Knot and link polynomials.

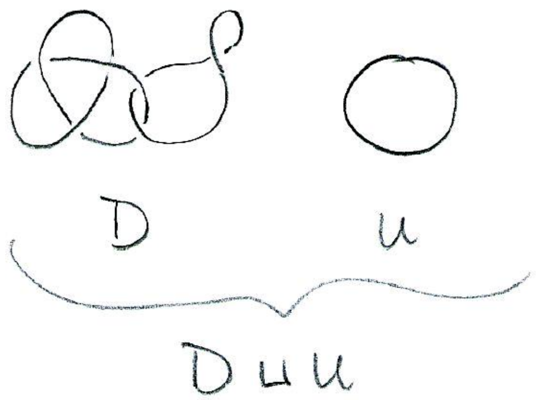
Knot theory was revolutionised in 1985, with the discovery of many new invariants that were simple to calculate and gave answers to many longstanding problems. This started with the Jones polynomial, but is was soon followed by many families of polynomials. We'll look at the Kaufman polynomial, which is a version of the Jones polynomial.

We start by defining the <u>bracket polynomial</u> of a diagram. It is a Laurent polynomial in one variable A, and we symbolize it $\langle D \rangle$. It is characterized by the following properties:

BP1 : $\mathcal{U}$ is the trivial diagram with one component :

$\mathcal{U}$:  . $\langle \mathcal{U} \rangle = 1$.

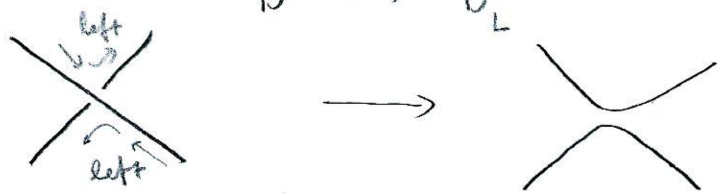
BP2 : If $D \sqcup \mathcal{U}$ is the diagram D with one extra component, with no extra crossings,



$$\langle D \sqcup \mathcal{U} \rangle = -(A^2 + A^{-2}) \langle D \rangle .$$

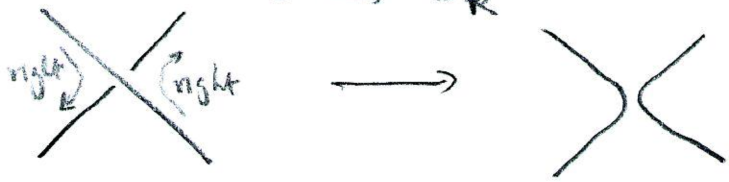
We consider a crossing on the diagram D. Then we obtain 2 new diagrams D_L and D_R by replacing the crossing by :

$D \to D_L$



"splitting to the left", "positive state".

$D \to D_R$

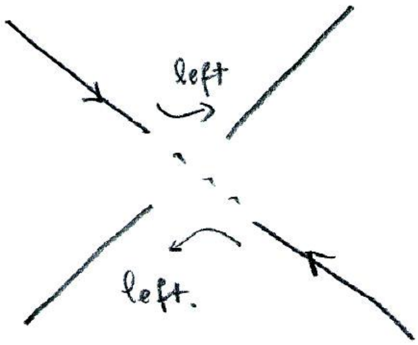


"splitting to the right", "negative state".

BP3 : When we "split" a crossing of D, to obtain D_L and D_R, we have:

