$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$= A \Big\langle$  $\Big\rangle - A^{-1}\left(A^{2}+A^{-2}\right) \Big\langle$  $\Big\rangle$

$= A^{-3} \Big\langle$  $\Big\rangle$.

Similarly

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$= \left(-A\left(A^{2}+A^{-2}\right) + A^{-1}\right) \Big\langle$  $\Big\rangle = -A^{3} \Big\langle$  $\Big\rangle$.

//

Lemma The bracket polynomial does not depend on the order in which we split the crossings.

Proof We define a state (κατάσταση) of the diagram D to be a mapping from the crossings of D to $\{-1,+1\}$. If D has n crossings, there are 2^n states.

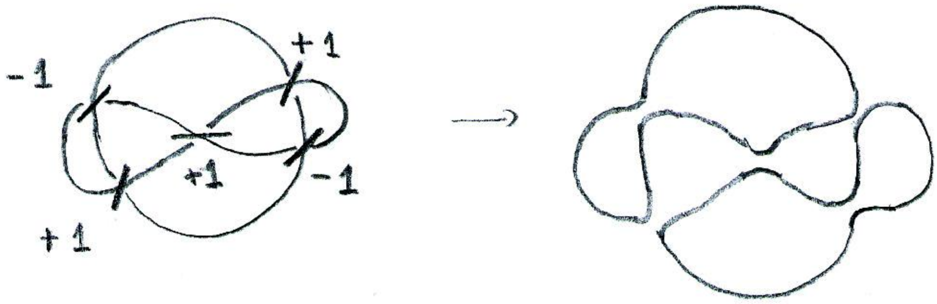
Each state determines a way to split all the crossings of D: if a crossing is positive we split it to the left, if a state is negative we split it to the right. We mark this on the diagram with a dash in the direction in which we split.

+1 

−1 

When we have split all the crossings according to a state, we end up with a diagram of k disjoint circles, since there are no more crossings.

Ex



In this way, each state determines a

polynomial $\quad A^m \langle U_k \rangle$,

where m is the sum of all the signs of the state, and k the number of components.

The sum of these polynomials for all the states is equal to the bracket polynomial for D obtained by an ordering of the crossings.

But this sum is independent of the ordering. So $\langle D \rangle$ depends on the diagram and not on the order in which we split the crossings. //.

The Kauffman polynomial.

Recall the writhe of an oriented link diagram:

$+$ $\qquad\qquad$ $-$

sum of the signs of the crossings on an oriented diagram.

$w\Bigg($  $\Bigg) = 3 - 2 = 1.$

Defn If D is an oriented link diagram, the Kauffman polynomial of D in one variable A is

$$f[D] = (-A)^{-3w(D)} \langle D \rangle .$$

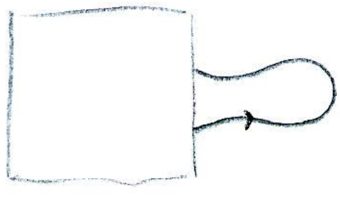
Theorem. The Kauffman polynomial is an isotopy invariant of oriented knots.

Proof. The writhe $w(D)$ and the bracket polynomial $\langle D \rangle$ are invariant by R0, R2, R3.

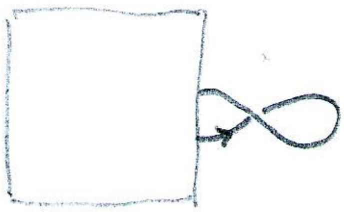
So $f[D]$ is also invariant by these moves.

We only need to check R1.

Assume D is an oriented link diagram, and D' is the link diagram resulting from D after a move of type R1:



D



D'

Then $w(D') = w(D) - 1$

$$\langle D' \rangle = -A^{-3} \langle D \rangle .$$

So $f[D'] = (-A)^{-3\,w(D')} \langle D' \rangle$

$$= (-A)^{-3\,w(D)+3} \left(-A^{-3} \langle D \rangle \right)$$

$$= (-A)^{-3w(D)} (-A)^{3} (-A)^{-3} \langle D \rangle$$

$$= (-A)^{-3w(D)} \langle D \rangle = f[D].$$

//

Example

$w(H) = -1 - 1 = -2.$

$\langle H \rangle = -(A^4 + A^{-4}).$

$$f[H] = (-A)^{-3(-2)} (-A^4 - A^{-4})$$

$$= A^6 (-A^4 - A^{-4}) = -A^{10} - A^2.$$

Example

T

$\overline{T}$

$w(T) = -3$

$\langle T \rangle = A^7 - A^3 - A^{-5}$

$w(\overline{T}) = 3$

$\langle \overline{T} \rangle = A^{-7} - A^{-3} - A^5$

$$f[T] = (-A)^{-3(-3)} (A^7 - A^3 - A^{-5})$$

$$= -A^{16} + A^{12} + A^4$$

$$f[\overline{T}] = (-A)^{-3(3)} (A^{-7} - A^{-3} - A^5)$$

$$= -A^{-16} + A^{-12} + A^{-4}.$$

Corollary $T \not\simeq \overline{T}$.

