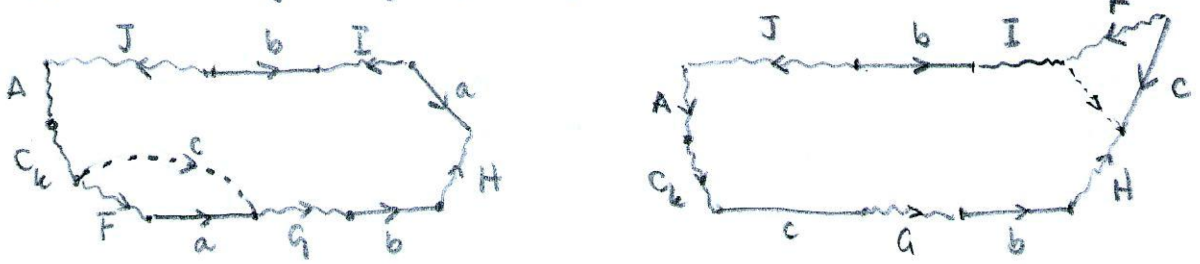
$$A_k D x E x F \longrightarrow A_k y D^{-1} y E^{-1} F \longrightarrow A_k z z D E^{-1} F .$$

We repeat this step until there are no similar pairs in $DE^{-1}F$.

Step 2b. We collect all interlocked reversed pairs immediately after the similar pairs. We replace AB with a symbol ACD, where C is of the form $y_1 z_1 y_1^{-1} z_1^{-1} \cdots y_s z_s y_s^{-1} z_s^{-1}$, and D does not contain interlocked reversed pairs.

Inductively, assume $C_k = y_1 z_1 y_1^{-1} z_1^{-1} \cdots y_k z_k y_k^{-1} z_k^{-1}$, and $D = F a G b H a^{-1} I b^{-1} J$.

First we apply type 2 by cutting from the end of C_k to the beginning of G,



$$A C_k F a G b H a^{-1} I b^{-1} J \longrightarrow A C_k c G b H c^{-1} F I b^{-1} J.$$

Then we cut from the end of G to the beginning of F and apply type 2, pasting along b, to obtain

$$A C_k c G d F I H c^{-1} d^{-1} J$$

Thirdly, we cut from the end of C_k to the beginning of F, and apply type 2, pasting along c, to obtain

$$A C_k e F I H G d e^{-1} d^{-1} J$$

Finally, we cut from the beginning of F to the end of d, and paste along d, to obtain