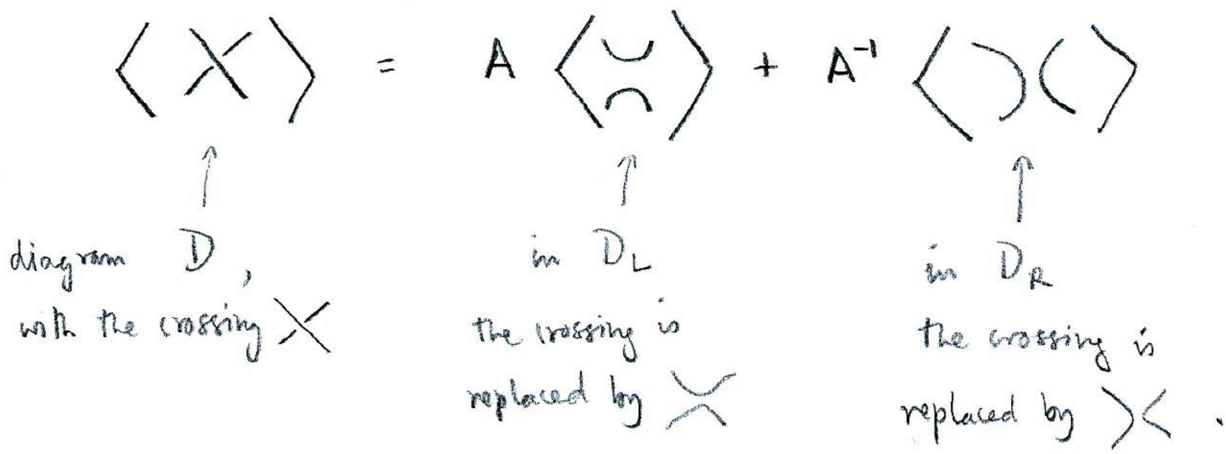
$$\langle D \rangle = A \langle D_L \rangle + A^{-1} \langle D_R \rangle .$$

splitting to the <u>left</u> (right): cut the strings at the crossing and join each string of the overpass to the string of the underpass to its <u>left</u> (right) as we approach the crossing.



Splitting to the left or to the right is <u>not</u> an isotopy. It replaces the diagram with two diagrams with fewer crossings and possibly more components. By repeating this at all crossings, we get diagrams with no crossings, whose bracket polynomial is given by BP1 and BP2.

We symbolize BP3 as follows:



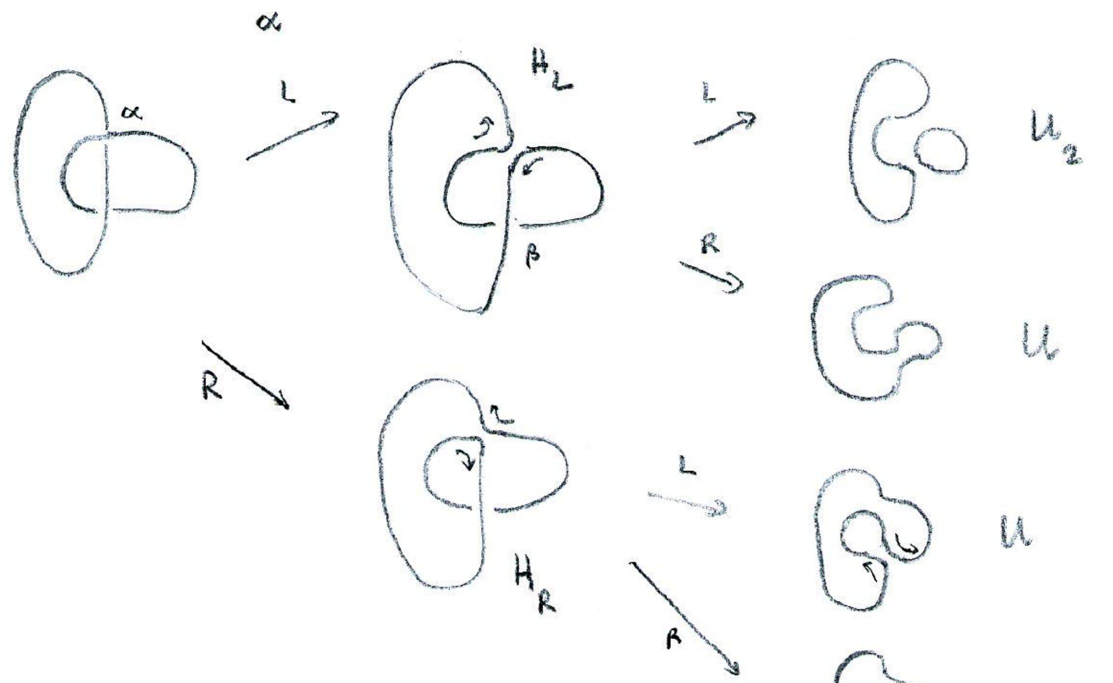
<u>Example.</u> U_k is the disjoint union of k circles, with no crossings. Using BP1 and BP2 we have by induction:

