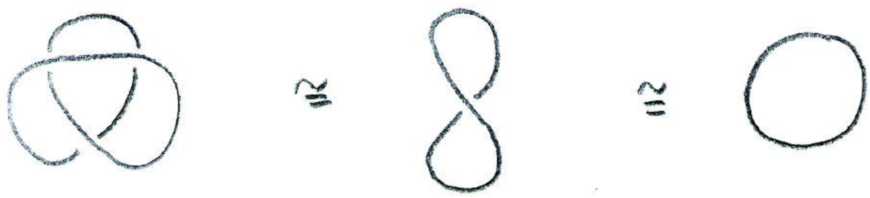
Reidemeister moves

It is obvious that different diagrams may represent the same knot or link. For the moment we think of "the same knot" intuitively; two knots are the same if I can deform the one into the other by pulling on the string.



We'll define an equivalence relation on diagrams so that equivalent diagrams obviously represent the same knot.

Later we'll give a mathematical defn of equivalence between knots and links. It is a deep theorem of Reidemeister, that equivalent knots or links have equivalent diagrams.

We start by defining 4 different <u>moves</u> we can make on a diagram.

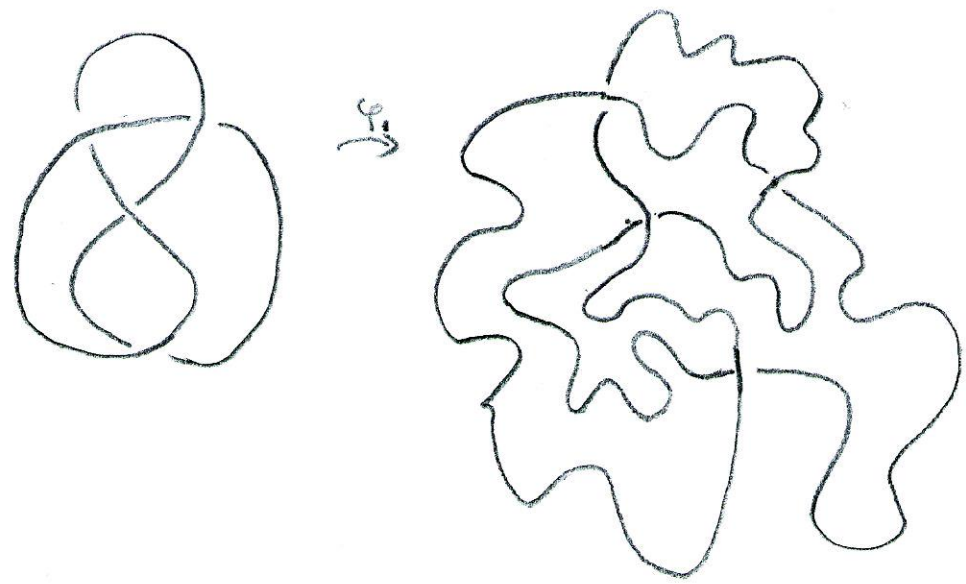
<u>RO</u>. Two diagrams D, F are related by a move of type RO if there is a <u>continuous deformation of the plane</u> taking the one diagram to the other.

This is a cts mapping $\varphi : \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2$ s.t.

a). For every $t \in [0,1]$, $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto \varphi(x,y,t)$ is a homeomorphism.

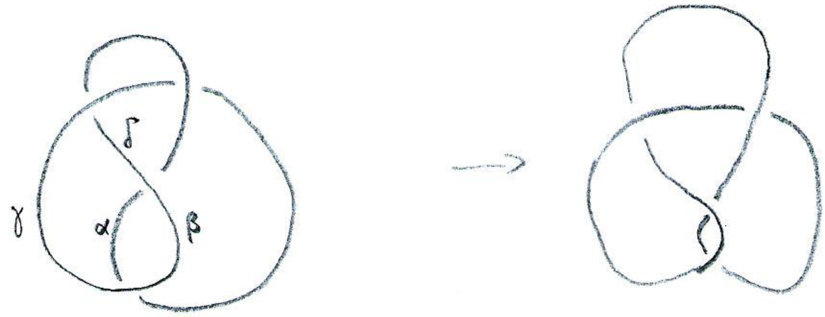
b) For every $(x,y) \in \mathbb{R}^2$, $\varphi(x,y,0) = (x,y)$

c) For every $(x,y) \in D$, $\varphi(x,y,1) \in F$.



