<u>Lemma</u> The composition of cts maps is cts.

<u>Proof.</u> If $f : X \to Y$, $g : Y \to Z$ are cts, and A is open in Z, then $g^{-1}(A)$ is open in Y, and $f^{-1}(g^{-1}(A))$ is open in X. But this is $(g \circ f)^{-1}(A)$. ⁄⁄.

Closed sets

X top. space. A subset $F \subseteq X$ is <u>closed</u> if $X \setminus F$ is open.

<u>Proposition</u>

The family of closed subsets of X has the properties:

1) X and $\emptyset$ are closed.
2) The union of a finite family of closed sets is closed.
3) The intersection of any family of closed sets is closed.

<u>Proof.</u> De Morgan rules.

<u>Exercise</u> $f: X \to Y$ is cts <u>iff</u> the inverse image of any closed subset of Y is closed in X.

Neighbourhoods

X top. space, $x \in X$. A subset $U \subseteq X$ is a <u>neighbourhood</u> of x if there is an open subset $A \subseteq X$ s.t. $x \in A \subseteq U$.

<u>Example</u> An open subset is a nbd of all its points.

<u>Defn</u> A function $f: X \to Y$ is <u>cts at the point $x \in X$</u> if for every nbd V of $f(x)$, there is a nbd U of x s.t. $f(U) \subseteq V$.

<u>Proposition</u> $f: X \to Y$ is continuous iff f is continuous at every point of X.

<u>Proof</u> $\Rightarrow$. If $x \in X$ and V is a nbd of $f(x)$, there is open B of Y s.t. $f(x) \in B \subseteq V$. Then $f^{-1}(B)$ is open in X. But $x \in f^{-1}(B)$, hence $U = f^{-1}(B)$ is a nbd of x s.t. $f(U) \subseteq V$.

$\Leftarrow$. Let B be open in Y. For each $x \in f^{-1}(B)$, B is a nbd of $f(x)$. Hence there is nbd U_x of x s.t. $f(U_x) \subseteq B$. Hence there is open $A_x \subset U_x$. But then
$$f^{-1}(B) = \bigcup_{x \in f^{-1}(B)} A_x,$$
which is open. //.

Relative topology

(X, τ) top. space, $Y \subseteq X$. Then

$$\tau_Y = \{ A \subseteq Y : \text{there is } B \in \tau \text{ s.t. } A = B \cap Y \}$$

is a topology on Y, the *relative* topology, or *subspace* topology.

Exercise Show that τ_Y satisfies the axioms for a topology.

If Y has the relative topology as a subset of X, the *inclusion* (*ένθεση*) $\iota : Y \to X : y \mapsto y$ is a cts mapping: if A is open in X, $\iota^{-1}(A) = A \cap Y$, which is open in Y.

If $f: X \to Z$ is cts, then the restriction of f to a subset $Y \subseteq X$, is a cts mapping w.r.t. the relative topology of Y: if A open in Z, $f^{-1}(A)$ is open in X and $f^{-1}|_Y(A) = f^{-1}(A) \cap Y$ is open in Y.

Homeomorphism

A mapping between two topological spaces $f: X \to Y$ is a homeomorphism if f is bijective and continuous, and the inverse function is also continuous.

If there exists a homeomorphism we say X and Y are homeomorphic, $X \cong Y$.

Examples 1. $f: (-1,1) \to \mathbb{R}$, $f(x) = \dfrac{x}{1+|x|}$ is a homeomorphism. So $B^1 \cong \mathbb{R}$.

Similarly, $B^n \cong \mathbb{R}^n$.

Another homeomorphism from $(-1,1)$ to $\mathbb{R}$ is $g(x) = \tan \frac{x\pi}{2}$.

2. $\mathbb{R}^2$ is homeomorphic to $\mathbb{C}$: a set A open in $\mathbb{C}$ if for every $w \in A$, there is $\varepsilon > 0$ s.t.
$\{ z \in \mathbb{C} : |z - w| < \varepsilon \} \subseteq A$.