$$\begin{aligned}\langle U_k \rangle &= -(A^2 + A^{-2}) \langle U_{k-1} \rangle \\ &= (-1)^{k-1} (A^2 + A^{-2})^{k-1} \langle U_1 \rangle \\ &= (-1)^{k-1} (A^2 + A^{-2})^{k-1}.\end{aligned}$$

So, $\mathcal{U}_3 = A^4 + 2 + A^{-4}$

$$\mathcal{U}_4 = -\left(A^6 + 3A^2 + 3A^{-2} + A^{-6}\right).$$

<u>Example</u> We shall compute the bracket polynomial of the Hopf link: starting with crossing α.



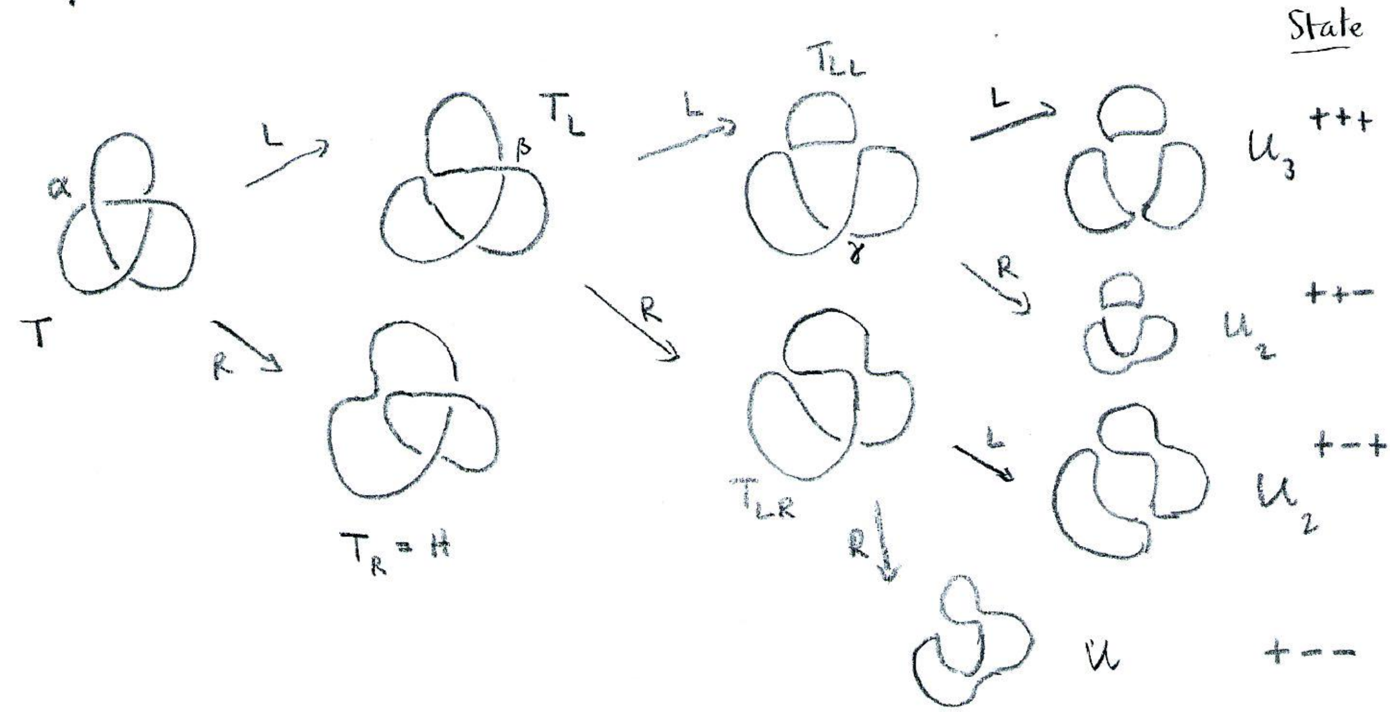
$$\langle H \rangle = A\langle H_L \rangle + A^{-1}\langle H_R \rangle$$