Such a deformation is also called an <u>ambient isotopy</u>.

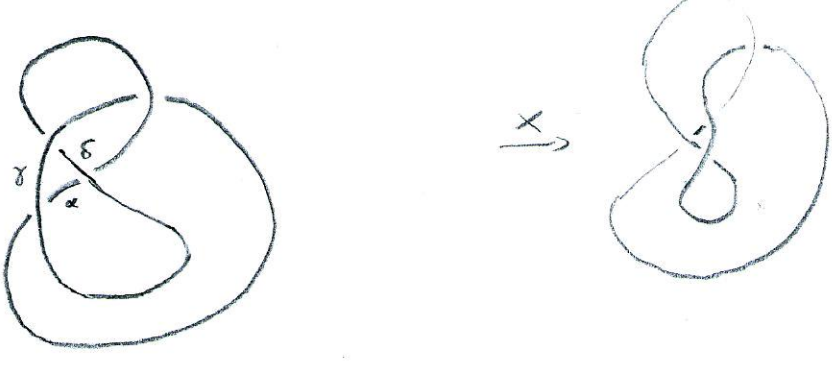
Let us look at what we can do with an ambient isotopy and what we cannot do.



we can shorten the lengths of edges α and β.

but we cannot make them disappear.

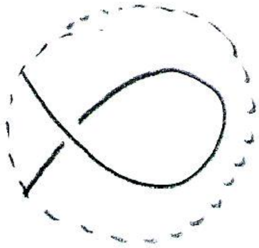
Or we can shorten the size of triangle α, γ, δ but cannot move it to the other side of the crossing.

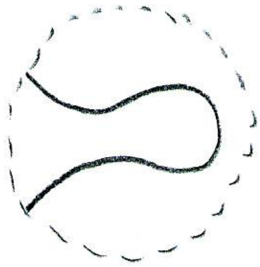


We can shorten a twist, like ε, but we

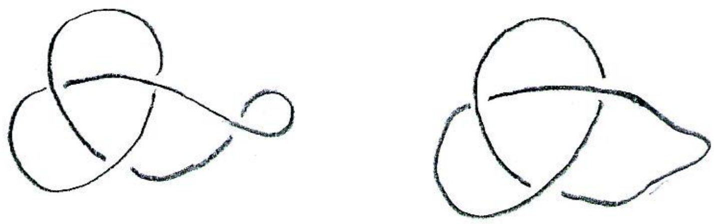
cannot make it disappear. However, when we manipulate a knot diagram to get a different diagram of the same knot we need to be able to do such moves.

R1 Two diagrams D and F are related by a move of type R1 if they are the same in the whole plane, except in the interior of a disc, where D contains α and F contains β.

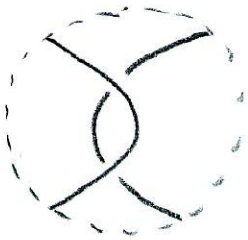
α 

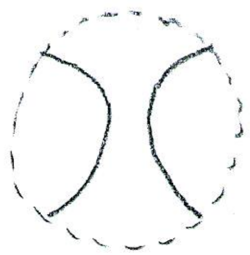
 β

Ex

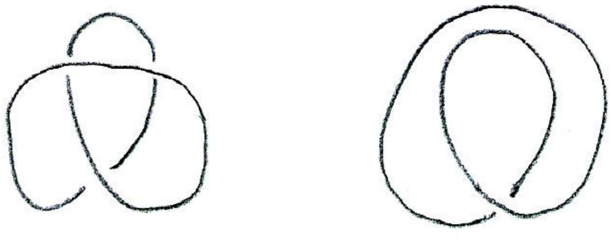


R2. Two diagrams D and F are related by a move of type R2 if they are the same in the whole plane, except in the interior of a disc, where D contains α and F contains β.

