# Diagrams of Knots.

A <u>knot</u> is an embedding $S^1 \longrightarrow \mathbb{R}^3$.

A <u>link</u> (σύνδεσμος) is an embedding from the

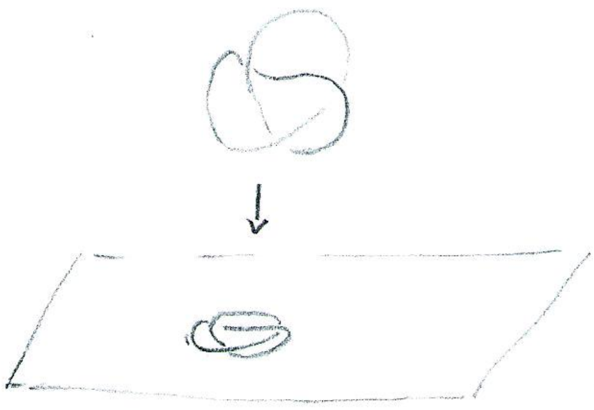
$$S_1 \sqcup S_1 \cdots \sqcup S_1 \longrightarrow \mathbb{R}^3.$$

$$\bigcirc\ \bigcirc \cdots \bigcirc \longrightarrow \mathbb{R}^3$$

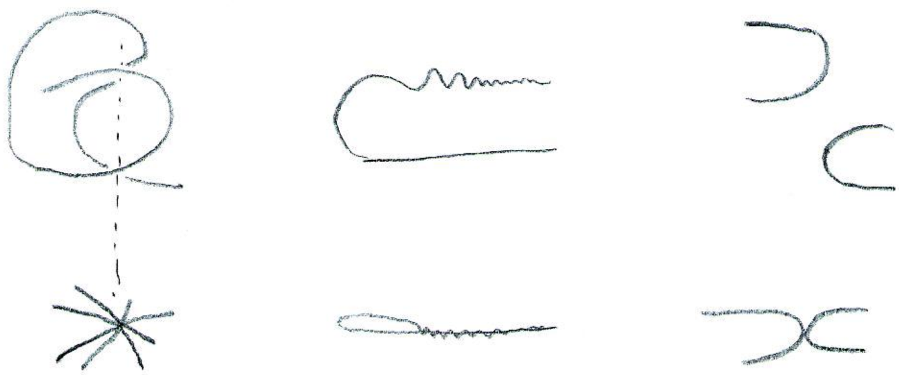
disjoint union of $k$ circles to $\mathbb{R}^3$.

Usually we are not interested in the parametrisation, so we think of a knot or a link as the subset of $\mathbb{R}^3$ which is homeomorphic to $S^1$ or to $\bigsqcup S^1$.

We begin the study of knots and links by looking at their <u>projections</u> on the plane.



We consider projections "in general position", that is we assume that any point on the projection corresponds to one or two points of the knot, that there are only finitely many double points, and that at any double point the strings cross transversely.



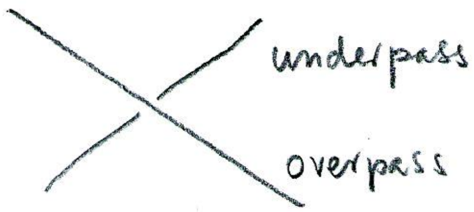
multiple points    accumulating double points    non transverse double points.

A projection satisfying these restrictions is described by a <u>link diagram</u> or a knot diagram.

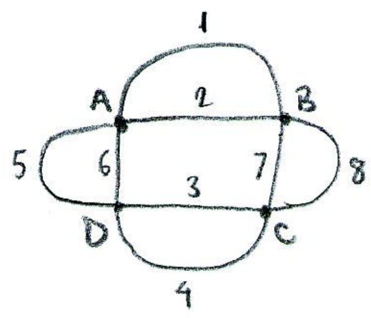
A <u>link diagram</u> is a graph on the plane consisting of $n$ vertices (κορυφές) and $2n$ edges (ακμές) so that at each vertex meet exactly 4 ends of edges, with the following extra structure:

at each edge, one pair of opposite edges is the <u>overpass</u> (πάνω πέρασμα) and the other pair of opposite edges is the <u>underpass</u> (κάτω πέρασμα). Each edge with this extra structure is called a <u>crossing</u> (διασταύρωση).