$$\begin{aligned}
\langle H_L \rangle &= A\langle \mathcal{U}_2 \rangle + A^{-1}\langle \mathcal{U} \rangle \\
&= -A\left(A^2 + A^{-2}\right) + A^{-1}\,1 \\
&= -A^3 - A^{-1} + A^{-1} = -A^3.
\end{aligned}$$

$$\begin{aligned}
\langle H_R \rangle &= A\langle \mathcal{U} \rangle + A^{-1}\langle \mathcal{U}_2 \rangle \\
&= A\cdot 1 - A^{-1}\left(A^2 + A^{-2}\right) \\
&= A - A - A^{-3} = -A^{-3}.
\end{aligned}$$

So $\langle H \rangle = -\left(A^4 + A^{-4}\right)$.

<u>Example</u> We shall compute the bracket polynomial of the trefoil knot T.



So $\langle T \rangle = A\langle T_L \rangle + A^{-1}\langle H \rangle$.

$$\langle T_L \rangle = A\langle T_{LL} \rangle + A^{-1}\langle T_{LR} \rangle$$

$$= A\left(A\langle U_3 \rangle + A^{-1}\langle U_2 \rangle\right) + A^{-1}\left(A\langle U_2 \rangle + A^{-1}\langle U \rangle\right)$$

$$= A^2\langle U_3 \rangle + \langle U_2 \rangle + \langle U_2 \rangle + A^{-2}\langle U \rangle$$

$$= A^2\left(A^4 + 2 + A^{-4}\right) - 2\left(A^2 + A^{-2}\right) + A^{-2}$$

$$= A^6 .$$

And

$$\langle T \rangle = A^7 - A^3 - A^{-5} .$$

Ex. Compute the bracket polynomial of the reflected trefoil knot $\overline{T}$: 

Lemma The bracket polynomial is invariant by R0, R2, R3.

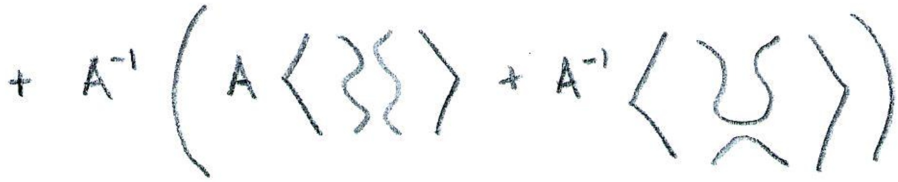
Proof R0 do not change any crossings, hence they do not affect the computation of $\langle D \rangle$.

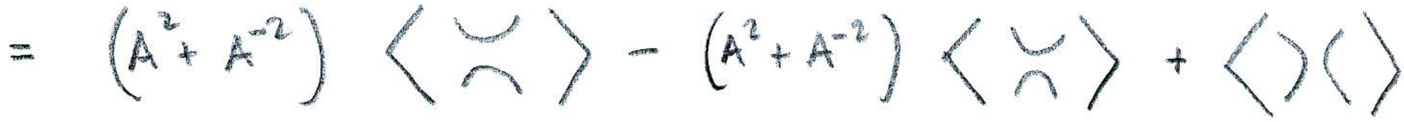
R2 : We consider a diagram D which contains a disc with  .

let  symbolize the bracket polynomial of D, and $\langle K \rangle$ the bracket polynomial of D when we replace  by a disc containing K.

