Free product of groups.

Let G and H be two groups, with identity elements 1_G and 1_H.
We consider the set of all finite sequences $g_1 h_1 g_2 h_2 \cdots g_k h_k$, for $k \in \mathbb{N}$, $g_i \in G$, $h_i \in H$ and $g_i \neq 1_G$ for $i > 1$, $h_i \neq 1_H$ for $i < k$.

We define a multiplication on this set as follows:
Let $\alpha = g_1 h_1 \cdots g_k h_k$, $\beta = p_1 q_1 \cdots p_\ell q_\ell$. To obtain the product $\alpha\beta$ we write $g_1 h_1 \cdots g_k h_k p_1 q_1 \cdots p_\ell q_\ell$ and then carry out any cancellation possible:

i. We delete any 1_G or 1_H which appear in the interior of the word

ii. If two elements of the same group appear side by side, we replace them by their product in the group.

iii. We repeat i and ii until we have brought $\alpha\beta$ to canonical form.

We can show that this product is well defined, associative and has an inverse. So we have a group structure, with identity element $1_G 1_H$.

This group is the <u>free product</u> of G and H, denoted $G * H$.

The free product of n copies of $\mathbb{Z}$ is the <u>free group</u> on n generators, denoted $F_n = \mathop{\ast}\limits_1^n \mathbb{Z}$, $n \geq 1$.

For more information on the free product and on free groups, look at Massey, Ch III.

Seifert – van Kampen Theorem.

The Seifert - van Kampen Theorem gives a way to calculate the fundamental group of a topological space in terms of the fundamental groups of certain open subspaces. We shall only consider two special cases of the Theorem. For the general theorem and detailed proofs look at Massey, Ch IV.

Theorem Let X be a space, U and V open subsets of X such that $X = U \cup V$, and X, U, V, $U \cap V$ are path connected. Let $x_0 \in U \cap V$.

1) If $\pi(U \cap V, x_0) = 1$, then

$$\pi(X, x_0) \cong \pi(U, x_0) * \pi(V, x_0)$$

2) If $\pi(V, x_0) = 1$, then

$$\pi(X, x_0) \cong \pi(U, x_0) / N$$

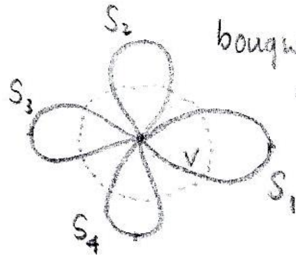
where N is the smallest normal subgroup of $\pi(U, x_0)$ containing the image of the homomorphism

$$\iota_* : \pi(U \cap V, x_0) \longrightarrow \pi(U, x_0)$$

induced by the inclusion $\iota : U \cap V \hookrightarrow U$.

//.

<u>The bouquet of n circles.</u>



bouquet of 4 circles

Let $S_i = S^1 \times \{i\}$, for $i = 1, \dots, n$.

On $\bigcup_1^n S_i$ define the equivalence relation generated by $(1,i) \sim (1,j)$ for $i,j = 1, \dots, n$.

The space $X_n = \bigcup S_i /_{\sim}$ is the <u>bouquet</u> of <u>n circles</u>.

Let $V = \{ [z,i] \in X_n : |z-1| < 1 \}$, $U_i = S_i \cup V$ for $i = 1, \dots, n$, and $x_0 = [1,i]$. Then

a) $\pi(V, x_0) = 1$ and b) $\pi(U_i, x_0) \cong \pi(S_i, x_0)$.

a) Consider the mapping $F: V \times I \to V$ with
$F([e^{2\pi i s}, j], t) = [e^{2\pi i s t}, j]$. Then F is cts.

$F([z,j], 1) = [z,j]$ and $F([z,j], 0) = x_0$.

Hence $F(\sigma(s), t)$ gives a homotopy from the constant path x_0 to any path σ on V.

b) Consider the mapping $G: U_j \times I \to U_j$ with

$$G([z,k], t) = \begin{cases} [z,j] & \text{if } k = j \\ F([z,k], t) & \text{if } k \neq j \end{cases}$$

G is continuous, and if σ is a path in U_j, then $s \longmapsto F(\sigma(s), 1)$ is a path in U_j homotopic to σ, which is contained entirely in S_j. This defines an isomorphism $\pi(U_j, x_0) \cong \pi(S_j, x_0)$.

Now we prove by induction on n that $\pi(X_n, x_0)$ is a free group on n generators.

We have $X_1 = S^1$ and $\pi(S^1, 1) \cong \mathbb{Z}$.

In X_n, consider $A = \bigcup_1^{n-1} U_i$ and $B = U_n$. Then A and B are open in X_n, $A \cup B = X_n$ and $A \cap B = V$. Since $\pi(V, x_0) = 1$, by the S-vK Theorem we have

$$\pi(X_n, x_0) = \pi(A, x_0) * \pi(B, x_0).$$

But $\pi(B, x_0) \cong \pi(S^1, 1) \cong \mathbb{Z}$, and by a similar argument, any closed path in A based at x_0 is homotopic to a closed path in X_{n-1}. Hence

$$\pi(A, x_0) \cong \pi(X_{n-1}, x_0) \cong \mathop{\ast}_{1}^{n-1} \mathbb{Z}.$$

By induction, $\pi(X_n, x_0) \cong \mathop{\ast}_{1}^{n} \mathbb{Z}$.

