<u>Proposition</u> The linking number is an invariant of an oriented link.

<u>Proof</u>. We look at the linking numbers l_{ij} between two components.

R0 does not change the signs of the crossings.

R1 affects only one component, so it does not change the crossings between two different components.

R2

R2 moves create or eliminate two crossings of opposite sign.

So l_{ij} remains unchanged.

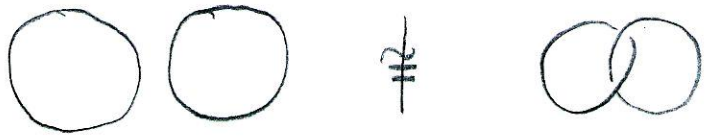
R3 moves do not change the number or the signs of the crossings, hence they do not change l_{ij}.

- Check for different orientations.

//.

We have seen that the Hopf link has linking number +1 or −1, depending on the orientation, $|\ell(H)| = 1$.

Since the unlink with two components, $\bigcirc\bigcirc$, has linking number 0, we conclude that

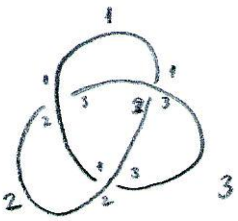


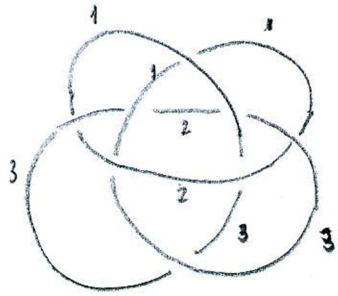
A simple invariant that can be used to study knot diagrams is <u>3-colourability</u>.

A knot diagram is <u>3-colourable</u> if we can assign colours to its arcs, so that:

1) Each arc is assigned one colour.
2) Exactly 3 colours are used
3) At each crossing, either all arcs have the same colour, or arcs of all three colours meet.

Ex

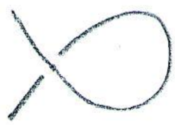




Theorem 3-colourability is an isotopy invariant.

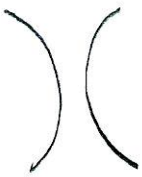
- R0 does not change any crossing. Hence does not affect 3-c.

- R1.

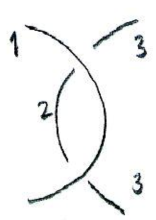


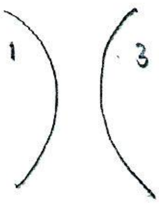
necessarily 1 colour

R2.



one colour





three colours.

Question: Is there a third colour on the right?
Since it is a diagram with one component, the arcs coloured 1 and 3 must meet at some crossing, and there will be a 3rd colour.

R3. We only need to check R3, since R3′, R3″, R3‴ can be decomposed into the other moves.

The colours of the top 3 arcs determine the others.

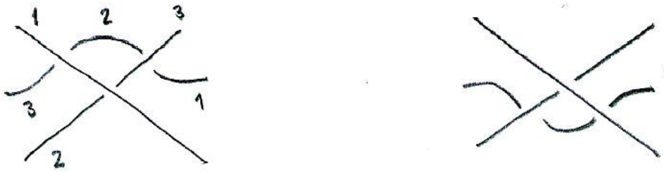
1,1,1

1,1,2

1,2,1

1,2,2

1,2,3



In each case, check that the arcs coming out are the same colour after the move. So the move does not offect the 3-colourability of the diagram.

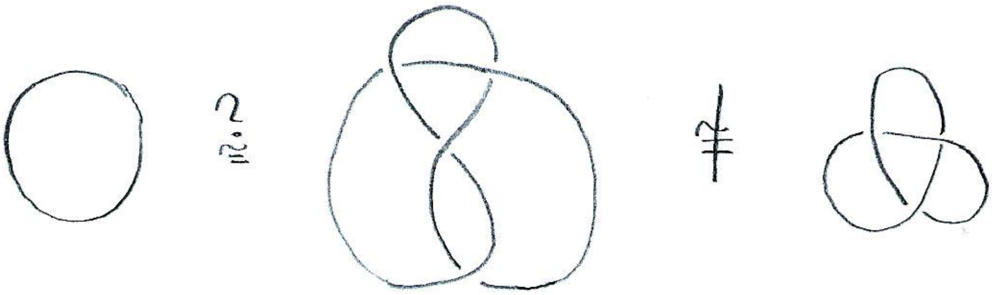
//.

The unknot is not 3-colourable: it can only have 1 colour.

