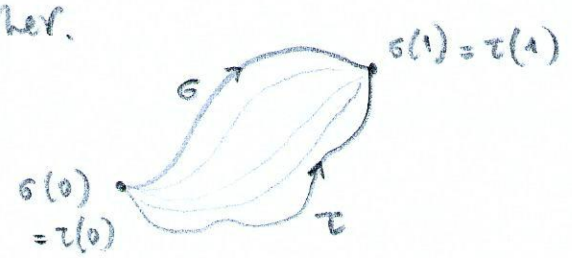
Homotopy of paths.

Two paths in a topological space X, $\sigma : I \to X$, $\tau : I \to X$ are homotopic if we can deform continuously the one into the other.



<u>Defn</u> Two paths $\sigma : I \to X$, $\tau : I \to X$ are <u>homotopic</u> (<u>relative to their endpoints</u>) if there is a continuous mapping $H : I \times I \to X$ s.t.

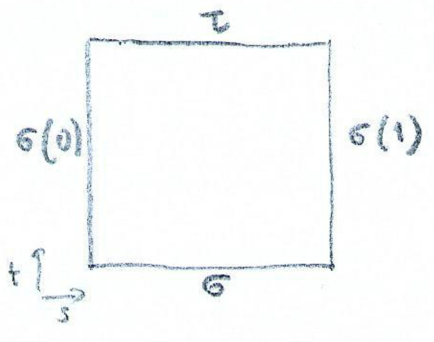
1) $H(s,0) = \sigma(s)$

2) $H(s,1) = \tau(s)$

3) $H(0,t) = \sigma(0) = \tau(0)$

4) $H(1,t) = \sigma(1) = \tau(1)$.

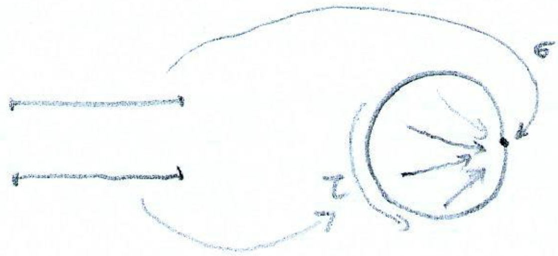
Schematically:



<u>Example</u> In $\mathbb{R}^n$, any two paths with the same endpoints are homotopic:

$$H(s,t) = (1-t)\,\sigma(s) + t\,\tau(s) .$$

Example. In S^1, the constant path $\sigma : I \to S^1$, $\sigma(s) = 1$, and the path $\tau : I \to S^1 : s \mapsto e^{2\pi i s}$ are not homotopic. Any deformation of τ to σ would have to move outside the circle.



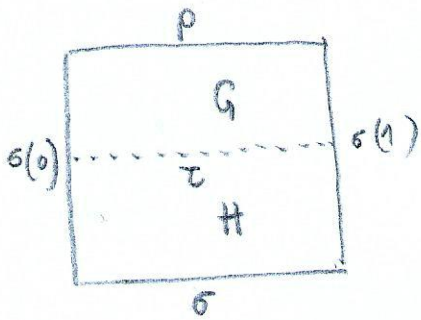
Lemma. Path homotopy is an equivalence relation on the set of paths in a topological space.

Proof. Reflexivity, symmetricity are obvious.

Transitivity: If $\sigma \sim \tau$, $\tau \sim \rho$, there exist

$H : I \times I \to X$, $H : \sigma \sim \tau$ and

$G : I \times I \to X$, $G : \tau \sim \rho$.



Define $F : I \times I \to X$ by:

$$F(s,t) = \begin{cases} H(s, 2t), & 0 \le t \le \frac{1}{2} \\ G(s, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

Then F is cts, since $H(s,1) = \tau(s) = G(s,0)$,

$F(s,0) = \sigma(s)$, $F(s,1) = \rho(s)$,

$F(0,t) = \sigma(0) = \tau(0) = \rho(0)$

$F(1,t) = \sigma(1) = \tau(1) = \rho(1)$. //

We denote $\Pi(X)$ the set of homotopy classes (relative to the endpoints) of paths in a topological space X.

On $\Pi(X)$ we define a composition: travelling along path σ and then along path τ.

If $\sigma: I \to X$ and $\tau: I \to X$ are paths in X, and $\sigma(1) = \tau(0)$, we define the path

$$\sigma \cdot \tau : I \longrightarrow X$$

by

$$\sigma \cdot \tau(s) = \begin{cases} \sigma(2s) & 0 \leq s \leq \frac{1}{2} \\ \tau(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

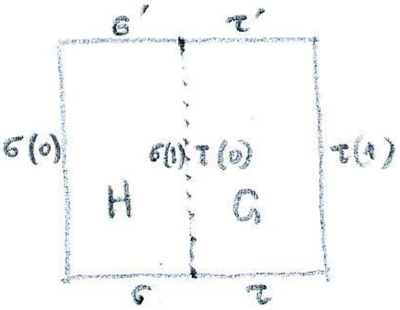
$\sigma \cdot \tau$ is continuous since $\sigma(1) = \tau(0)$.