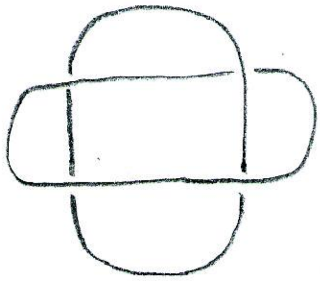
We represent an overpass by joining the two edges, and an underpass by not joining the edges.




graph



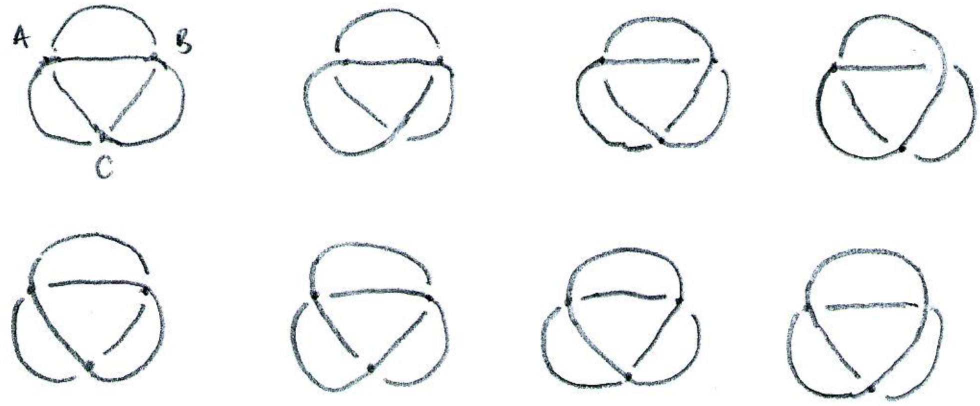

link diagram



At vertex A, the edges 2 and 5 form an overpass, the edges 1 and 6 an underpass.

From a graph with $n$ vertices we obtain $2^n$ different diagrams.

Diagrams with 3 crossings.




Which of them are knotted?

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Two edges of the graph belong to the same <u>arc</u> (τόξο) if their ends form an overpass at a crossing.

This relation extends to an equivalence relation on the set of edges, and divides the edges into <u>arcs</u>.

In the example, the arcs are:

$$\{1,7\},\ \{2,5,3,8\},\ \{4\},\ \{6\}.$$

Two edges belong to the same <u>component</u> (συνιστώσα) if their ends form an overpass or an underpass.

This extends to an equivalence relation on the edges and on the arcs of the diagram, and divides the diagram into <u>components</u>.

In the example, the components are

$$\{2,5,3,8\},\ \{1,7,4,6\}.$$

# Equivalence relation generated by a relation R.

Let $\sim$ be a relation on a set $A$ (that is $\sim \subseteq A \times A$)

We define an equivalence relation $\sim$ on $A$ as follows:

- First we make $\sim$ reflexive, by setting $a \sim a$ for all $a \in A$, and symmetric by setting $a \sim b$ if $b \sim a$.
- If there is a sequence $a_1, a_2, \dots, a_k$ s.t.

  $a_i \sim a_{i+1}$ for $i = 1, \dots, k-1$, then $a_1 \sim a_k$.

Prove that $\sim$ is an equivalence relation (reflexive, symmetric, transitive).

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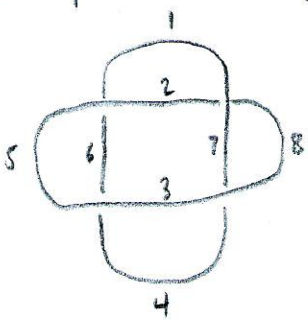
We start with the relation ("two edges belong to the same arc" if their ends form an overpass at a crossing.

So for




we have the relation:

$1 \sim 7$, $7 \sim 1$, $2 \sim 5$, $5 \sim 2$, $5 \sim 3$, $3 \sim 5$, $3 \sim 8$, $8 \sim 3$.

This is symmetric. To make it reflexive we add $1 \sim 1$, $2 \sim 2$, $3 \sim 3$, $4 \sim 4$, $5 \sim 5$, $6 \sim 6$, $7 \sim 7$, $8 \sim 8$.

Now we extend it to an equivalence relation:

$2 \sim 5$ and $5 \sim 3$, hence $2 \sim 3$

$5 \sim 3$ and $3 \sim 8$, hence $5 \sim 8$

Similarly $2 \sim 8$.

In the end, we have equivalence classes $\{2, 5, 3, 8\}$, $\{1, 7\}$, $\{6\}$, $\{4\}$.

