The fundamental group of a closed surface.

The sphere. $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$, $x_0 = (1,0,0) \in S^2$.

Let $\mathcal{U} = \{ (x,y,z) \in S^2 : z > -\frac{1}{2} \}$ and

$$V = \{ (x,y,z) \in S^2 : z < \tfrac{1}{2} \}.$$

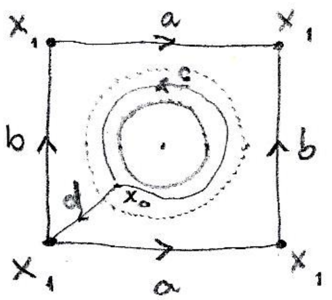
Then $\mathcal{U}$ and V are both homeomorphic to a disc, and hence

$$\pi(\mathcal{U}, x_0) \cong 1 \,, \quad \pi(V, x_0) \cong 1.$$

Hence by S-vK Theorem, $\pi(S^2, x_0) \cong \pi(\mathcal{U}, x_0)/N \cong 1$.

The torus. We know $T = S^1 \times S^1$ and hence $\pi(T, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}$.
We'll prove this result using the S-vK Theorem, as an introduction to the general method.

The torus is obtained from $X = [-1,1] \times [-1,1]$, by identifying the sides according to the symbol $a b a^{-1} b^{-1}$,



The identification $p(s,t) = (e^{\pi i s}, e^{\pi i t})$ maps the sides a and b to closed paths based at $x_1 = (-1,-1)$.

Let $\tilde{\mathcal{U}} = \{ x \in \mathbb{R}^2 : |x| > \frac{1}{3} \}$ and $\tilde{V} = \{ x \in \mathbb{R}^2 : |x| < \frac{2}{3} \}$,

and define $\mathcal{U} = p(\tilde{\mathcal{U}})$ and $V = p(\tilde{V})$.

Then $T = \mathcal{U} \cup V$ and $\mathcal{U} \cap V$ is homeomorphic to the annulus $\tilde{\mathcal{U}} \cap \tilde{V}$.

Let x_0 be a point in $\mathcal{U} \cap V$, and choose a path d from x_0 to x_1.

V is homeomorphic to a disc, hence $\pi(V, x_0) \cong 1$.

The annulus $U \cap V$ is homeomorphic to $S^1 \times I$, hence $\pi(U \cap V, x_0) \cong \mathbb{Z}$ with generator the homotopy class γ of the path c.

From the discussion on page 65, we have seen that $\pi(U, x_1) \cong \mathbb{Z} * \mathbb{Z}$, generated by the homotopy classes α, β of the paths $p \circ a$ and $p \circ b$. Let δ be the homotopy class of the path d from x_0 to x_1. Then there is an isomorphism $\delta_{\#}^{-1}: \pi(U, x_1) \to \pi(U, x_0)$, sending α and β to the homotopy classes $\bar{\alpha} = \delta \alpha \delta^{-1}$ and $\bar{\beta} = \delta \beta \delta^{-1}$. Hence $\pi(U, x_0) \cong \mathbb{Z} * \mathbb{Z}$, with generators $\bar{\alpha}$ and $\bar{\beta}$.

Inside U, the path $d a b a^{-1} b^{-1} d^{-1}$ is homotopic to the path c. So for $\iota: U \cap V \to U$, $\iota_*: \pi(U \cap V, x_0) \to \pi(U, x_0)$, we have $\iota_*([c]) = [d a b a^{-1} b^{-1} d^{-1}] = \delta \alpha \beta \alpha^{-1} \beta^{-1} \delta^{-1} = \bar{\alpha} \bar{\beta} \bar{\alpha}^{-1} \bar{\beta}^{-1}$.

By applying S-vK Theorem (2) we have

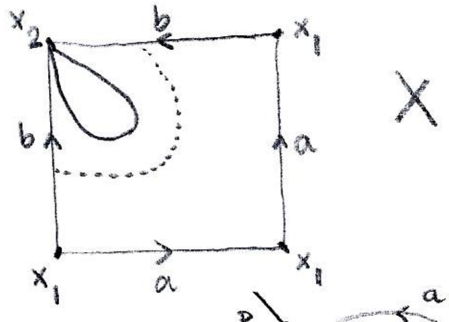
$$\pi(T, x_0) \cong \pi(U \cup V, x_0) \cong \pi(U, x_0)/N$$

where N is the smallest normal subgroup of $\pi(U, x_0)$ generated by $\bar{\alpha} \bar{\beta} \bar{\alpha}^{-1} \bar{\beta}^{-1}$.

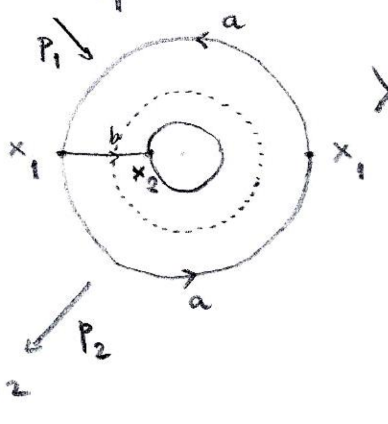
This is the commutator subgroups of $\mathbb{Z} * \mathbb{Z}$, and the quotient is the abelian group $\mathbb{Z} \oplus \mathbb{Z}$. //

The projective plane.