3. Consider $[0,1)$ with the topology as a subspace of $\mathbb{R}$, and S^1 with the topology as a subspace of $\mathbb{R}^2$.
The mapping $f : [0,1) \to S^1 : t \mapsto (\cos t, \sin t)$ is continuous and bijective, but is not a homeomorphism: the inverse mapping $g : S^1 \to [0,1)$ is not continuous, since $[0, \frac{1}{2})$ is open in $[0,1)$ but $g^{-1}([0,\frac{1}{2}))$ is not open in S^1 : it is a semicircle with one of its ends, 

A homeomorphism gives a bijective correspondence between the open sets of X and the open sets of Y.
Hence any property that can be expressed in terms of the open sets (any topological property) holds in X iff it holds in Y.

From the point of view of topology, two homeomorphic spaces are the same.

One of the problems of topology is to decide if two top. spaces are homeomorphic. This cannot be done algorithmically for all top spaces.

But there are important categories of top spaces where this is possible.

Poincaré proved this for surfaces.

Recently it has been proven for manifolds of dimension 3 (Thurston ... Perelman). → (*) Note on Invariants.

<u>Defn</u> A mapping $f : Y \longrightarrow X$ is an <u>embedding</u> (εμφύτευση) if f is cts injection, and $f : Y \to f(Y)$ is a homeomorphism w.r.t. the relative topology of $f(Y)$.

<u>Examples</u>

1. An inclusion is an embedding.
2. The mapping $f : \mathbb{R} \to \mathbb{R}^2 : t \longmapsto (t, t^2)$ is an embedding.
3. $f : [0,1) \longrightarrow \mathbb{R}^2 : t \longmapsto (\cos t, \sin t)$ is not an embedding, since $\operatorname{im} f = S^1$ is not homeomorphic to $[0,1)$.

4. A <u>knot</u> is an embedding from the circle S^1 to $\mathbb{R}^3$.

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* <u>Invariants.</u>

If we can find a homeo between two spaces, then they are homeomorphic.

But how can we show that two spaces are <u>not</u> homeomorphic?

We do this using <u>invariants</u>: we define a mathematical object, e.g. a <u>number</u> and we show that it does not change by a homeomorphism. Then, if two spaces have different values for some invariant, there cannot exist a homeo between them.

Think of geometry. If we want to show that two triangles are not equal, we do not have to try all possible euclidean isometries to see if we can move the one to coincide with the other. It is enough to find a geometric invariant, e.g. the size of an angle, which is different in the two triangles.

In topology, to prove that two spaces are not homeomorphic, we define <u>topological invariants</u> which are numbers, groups, vector spaces or other mathematical objects.

<u>Theorem of Invariance of Domain</u>

$$\mathbb{R}^m \cong \mathbb{R}^n \quad \text{iff} \quad m = n.$$

It is not possible to prove directly that if $m \neq n$ then $\mathbb{R}^m \ncong \mathbb{R}^n$. The usual proof uses invariants defined on the sets

$$C^m = \mathbb{R}^m \setminus \{0\}, \qquad C^n = \mathbb{R}^n \setminus \{0\}$$

These invariants, called the <u>Betti numbers</u>, β_k measure how a space is made up by gluing "k-simplices", that is

0-simplex • point

1-simplex —— interval

2-simplex  triangle

3-simplex  tetrahedron

We show that these are topological invariants (this is the difficult part in this case) and calculate that

$$\beta_k(C^m) = \begin{cases} 1 & \text{if} \quad m = k+1 \\ 0 & \text{if} \quad m \neq k+1. \end{cases}$$

so that $\beta_{m-1}(C^m) \neq \beta_{m-1}(C^n)$ if $m \neq n$.

So, removing a point from $\mathbb{R}^m$ gives a different space than removing a point from $\mathbb{R}^n$.

Hence $\mathbb{R}^m \not\cong \mathbb{R}^n$!

Back to page 15, defn of embedding.