<u>Lemma</u>. If $\sigma \sim \sigma'$ and $\tau \sim \tau'$, with $\sigma(1) = \tau(0)$, then $\sigma \cdot \tau \sim \sigma' \cdot \tau'$

<u>Proof</u> Let $H: \sigma \sim \sigma'$ and $G: \tau \sim \tau'$.

We define $F: I \times I \longrightarrow X$ by

$$F(s,t) = \begin{cases} H(2s,t) & 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$



F is continuous since $F(\frac{1}{2}, t) = \sigma(1) = \tau(0)$, and defines a homotopy $\sigma \cdot \tau \sim \sigma' \cdot \tau'$. //

So we can define the composition of homotopy classes of paths on X: If $[\sigma], [\tau] \in \Pi(X)$ and $\sigma(1) = \tau(0)$, then we define

$$[\sigma][\tau] = [\sigma \cdot \tau].$$

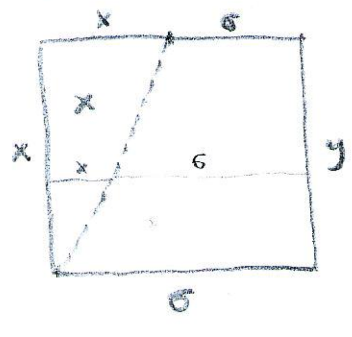
If $x \in X$, we denote $[x]$ the homotopy class of the constant path $s \longmapsto x$.

If $\sigma(0) = x$ and $\sigma(1) = y$, then

$$[\sigma] = [x][\sigma] = [\sigma][y].$$

A homotopy from σ to $x \cdot \sigma$ is given by

$$H(s,t) = \begin{cases} x & 0 \leq s \leq \frac{t}{2} \\ \sigma\left(\dfrac{s - t/2}{1 - t/2}\right) & \frac{t}{2} \leq s \leq 1 \end{cases}$$



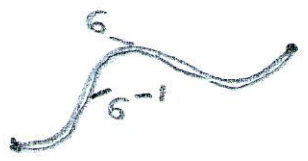
H is continuous, since $\dfrac{s - t/2}{1 - t/2} = 0$ when $s = t/2$, and $\sigma(0) = x$.

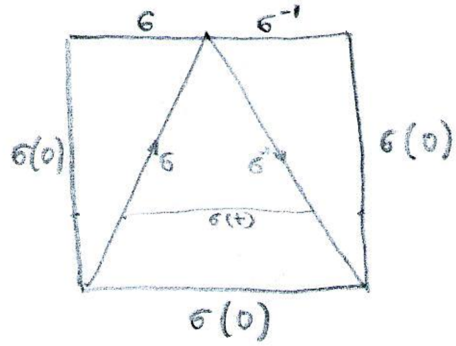
<u>Exercise</u> Find the homotopy from σ to $\sigma \cdot y$, when $\sigma(1) = y$.

For every path σ in X we define the <u>opposite</u> path, σ^{-1}.

$$\sigma^{-1}(s) = \sigma(1-s).$$

The opposite path is a one sided inverse for the composition of homotopy classes of paths. We construct a homotopy $H : \sigma \cdot \sigma^{-1} \sim \sigma(0)$.





$$H(s,t) = \begin{cases} \sigma(2s) & 0 \le s \le \frac{1}{2}t \\ \sigma(t) & \frac{1}{2}t \le s \le 1 - \frac{1}{2}t \\ \sigma^{-1}(2s-1) & 1 - \frac{1}{2}t \le s \le 1 \end{cases}$$

Note that $\sigma^{-1}(2s-1) = \sigma\big(1-(2s-1)\big) = \sigma(2-2s)$.

When $s = \frac{1}{2}t$, $\sigma(2s) = \sigma(t)$ and when $s = 1 - \frac{1}{2}t$,

$\sigma^{-1}(2s-1) = \sigma(2-2s) = \sigma(2-2+t) = \sigma(t)$.

Hence H is cts.

Similarly, we can construct homotopy $G : \sigma^{-1} \cdot \sigma \sim \sigma(1)$.

Proposition The inverse of a homotopy class of paths is unique, that is if $[\sigma \cdot \rho] = [\sigma(0)]$, then $\rho \sim \sigma^{-1}$.