<u>Theorem.</u> Let A be a subspace of the topological space X, and assume that there is a continuous mapping $F: X \times I \longrightarrow X$ such that

1) $F(x,0) = x$ for all $x \in X$.

2) $F(x,1) \in A$ for all $x \in X$

3) $F(a,t) = a$ for all $a \in A$ and all $t \in I$.

Then the inclusion $\iota : A \longrightarrow X$ induces an isomorphism $\iota_* : \pi(A,a) \longrightarrow \pi(X,a)$, for any $a \in A$.

<u>Proof</u> Let $r: X \to A$ be the mapping $r(x) = F(x,1)$. Then $r \circ \iota = id_A$, and hence $r_* \circ \iota_*$ is the identity homomorphism on $\pi(A,a)$. This implies that ι_* is injective and r_* is surjective.

To complete the proof we must show that $\iota_* \circ r_*$ is the identity homomorphism on $\pi(X,a)$. That is, we must show that, if $\sigma : I \to X$ is a path in X, with $\sigma(0)=\sigma(1)=a$, then $\sigma \sim \iota \circ r \circ \sigma$. Let the continuous mapping $H: I \times I \longrightarrow X$ be defined by $H(s,t) = F(\sigma(s),t)$.

Then
$$H(s,0) = F(\sigma(s),0) = \sigma(s)$$
$$H(s,1) = r(\sigma(s)) = \iota \circ r \circ \sigma(s)$$

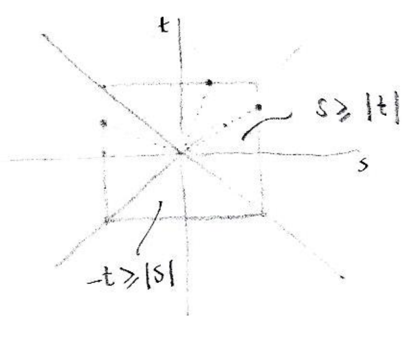
$H(0,t) = H(1,t) = F(a,t) = a$. Hence $H : \sigma \sim \iota \circ r \circ \sigma$.
We conclude that r_* is the inverse of ι_*, and hence that ι_* is an isomorphism. //

A mapping F satisfying the conditions of the theorem is called a (strong) deformation retraction, and the subspace A is a (strong) deformation retract of X.

Example 1. The boundary of a disc Δ is a strong deformation retract of the space $X = \Delta \setminus \{0\}$. Define $F : X \times I \to X$ by $(x,t) \mapsto (1-t)\,x + t\,\dfrac{x}{|x|}$, for $x \in \mathbb{R}^2$, $0 < |x| \leq 1$.
Check that the conditions of the theorem are satisfied. In particular, if $|x| = 1$, then $F(x,t) = x$ for all t.
We conclude that the homomorphism induced by the inclusion $\iota : S^1 \to \Delta \setminus \{0\}$ is an isomorphism

$$\pi(S^1, 1) \cong \pi(\Delta \setminus \{0\}, 1).$$

Example 2. Consider the square $X = [-1,1] \times [-1,1]$, and $Y = X \setminus \{(0,0)\}$. We define $r : Y \to Y$, on the subset $\{(s,t) \in Y : s \geq |t|\}$ by $r(s,t) = (1, \frac{t}{s})$,
and similarly on the subsets where $-s \geq |t|$, $t \geq |s|$ and $-t \geq |s|$.



Then r maps the square minus a point, to the boundary of the square, by sending each ray from 0 to its point on the boundary. We define $\tilde{F} : Y \times I \longrightarrow Y$ by

$$\tilde{F}((s,t),u) = (1-u)(s,t) + u\, r(s,t).$$

Check that $\tilde{F}$ is a deformation retraction of Y to $r(Y)$.

Example 3. The mapping $p(s,t) = (e^{\pi i s}, e^{\pi i t})$ maps $X = [-1,1] \times [-1,1]$ onto the torus $T = S^1 \times S^1$, and identifies the sides of the square according to the symbol $aba^{-1}b^{-1}$.

We let $T' = p(Y) = (S^1 \times S^1) \setminus \{(1,1)\}$, and define $F : T' \times I \longrightarrow T'$ by

$$F(p(s,t),u) = p(\tilde{F}((s,t),u)).$$

$$\begin{array}{ccc} Y \times I & \xrightarrow{\tilde{F}} & Y \\ {\scriptstyle p \times id_I} \downarrow & & \downarrow {\scriptstyle p} \\ T' \times I & \xrightarrow[F]{} & T' \end{array}$$

$T' \times I$ has the quotient topology from $p \times id_I$, and F is cts, since $p \circ \tilde{F}$ is cts.

Check that F is a deformation retraction of T' onto A, the bouquet of two circles joined at one common $(-1,-1)$,

$$A = (S^1 \times \{-1\}) \cup (\{-1\} \times S^1) \subset S^1 \times S^1.$$

We define the subset $\tilde{U} = \{ x \in X : |x| > 1/3 \}$, and the corresponding subset on the torus, $U = p(\tilde{U})$.

Then the restriction of F to $U \times I$ defines a deformation retraction $F : U \times I \longrightarrow U$, of U onto A. Then the inclusion $\iota : A \hookrightarrow U$ induces an isomorphism,

$$\iota_* : \pi(A, x_1) \longrightarrow \pi(U, x_1)$$

where $x_1 = p(-1,-1) = (e^{-\pi i}, e^{-\pi i}) = (-1,-1) \in S^1 \times S^1$.

Hence $\pi(U, x_1) \cong \mathbb{Z} * \mathbb{Z}$.