The trefoil knot is 3-colourable.

Exercise

The figure 8 is not 3-colourable. (Show that if 3 colours meet at some crossing the rules are violated at another crossing. Repeat for all crossings).



So the figure 8 knot is not isotopic to the trefoil knot. But is it knotted?

3-colourability is easy to determine, but it does not distinguish many knots.
Other invariants distinguish more knots but are difficult to calculate.

These are defined for the equivalence class of the knot or link, rather than for a particular diagram. So we must define the equivalence relation.

Defn. Two knots $K, M : S^1 \to \mathbb{R}^3$, are (ambient) isotopic if there is a continuous deformation of $\mathbb{R}^3$ taking the one to the other. That is, if there exists a mapping $\varphi : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$, where $\mathbb{R}^3 \times [0,1]$ has the relative topology from $\mathbb{R}^4$, and :

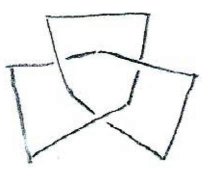
1) For each $t \in [0,1]$, φ restricted to $\mathbb{R}^3 \times \{t\}$ is a homeomorphism.

2) For each $x \in \mathbb{R}^3$, $\varphi(x, 0) = x$

3) For each $s \in S^1$, $\varphi(K(s), 1) = M(s)$.

To be able to develop a theory for knots we must restrict the definition of a knot.

A piecewise linear knot is one whose image $K(S^1) \subset \mathbb{R}^3$ is the union of a finite number of line segments.



piecewise linear image of trefoil.

A <u>tame</u> knot is a knot isotopic to a piecewise linear knot.

A knot that is not tame is called a <u>wild</u> knot.

The knot diagrams and all the properties we have described apply to tame knots, since we have assumed a finite number of crossings.

<u>Example of a</u> wild knot.

Now we can state the <u>theorem of Reidemeister</u>

<u>Theorem</u> Two knots are isotopic if and only if they project to isotopic diagrams.

Some quantities are by definition invariants of an isotopy class, but are difficult to compute.

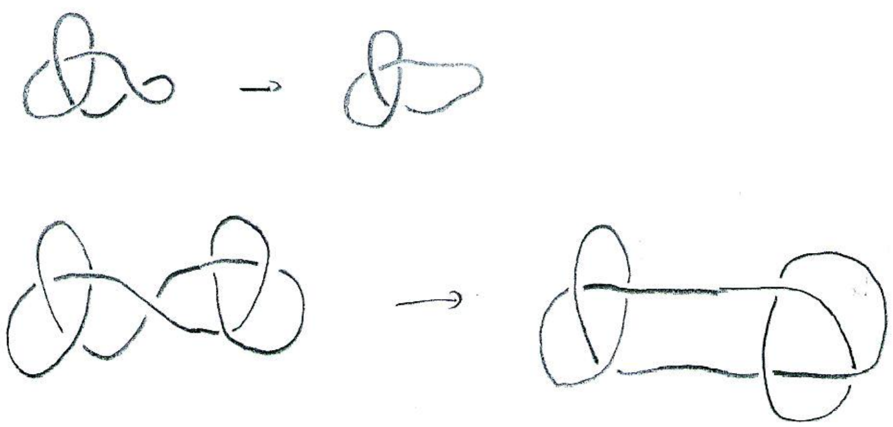
<u>Defn</u> The <u>crossing number</u> of a knot is the minimum number of crossings for any diagram for the isotopy class of the knot.

By definition, the crossing number is an invariant of the isotopy class. But it is not easy to calculate.

Some crossings can obviously be removed.

A <u>reduced</u> diagram(~~a projection~~) is a diagram without a crossing that can be removed by twisting part of the knot.

<u>Ex</u>.



not reduced ⟶ reduced

A knot or link projection is <u>alternating</u> if consecutive crossings on each component alternate between going over and going under.

A knot or link is <u>alternating</u> if it has an alternating projection.

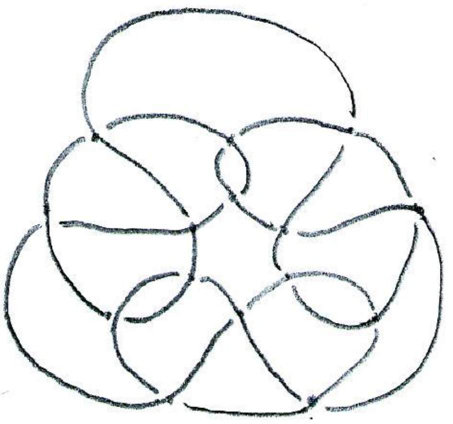
Most of the knots with small number of crossings are alternating. But the proportion of alternating knots tends to 0 as the number of crossings increases. (Check list of knots with up to 8 crossings)

For reduced alternating knots we have two important results, conjectured by Tait in 1870, and proven in 1986.

