The Fundamental Group.

A path $\sigma : I \to X$ is a closed path based at x_0 if $\sigma(0) = \sigma(1) = x_0$.
The set of homotopy classes of closed paths in X based at x_0 is denoted $\pi(X, x_0)$.

Thm. The set $\pi(X, x_0)$ with the operation of composition of homotopy classes of paths, is a group, called the fundamental group of X based at x_0.

Proof. The operation is well defined, since all the paths have the same ends. We have shown that it is associative. Since $\sigma(0) = \sigma(1)$, the path σ^{-1} is both a left and a right inverse:

$$\sigma \cdot \sigma^{-1} \sim \sigma(0) = \sigma(1) \sim \sigma^{-1} \cdot \sigma .$$

The identity element of the group is the homotopy class of the constant path, $[x_0]$.

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The fundamental group depends only on the path component containing the base point x_0. From now on we'll assume X is path connected.

Proposition. Let X be a path connected space, x_0 and x_1 points in X, path $\alpha : I \to X$, $\alpha(0) = x_0$, $\alpha(1) = x_1$. Then the mapping

$$\alpha_{\#} : \pi(X, x_0) \longrightarrow \pi(X, x_1) : [\sigma] \longmapsto [\alpha^{-1}][\sigma][\alpha]$$

is an isomorphism.

Proof. The mapping $\alpha_\#$ is well defined, since if $\rho \sim \sigma$, then $\alpha^{-1}\cdot\rho\cdot\alpha \sim \alpha^{-1}\cdot\sigma\cdot\alpha$.

It is a homomorphism, since

$$\begin{aligned}\alpha_\#([\sigma])\,\alpha_\#([\rho]) &= [\alpha^{-1}][\sigma][\alpha][\alpha^{-1}][\rho][\alpha]\\ &= [\alpha^{-1}][\sigma][\rho][\alpha]\\ &= [\alpha^{-1}][\sigma\cdot\rho][\alpha]\\ &= \alpha_\#([\sigma][\rho]).\end{aligned}$$

The homomorphism $\alpha_\#$ has inverse $(\alpha^{-1})_\#$, hence it is an isomorphism.

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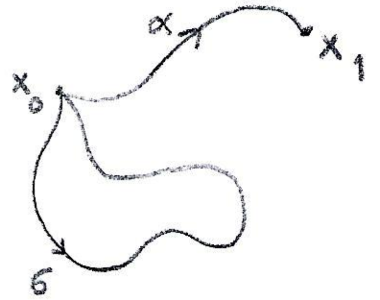


Proposition. If $f: X \longrightarrow Y$ is a cts mapping, and $f(x_0) = y_0$, then the mapping

$$f_*: \pi(X, x_0) \longrightarrow \pi(Y, y_0): [\sigma] \longmapsto [f\circ\sigma]$$

is a homomorphism of groups, with the following properties:

1) $(id_X)_* = id_{\pi(X,x_0)}$

2) If $g: Y \to Z$ is a continuous mapping, then
$$g_* \circ f_* = (g\circ f)_*$$

3) If f is a homeomorphism, then f_* is a group isomorphism.

Proof. f_* is well defined: if $H: I\times I \to X$ is a homotopy $\sigma \sim \rho$, then $f\circ H: I\times I \to Y$ is a homotopy $f\circ\sigma \sim f\circ\rho$.

f_* is a homomorphism: if $[\sigma], [\rho] \in \pi(X, x_0)$, then

$$\begin{aligned} f_*([\sigma][\rho]) = f_*([\sigma\cdot\rho]) &= [f\circ(\sigma\cdot\rho)] \\ &= [(f\circ\sigma)\cdot(f\circ\rho)] \\ &= [f\circ\sigma][f\circ\rho] \\ &= f_*([\sigma])\ f_*([\rho]). \end{aligned}$$

1) $$\begin{aligned} (id_X)_*([\sigma]) &= [id_X\circ\sigma] \\ &= [\sigma] = id_{\pi(X,x_0)}([\sigma]). \end{aligned}$$

2) $$\begin{aligned} (g_*\circ f_*)([\sigma]) &= g_*(f_*([\sigma])) \\ &= g_*([f\circ\sigma]) \\ &= [g\circ f\circ\sigma] = (g\circ f)_*([\sigma]). \end{aligned}$$

3) If f is a homeomorphism, with inverse f^{-1}, then $(f^{-1})_*$ is the inverse homomorphism of f_*:

$$(f^{-1})_*\circ f_*([\sigma]) = (f^{-1}\circ f)_*([\sigma]) = [\sigma]$$

and $$f_*\circ(f^{-1})_*([\sigma]) = (f\circ f^{-1})_*([\sigma]) = [\sigma].$$ //

Calculation of fundamental groups.