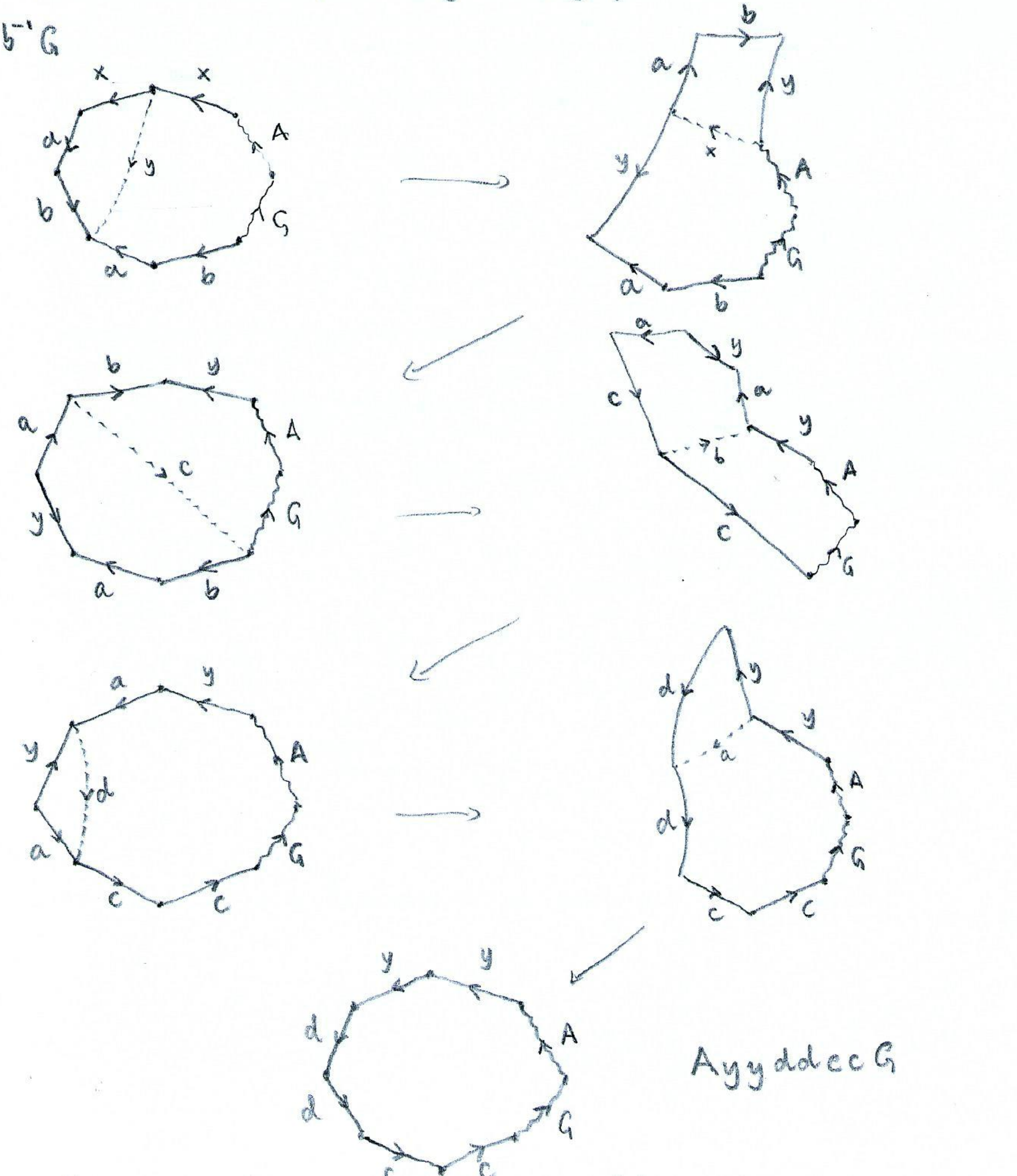
$$A C_k e f e^{-1} f^{-1} F I H G J.$$

This is of the form $A\, C_{k+1}\, D'$.

<u>Exercise</u> Draw all the transformations of type 2 in step 2b.

<u>Step 2c</u>. If A is not empty, that is if there exist similar pairs $\ldots x \ldots x \ldots$, in the following step we replace all strings of the form $aba^{-1}b^{-1}$ by $ccdd$. So the symbol ACD is replaced by $A'D'$.

$A x x a b a^{-1} b^{-1} G$



$A y y d d c c G$

These transformations are the inverse of type 1.
For example, in the first step

$$A\, x\, D B^{-1}\, x\, C E \longrightarrow A B y C^{-1} D y E$$

where $C = ab$ and B, D are empty.

<u>Exercise</u> Use the above procedure to show that

$$T \# \mathbb{RP}^2 \cong K \# \mathbb{RP}^2 .$$

At the end of Step 2b or 2c, we have a polygon with symbol AD or CD, and D contains only non interlocking reversed pairs. Let $\ldots x \ldots x^{-1} \ldots$ be the pair with the least number of letters between x and x^{-1}. If there is a letter y between x and x^{-1}, then y^{-1} must also be between x and x^{-1}, since the pairs are not interlocking. This would contradict the minimality of x. Hence there is no such y, and we can remove the pair $x x^{-1}$ by a transformation of type 3, provided we are left with 4 or more letters.

In the end, if A or C have at least 4 letters, we have a symbol of type M_g, $g \geq 1$, or N_h, $h \geq 2$. If A and C are empty, the symbol is M_0: $x x^{-1} y y^{-1}$ or $x y y^{-1} x^{-1}$, which represents the sphere.

If A has only two letters, then D also has two letters, and the symbol is $N_1 : xxyy^{-1}$, which represents a projective plane.

//.

This completes the proof that every closed surface is homeomorphic to one of S^2, $\#_1^g T$ for $g \geq 1$, or $\#_1^h \mathbb{RP}^2$, $h \geq 1$.

To complete the classification theorem, we must show that all these surfaces are not homeomorphic to one another.

To do this we'll need tools from Algebraic Topology.