So, if we split D at crossing α, we get

and then





So, when we replace the disc 

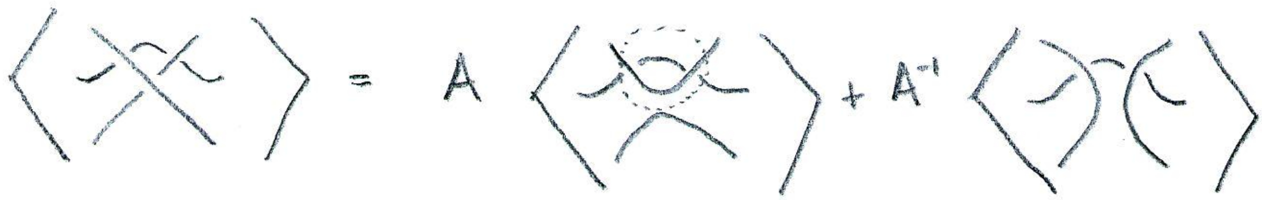
in D, by the disc , the

bracket polynomial does not change.

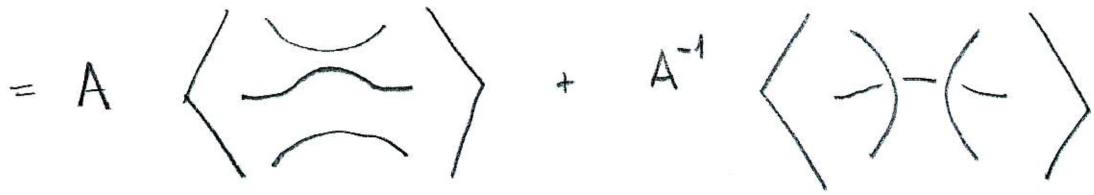
Hence $\langle D \rangle$ is invariant by move R2.

<u>R3</u> : We consider a diagram D which contains a disc

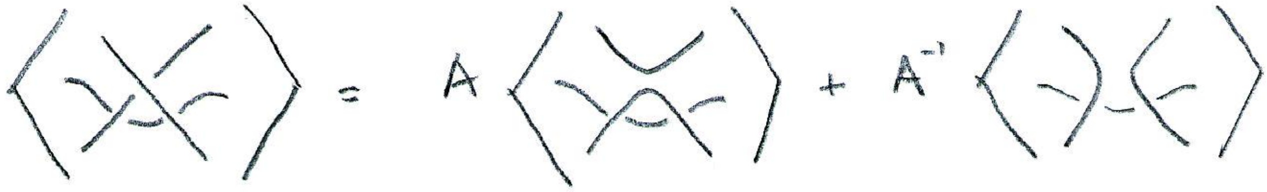
, and we split at crossing α.

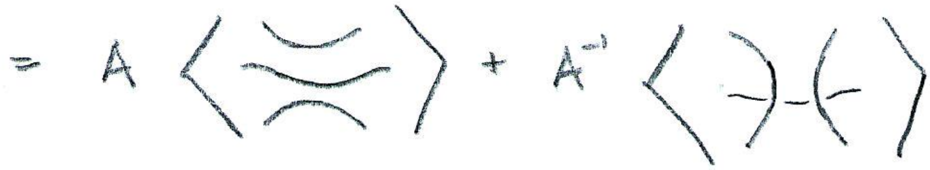


By a move R2 in the inner disc, we get



On the other hand,





So when we replace  by , $\langle D \rangle$ does not change.

Lemma. Movements of type R1 multiply the bracket polynomial by $-A^{-3}$ or $-A^{3}$.

Proof With the same notation, we compute the change in the bracket polynomial when we replace  by .

$$\langle \quad \rangle = A \langle \quad \rangle + A^{-1} \langle \quad \rangle$$

$$= A \langle \quad \rangle - A^{-1}(A^2 + A^{-2}) \langle \quad \rangle$$

$$= A^{-3} \langle \quad \rangle .$$

Similarly

$$\langle \quad \rangle = A \langle \quad \rangle + A^{-1} \langle \quad \rangle$$

$$= \left(-A(A^2 + A^{-2}) + A^{-1}\right) \langle \quad \rangle = -A^3 \langle \quad \rangle .$$

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