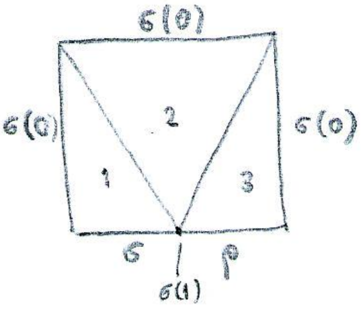
Proof. Assume that $H : \sigma \cdot \rho \sim \sigma(0)$.
We'll construct a homotopy $G : \sigma^{-1} \sim \rho$.

We have

$$H(s,0) = (\sigma \cdot \rho)(s)$$

$$H(s,1) = \sigma(0)$$

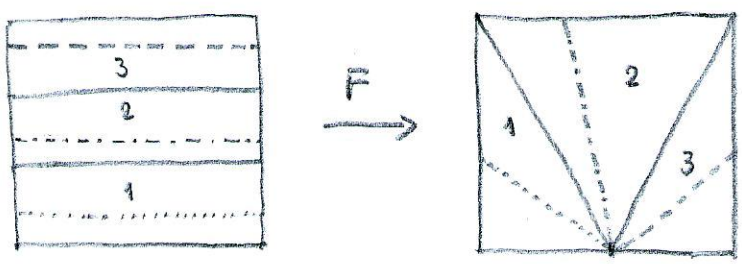
$$H(0,t) = H(1,t) = \sigma(0).$$



We divide the domain of H into 3 regions, and define

$F : I \times I \longrightarrow I \times I$ by

$$F(s,t) = \begin{cases} (1-s)\left(\tfrac{1}{2},0\right) + s\,(0,3t) & 0 \le t \le \tfrac{1}{3} \\ (1-s)\left(\tfrac{1}{2},0\right) + s\,(3t-1,1) & \tfrac{1}{3} \le t \le \tfrac{2}{3} \\ (1-s)\left(\tfrac{1}{2},0\right) + s\,(1,3-3t) & \tfrac{2}{3} \le t \le 1 \end{cases}$$



F is continuous.

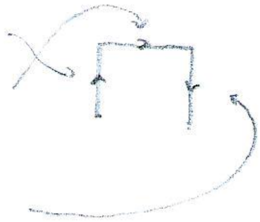
We define $G = H \circ F : I \times I \longrightarrow X$.

Then $G(s,0) = H\left(\tfrac{1}{2} - \tfrac{s}{2}, 0\right) = \sigma(1-s) = \sigma^{-1}(s)$

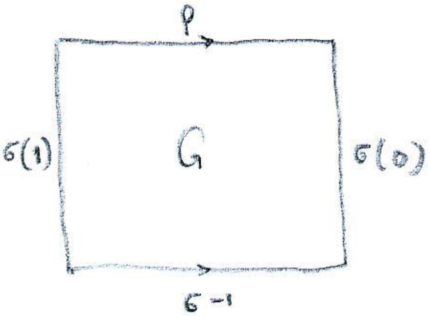
$G(s,1) = H\left(\tfrac{1}{2} + \tfrac{s}{2}, 0\right) = \rho(s)$

$G(0,t) = H\left(\tfrac{1}{2}, 0\right) = \sigma(1)$

$$G(1,t) = \begin{cases} H(0,3t) = \sigma(0) & 0 \leq t \leq \frac{1}{3} \\ H(3t-1,1) = \sigma(0) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ H(1,3-3t) = \sigma(0) & \frac{2}{3} \leq t \leq 1 \end{cases}$$



Altogether, G is the homotopy



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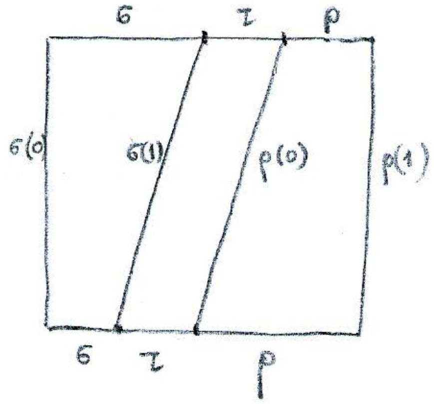
<u>Lemma</u>. The composition of homotopy classes of paths is associative: if $\sigma(1) = \tau(0)$ and $\tau(1) = \rho(0)$, then

$$(\sigma \cdot \tau) \cdot \rho \sim \sigma \cdot (\tau \cdot \rho) .$$

<u>Proof</u>

$$H(s,t) = \begin{cases} \sigma\left(\frac{4s}{1+t}\right) & 0 \leq s \leq \frac{1+t}{4} \\ \tau(4s-1-t) & \frac{1+t}{4} \leq s \leq \frac{2+t}{4} \\ \rho\left(\frac{4s-2-t}{2-t}\right) & \frac{2+t}{4} \leq s \leq 1 \end{cases}$$



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