The projective plane can be obtained from $X = [-1,1] \times [-1,1]$ by identifying the sides according to the symbol $a\,a\,b\,b^{-1}$.



We do the identification in two steps, p_1 identifying the two sides labeled b, to give a disc X', and then p_2 identifying the sides labeled a, to obtain $\mathbb{R}P^2$,



$p_1 : X \longrightarrow X'$, $p_2 : X' \longrightarrow \mathbb{R}P^2$.

In X' consider the discs $\Delta_{1/3} = \{ |x| \leq 1/3 \}$ and $D_{2/3} = \{ |x| < \frac{2}{3} \}$, and set $\mathcal{U} = p_2(X' \setminus \Delta_{1/3})$ and $V = p_2(D_{2/3})$.

Then $\mathbb{R}P^2 = \mathcal{U} \cup V$ and $\mathcal{U} \cap V$ is homeomorphic to $S^1 \times I$.

The side a, under the identification, becomes a closed path based at a point $x_1 \in \mathcal{U}$. Choose a point $x_0 \in \mathcal{U} \cap V$ and a path d from x_0 to x_1.

We consider the disc X' as the unit disc in $\mathbb{C}$, and define the cts mapping

$$\tilde{F} : (X' \setminus \Delta_{1/3}) \times I \longrightarrow (X' \setminus \Delta_{1/3})$$

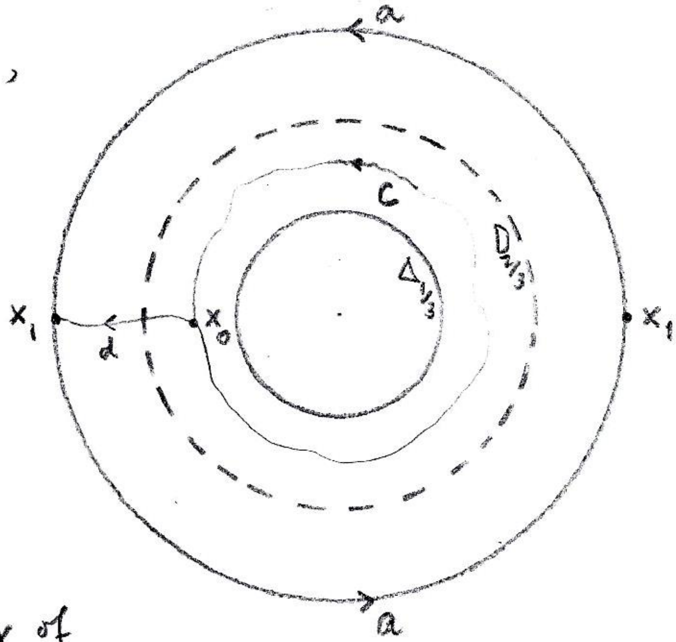
by

$$\tilde{F}(z, u) = (1-u)z + u\frac{z}{|z|}.$$

If $|z|=1$, $\tilde{F}(z,u) = z$ for every $u \in [0,1]$. We can define $F : \mathcal{U} \times I \to \mathcal{U}$, by

$$F(p_2(z), u) = p_2(\tilde{F}(z,u))$$

F is continuous, $F(x,0) = x$ for every $x \in \mathcal{U}$, and if $|z|=1$, $F(p_2(z), u) = p_2(z)$ for all $u \in [0,1]$. Hence F is a deformation retraction onto the image of S^1 under the identification p_2, which is again homeomorphic to S^1. Hence $\pi(\mathcal{U}, x_1) \cong \mathbb{Z}$. Then $\pi(\mathcal{U}, x_0)$ is also $\cong \mathbb{Z}$,



with generator the class α of the path $p_2 \circ (d \cdot a \cdot d^{-1})$.

The group $\pi(\mathcal{U} \cap V, x_0)$ is also isomorphic to $\mathbb{Z}$, with generator the class γ of the path $p_2 \circ c$. Inside $\mathcal{U}$, the path $p_2 \circ c$ is homotopic to the path $p_2 \circ (d\, a\, a\, d^{-1})$, that goes twice round α. Hence, for $i : \mathcal{U} \cap V \to \mathcal{U}$, the homomorphism $i_* : \pi(\mathcal{U} \cap V, x_0) \to \pi(\mathcal{U}, x_0)$ maps γ to α^2.