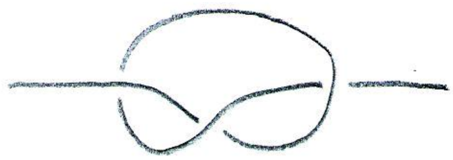
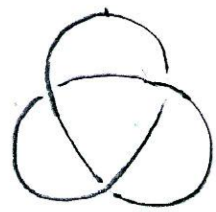
A link diagram with one component is a <u>knot diagram</u>

<u>Examples</u>


un-knot


trefoil

are they the same?

trefoil

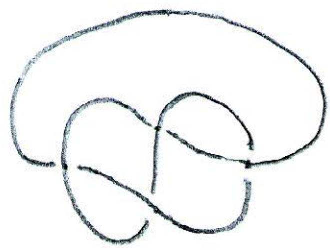
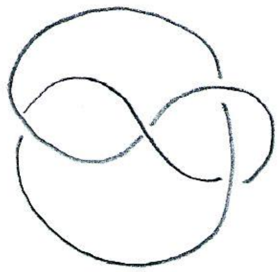
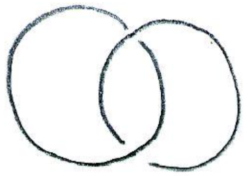
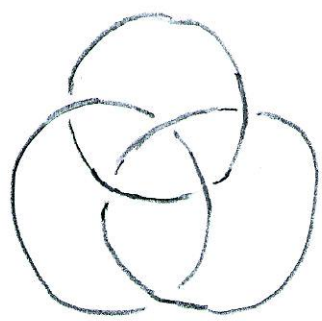



figure 8





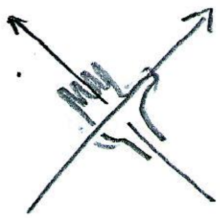
Hopf



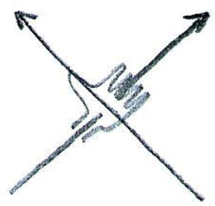
Borromean

On each component we can choose one of the two possible orientations, and then we have an <u>oriented</u> diagram.

On an oriented diagram we distinguish two types of crossings, positive and negative.



positive
(right handed)



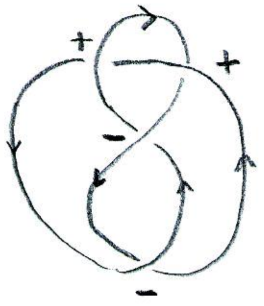
negative
(left handed)

If we change the orientation of all components, the signs of the crossings do not change.

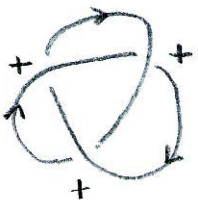
positive

negative.

The sum of the signs of all crossings in a diagram (verwringung) is the <u>writhe</u> (περιέλιξη) of the diagram.
For a knot it does not depend on the orientation.
For a link it depends on the orientation of each component.



$w = 0.$

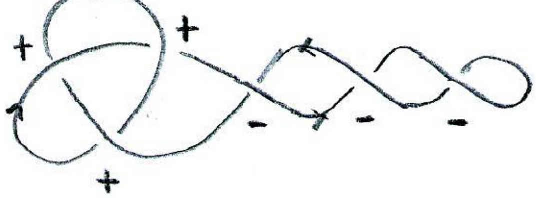


$w = 3$

$w = -3$

Does this show that $T^+$ and $T^-$ are different?

No, writhe is not an invariant of the <u>knot</u>.



$w = 0.$

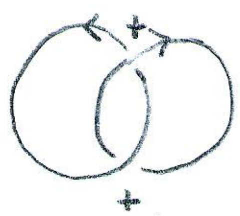
For a link diagram with $m$ components $C_1, \dots, C_m$ we define the <u>linking number</u> (αριθμός σύνδεσης) $l_{ij}$ of component $C_i$ with component $C_j$, for $i \neq j$:

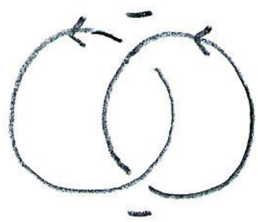
$$l_{ij} = \frac{1}{2} \left( \text{sum of signs of crossings between } C_i \text{ and } C_j \right)$$

The <u>linking number</u> of the diagram is

$$l(D) = \sum_{1 \leq i < j \leq m} l_{ij} .$$

Writhe and linking number of Hopf link depends on orientation.

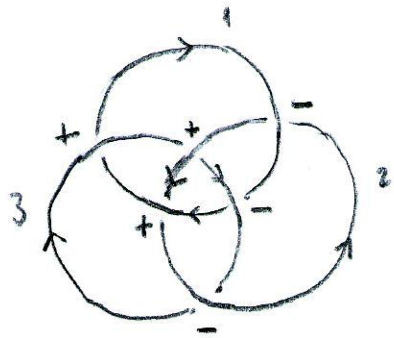




$w = 2$ $\qquad\qquad\qquad$ $w = -2$

$l = \frac{1}{2}(1+1) = 1$ $\qquad\qquad$ $l = \frac{1}{2}(-1-1) = -1.$

For the Borromean link, they do not depend on orientation of the components.



$w = 0$

$$l = l_{12} + l_{13} + l_{23}$$
$$= \frac{1}{2}(1-1) + \frac{1}{2}(1-1) + \frac{1}{2}(1-1) = 0.$$