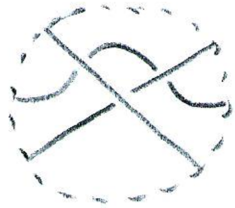
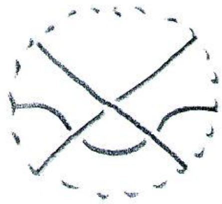
α 

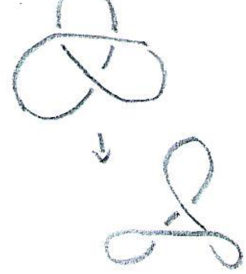


Ex



R3 Two diagrams D and F ...
a move of type R3 if ...

α   β

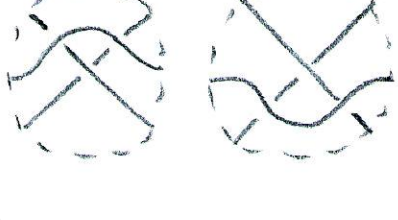
Ex 

Defn Two knot or link diagrams are isotopic if there is a sequence of link diagrams $D_1, \ldots, D_k$ such that $D = D_1$, $F = D_k$ and for each $i = 1, \ldots, k-1$ the diagram D_i and D_{i+1} are related by R0 or a move of type R1, R2, R3.

It is clear that isotopy is an equivalence relation on the set of all link diagrams.

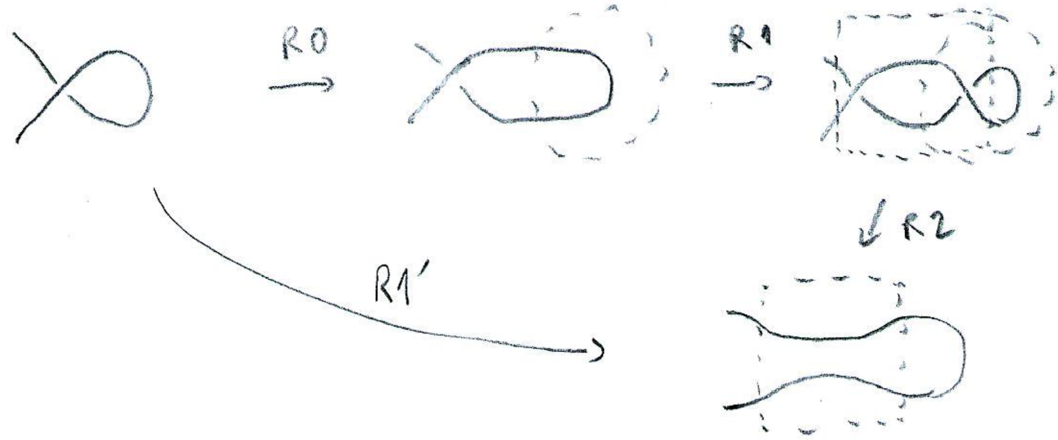
It seems that we have left out some obvious moves, like