Since $\pi(V, x_0) = 1$, by S-vK Theorem 2,

Since $\pi(V, x_0) = 1$, by S-vK Theorem, 2,

$$\pi(\mathbb{R}P^2, x_0) \cong \pi(U, x_0) / N$$

where $\pi(U, x_0) \cong \mathbb{Z}$ with generator α and N is the (normal) subgroup generated by α^2. In other words,

$$\pi(\mathbb{R}P^2, x_0) \cong \mathbb{Z}_2$$

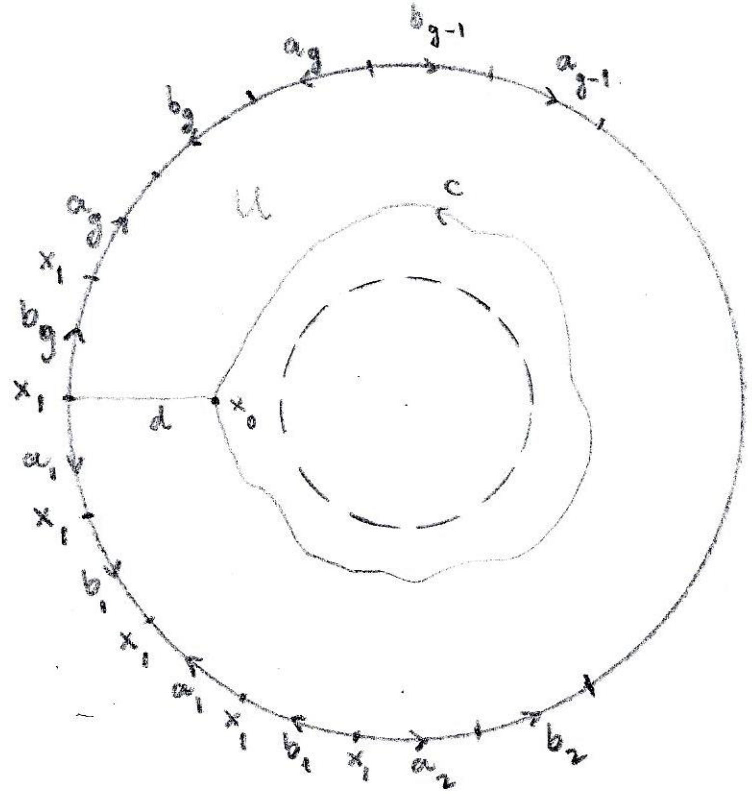
with generator $\bar{\alpha}$ s.t. $\bar{\alpha}^2 = 1$. //.

Similarly we show that the fundamental group of the connected sum of g tori is the quotient of the free group with $2g$ generators, α_i, β_i, $i = 1, \dots, g$, by the normal subgroup generated by the element

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}.$$

The fundamental group of the connected sum of h projective planes is the quotient of the free group with h generators $\alpha_1, \dots, \alpha_h$, by the normal subgroup generated by the element

$$\alpha_1^2 \alpha_2^2 \cdots \alpha_h^2.$$


a_g
b_{g-1}
a_{g-1}
b_g
U
c
a_g
x_1
b_g
x_1
d
x_0
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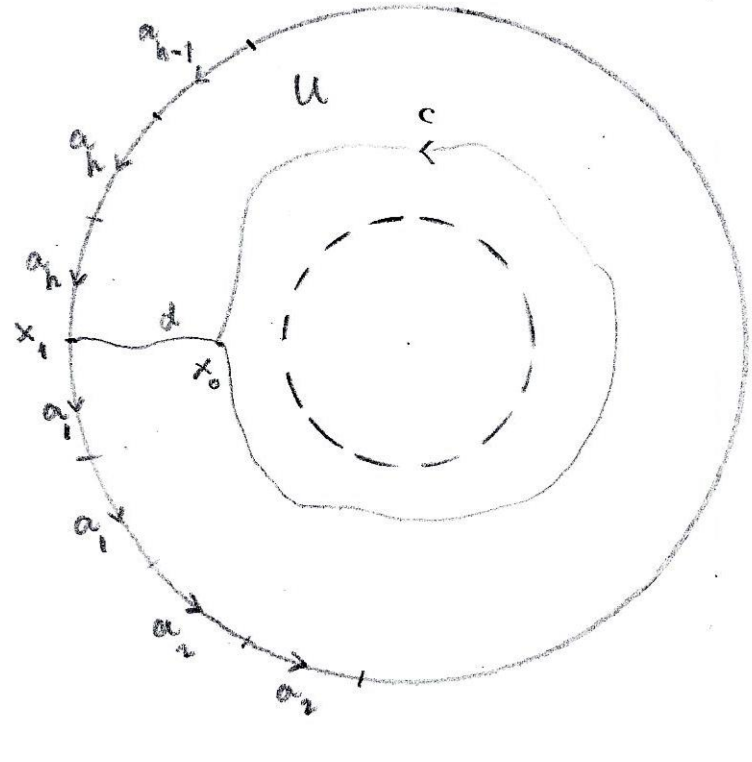
a_{h-1}
U
c
a_h
a_h
d
x_1
x_0
a_1
a_1
a_2
a_2

From this description it is not transparent that the fundamental group of each surface type M_g, $g = 0, 1, \ldots$ and N_h, $h = 1, 2, \ldots$ is different from that of any other surface (and hence that the surface types are not homeomorphic).

One way to show this is to calculate the abelianization of each group, that is the quotient of the group G its commutator subgroup $[G\ G]$. It is then easy to use the classification of abelian groups to show that the abelianizations of the groups corresponding to different surface types are all different, and hence the surface types are not homeomorphic.

This concludes the proof of the classification theorem.