<u>Theorem</u> A reduced alternating projection of a knot or link has the minimum number of crossings of any projection of the knot or link.

So a knot like

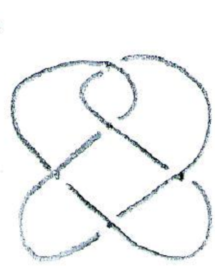
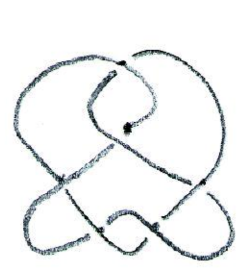
K



$c(K) = 15$

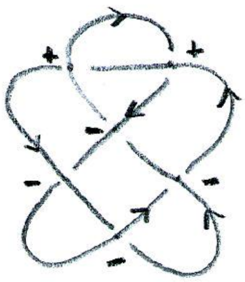
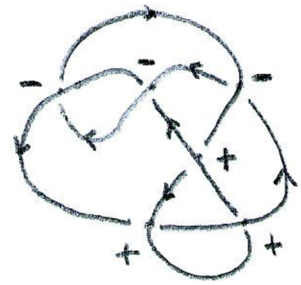
is reduced and alternating, and its crossing number is 15.

<u>Ex.</u>

5_2  $\neq$  6_1

<u>Theorem</u> All reduced alternating projections of a given oriented knot have the same writhe.

<u>Ex</u>.

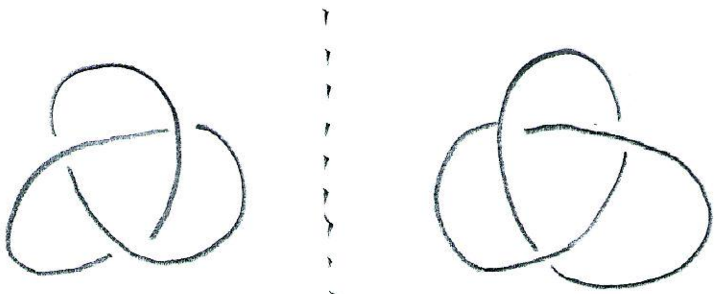
6_2   6_3

Give an orientation and calculate the writhe.

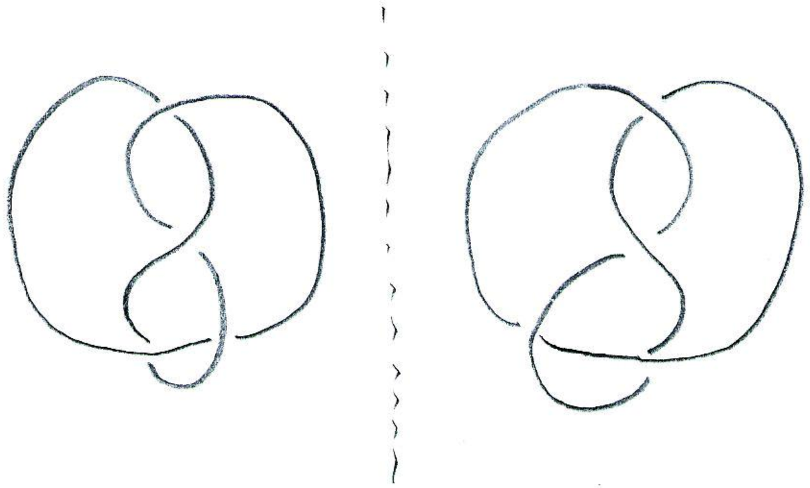
$w(6_2) = -2$ $\qquad\qquad$ $w(6_3) = 0$.

Hence they are not isotopic.

If D is a link diagram, $\overline{D}$ is the diagram in which we have changed every overpass to an underpass. $\overline{D}$ is obtained from D by reflection in an axis in the plane of the diagram:



$\overline{T}$ $\qquad\qquad$ T



8 $\qquad$ $\bar{8}$

<u>Exercise</u>. Show that $8 \cong \bar{8}$.

We'll show later that $T \not\cong \bar{T}$.

If D is an oriented diagram, we denote rD the diagram with the reverse orientation.

T and rT are isotopic, by a rotation in an axis in $\mathbb{R}^3$.

T $\qquad$ rT