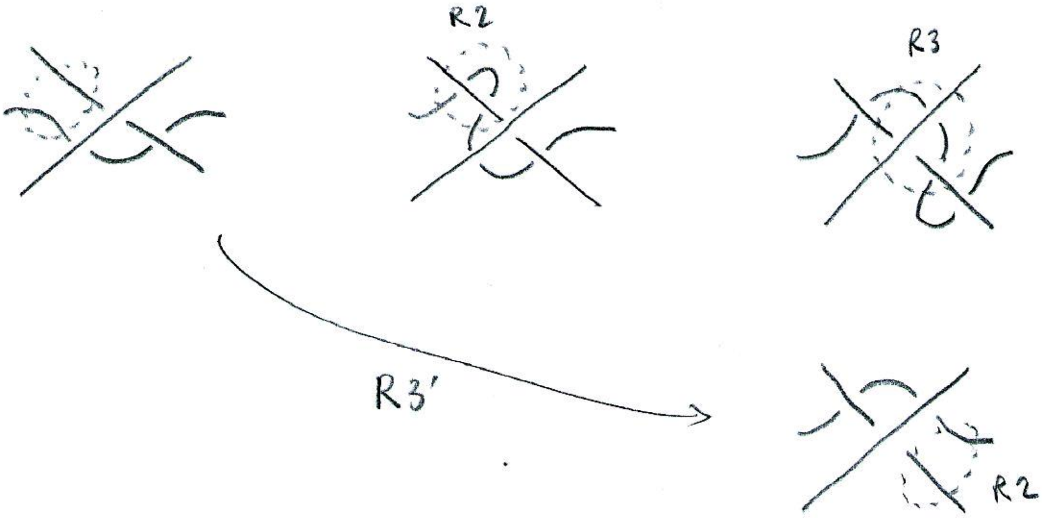
R1′ 

R3′  , R3″ 

R3‴ 

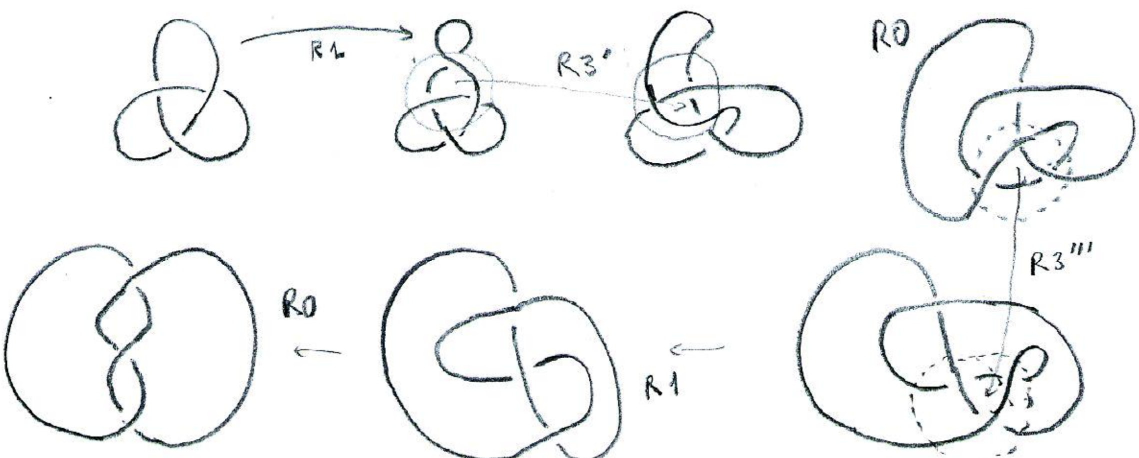
But we can find a sequence of moves R1, R2, R3 that give each of those. For example:





<u>Exercise</u> Decompose R3" into the 3 moves.

Using these moves we can see that two different projections of the trefoil knot are isotopic:



Of course it would be simpler to see that these diagrams represent the same knot without using Reidemeister moves.

Why are Reidemeister moves important?

Because they allow us to find invariants!

Since any isotopy can be decomposed into Reidemeister moves, if we can show that a property of a link diagram is not changed by the 3 Reidemeister moves, then it will be an invariant of the isotopy class of the diagram.

Two diagrams that differ in this invariant will not be isotopic.

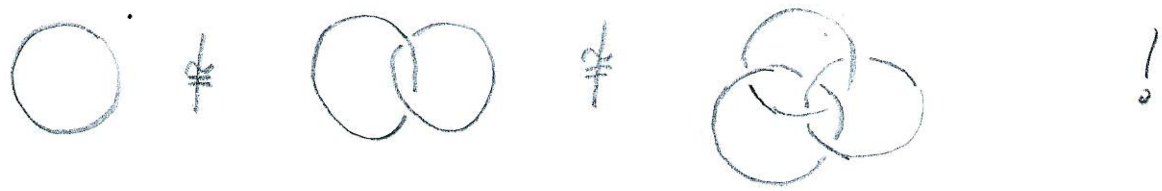
<u>Example</u> The number of components of a diagram, does not change by R0, R1.

In R2, if the two arcs belong to the same component they remain in the same component.

In R3, the arcs can belong to 1, 2 or 3 components, but this again does not change.

We conclude that number of components is an <u>isotopy invariant</u> of a link diagram.

In particular, the diagrams of the Hopf link, with 2 components, cannot be isotopic to the diagram of the Borromean link, with 3 components.



We want to look for more invariants, to be able to distinguish more link or knot diagrams.

If two link diagrams D and F are oriented, an <u>orientation preserving isotopy</u> is an isotopy such that the orientation of each component of D is the same as the orientation of the corresponding component of F.

We can look for invariants of oriented diagrams under orientation preserving isotopies

<u>Ex</u> The writhe is not an invariant: movements of type R1 change it.