1. Let X be the space with one point, $X = \{x_0\}$.

Then the only path on X is the constant path $\sigma(t) = x_0$, and $\pi(X, x_0)$ is isomorphic to the trivial group,

$$\pi(X, x_0) \cong 1.$$

2. Let X be a convex subset of $\mathbb{R}^n$, and $\sigma : I \longrightarrow X$ a path with $\sigma(0) = \sigma(1) = x_0 \in X$. Then

$$H(s,t) = (1-t)\,\sigma(s) + t\,x_0$$

is a homotopy from σ to the constant path x_0, which leaves the endpoints fixed. Hence $\sigma \sim x_0$, and

$$\pi(X, x_0) \cong 1.$$

3. We'll show that $\pi(S^1, 1) \cong \mathbb{Z}$.

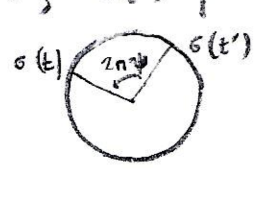
Let $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ and define the mapping $\exp : \mathbb{R} \longrightarrow S^1$ by $\exp(t) = e^{2\pi i t}$.

Define $\alpha : S^1 \setminus \{-1\} \longrightarrow (-\tfrac{1}{2}, \tfrac{1}{2})$ such that $\exp(\alpha(z)) = z$.

Lifting lemma. If σ is a path in S^1, $\sigma : I \longrightarrow S^1$, and $\sigma(0) = 1$, there is a unique path σ' in $\mathbb{R}$, $\sigma' : I \to \mathbb{R}$, with $\sigma'(0) = 0$, such that $\exp \circ \sigma' = \sigma$.

Proof. Since I is a compact subset of $\mathbb{R}$, σ is uniformly continuous and there exists $\varepsilon > 0$ such that if $|t - t'| < \varepsilon$ then $|\sigma(t) - \sigma(t')| < 1$. In particular, for such t, t', $\sigma(t) \neq -\sigma'(t)$.

We can define $\psi(t,t') = \alpha\left(\sigma(t) / \sigma(t') \right)$.



Choose $N > \frac{1}{\varepsilon}$. Define

$$\sigma'(t) = \psi\left(t, \tfrac{N-1}{N}t\right) + \psi\left(\tfrac{N-1}{N}t, \tfrac{N-2}{N}t\right) + \cdots$$
$$\cdots + \psi\left(\tfrac{2}{N}t, \tfrac{1}{N}t\right) + \psi\left(\tfrac{1}{N}t, 0\right).$$

Then $\sigma' : I \longrightarrow \mathbb{R}$ is continuous, $\sigma'(0) = N\psi(0,0) = 0$, and

$$\exp(\sigma'(t)) = \exp\circ\psi\left(t, \tfrac{N-1}{N}t\right)\exp\circ\psi\left(\tfrac{N-1}{N}t, \tfrac{N-2}{N}t\right)\cdots$$
$$\cdots\exp\circ\psi\left(\tfrac{2}{N}t, \tfrac{1}{N}t\right)\exp\circ\psi\left(\tfrac{1}{N}t, 0\right)$$

$$= \exp\circ\alpha\left(\frac{\sigma(t)}{\sigma\left(\frac{N-1}{N}t\right)}\right)\exp\circ\alpha\left(\frac{\sigma\left(\frac{N-1}{N}t\right)}{\sigma\left(\frac{N-2}{N}t\right)}\right)\cdots$$
$$\cdots\exp\circ\alpha\left(\frac{\sigma\left(\frac{2t}{N}\right)}{\sigma\left(\frac{t}{N}\right)}\right)\exp\circ\alpha\left(\frac{\sigma(t/N)}{\sigma(0)}\right)$$

$$= \frac{\sigma(t)}{\sigma\left(\frac{N-1}{N}t\right)}\,\frac{\sigma\left(\frac{N-1}{N}t\right)}{\sigma\left(\frac{N-2}{N}t\right)}\cdots\frac{\sigma\left(\frac{2t}{N}\right)}{\sigma\left(\frac{t}{N}\right)}\,\frac{\sigma\left(\frac{t}{N}\right)}{\sigma(0)}$$

$$= \sigma(t).$$

To show uniqueness, assume $\tilde{\sigma} : I \longrightarrow \mathbb{R}$ continuous mapping with $\tilde{\sigma}(0) = 0$ and $\exp\circ\tilde{\sigma} = \sigma$. Then $t \longmapsto \sigma'(t) - \tilde{\sigma}(t)$ is a cts mapping on I with image in $\mathbb{Z}$,

since $\exp(\sigma'(t) - \tilde{\sigma}(t)) = 1$. But $\mathbb{Z}$ is discrete, I is connected, hence $\sigma' - \tilde{\sigma}$ is constant. Since $\sigma'(0) = \tilde{\sigma}(0)$, we have $\sigma'(t) = \tilde{\sigma}(t)$ for all $t \in I$.

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Homotopy lifting lemma. If σ and τ are paths in S^1, with $\sigma(0) = \tau(0) = 1 \in S^1$, and F is a homotopy from σ to τ, there is a unique homotopy F' from σ' to τ' relative to the endpoints, so that $F = \exp \circ F'$.

Proof. $F : I \times I \to S^1$ is uniformly cts, so there is $\varepsilon > 0$ s.t. if $(s-s')^2 + (t-t')^2 < \varepsilon^2$, then $|F(s,t) - F(s',t')| < 1$ and hence $\varphi((s,t),(s',t')) = \alpha\left(F(s,t)/F(s',t')\right)$ is defined.

Choose $N > \frac{1}{\varepsilon}$ and define, for $(s,t) \in I \times I$,

$$F'(s,t) = \varphi\left((s,t), \tfrac{N-1}{N}(s,t)\right) + \cdots + \varphi\left(\tfrac{1}{N}(s,t), (0,0)\right).$$

Then $F' : I \times I \to \mathbb{R}$ is cts, $F'(0,0) = 0$ and

$$\exp(F'(s,t)) = \exp \circ \varphi\left((s,t), \tfrac{N-1}{N}(s,t)\right) \cdots \exp \circ \varphi\left(\tfrac{1}{N}(s,t),(0,0)\right)$$

$$= \exp \circ \alpha\left(F(s,t)/F\left(\tfrac{N-1}{N}(s,t)\right)\right) \cdots \exp \circ \alpha\left(F\left(\tfrac{1}{N}(s,t)\right)/F(0,0)\right)$$

$$= \frac{F(s,t)}{F\left(\frac{N-1}{N}(s,t)\right)} \cdot \cdots \cdot \frac{F\left(\frac{1}{N}(s,t)\right)}{F(0,0)} = F(s,t).$$

The uniqueness of F' follows again from the discreteness of $\mathbb{Z}$. //

Corollary The endpoint of σ', $\sigma'(1)$, depends only on the homotopy class of σ.

Proof If τ is homotopic to σ, then τ' is homotopic to σ' relative to the endpoints, and hence $\tau'(1) = \sigma'(1)$. //

Theorem $\pi(S^1, 1) \cong \mathbb{Z}$.

Proof. Define $\chi : \pi(S^1,1) \to \mathbb{Z}$ by $\chi([\sigma]) = \sigma'(1)$. We have shown χ is well defined.

To show it is a homomorphism, consider $[\sigma], [\tau] \in \pi(S^1, 1)$,
and let $m = \sigma'(1)$, $n = \tau'(1)$.
Define $\tilde{\tau} : I \to \mathbb{R}$ by $\tilde{\tau}(s) = \tau'(s) + m$.
Then $\sigma' \cdot \tilde{\tau}$ is a path in $\mathbb{R}$ from 0 to $m+n$,
and $\exp \circ (\sigma' \cdot \tilde{\tau}) = \sigma \cdot \tau$. Hence $[\sigma' \cdot \tilde{\tau}] = [(\sigma \cdot \tau)']$
and $\chi([\sigma][\tau]) = \chi[\sigma \cdot \tau] = (\sigma \cdot \tau)'(1) = \sigma' \cdot \tilde{\tau}(1) = m+n$.
So $\chi([\sigma][\tau]) = \chi([\sigma])\,\chi([\tau])$.

To show χ is surjective, we define a right inverse:
For $n \in \mathbb{Z}$, define $\sigma_n' : I \to \mathbb{R}$ by $\sigma_n'(s) = ns$, and let
$\sigma_n = \exp \circ \sigma_n'$. Then $\chi([\sigma_n]) = n$.
To show χ is injective, we show $\ker \chi$ is trivial:
If $\chi([\sigma]) = 0$, then σ' is a closed path in $\mathbb{R}$,
and since $\mathbb{R}$ is convex, $\sigma' \sim 0$. But then $\sigma \sim \exp \circ 0$,
so σ is homotopic to the constant path 1.
This concludes the proof: $\chi : \pi(S^1, 1) \to \mathbb{Z}$ is an isomorphism. //

4. If X, Y are connected top. spaces, $x_0 \in X$, $y_0 \in Y$, then
$$\pi(X \times Y, (x_0, y_0)) \cong \pi(X, x_0) \oplus \pi(Y, y_0).$$

<u>Proof</u> let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections, and $\sigma : I \to X \times Y$ be a path in $X \times Y$,
$\sigma(0) = \sigma(1) = (x_0, y_0)$.
Define $f : \pi(X \times Y, (x_0, y_0)) \longrightarrow \pi(X, x_0) \oplus \pi(Y, y_0)$
by $f([\sigma]) = (p_{1*}([\sigma]), p_{2*}([\sigma]))$.
Show that f is a homomorphism.

Define f^{-1}: if $[\tau] \in \pi(X, x_0)$, $[\rho] \in \pi(Y, y_0)$, define $(\tau, \rho) : I \longrightarrow X \times Y$ by $(\tau, \rho)(s) = (\tau(s), \rho(s))$. Check that $([\tau], [\rho]) \longmapsto [(\tau, \rho)]$ is well defined and is the inverse of f.

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<u>Corollary.</u> $\pi(S^1 \times S^1, (1,1)) \cong \mathbb{Z} \oplus \mathbb{Z}$. σπείρα

$\pi(S^1 \times I, (1,0)) \cong \mathbb{Z}$ κύλινδρος