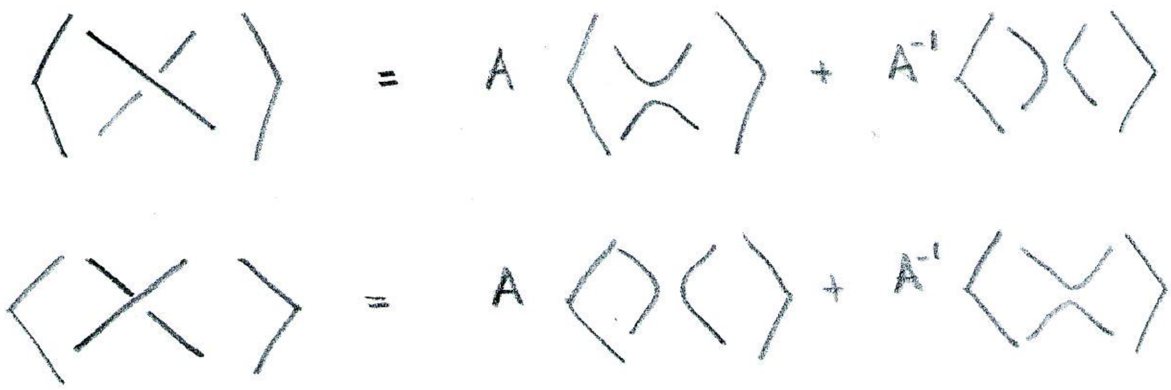
<u>Proposition</u>. If D is an oriented link diagram and $\bar{D}$ the reflected link diagram with the same orientation, then

$$f[D](A) = f[\bar{D}](A^{-1}).$$

<u>Proof</u> $\bar{D}$ is isotopic to D with all overpasses changed to underpasses. This changes the sign of each crossing (with respect to the same orientation) and hence multiplies $w(D)$ with -1.

$$w(\bar{D}) = -w(D).$$

In the bracket polynomial, a change in a crossing



changes A into A^{-1} and A^{-1} into A.

Rule BP2 is not affected by this change. So

$$\langle \bar{D} \rangle (A) = \langle D \rangle (A^{-1}).$$

Altogether,
$$\begin{aligned} f[\bar{D}](A) &= (-A)^{-3w(\bar{D})} \langle \bar{D} \rangle (A) \\ &= (-A)^{-3(-w(D))} \langle D \rangle (A^{-1}) \\ &= (-A^{-1})^{3w(D)} \langle D \rangle (A^{-1}) = f[D](A^{-1}). \end{aligned}$$

The Jones Polynomial

The Jones polynomial was discovered in 1985, by Vaughan Jones, who was working on Banach algebras, and noticed that certain relations which appeared in his work were similar to relations in braid theory, which is an algebraic description of knots.

Defn The Jones polynomial of an oriented link D diagram is the Laurent polynomial in one variable t,

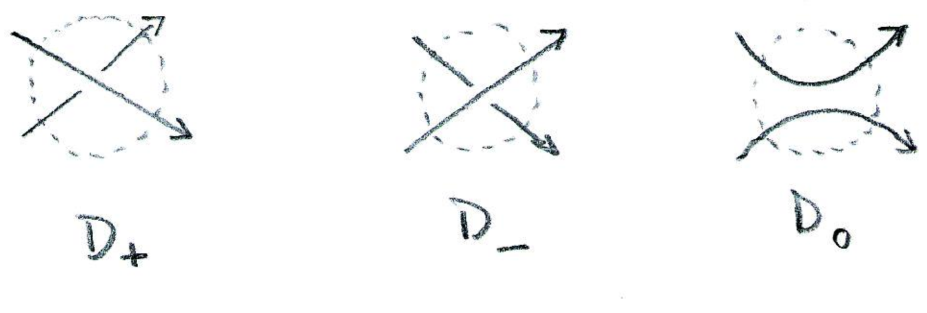
correction

$$V_D(t) = f[D](t^{-1/4}).$$

correction

Ex. $V_T(t) = -t^{-4} + t^{-3} + t^{-1}$, $V_{\bar{T}}(t) = -t^4 + t^3 + t^1$.

If D is a diagram, and we fix a crossing of D, then we denote D_+, D_-, D_0 the diagrams



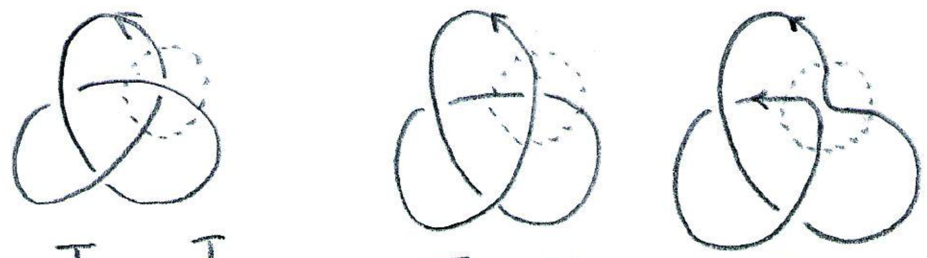
Lemma. The Jones polynomial is determined by two relations:

$$\text{JP1}: \quad V_{\mathcal{U}} = 1.$$

JP2: For any oriented diagram D, let V_+, V_-, V_0 denote the Jones polynomials of D_+, D_-, D_0. Then

$$t^{-1} V_+(t) - t\, V_-(t) = \left(t^{1/2} - t^{-1/2}\right) V_0(t).$$

//.



$T_- = T$ $\qquad$ $T_+ \cong \mathcal{U}$ $\qquad$ $T_0 = H$.

$$\begin{aligned}\left(t^{1/2} - t^{-1/2}\right) V_H &= t^{-1} V_{\mathcal{U}} - t\, V_T \\ &= t^{-1} - t\left(-t^{-4} + t^{-3} + t^{-1}\right) \\ &= -1 + t^{-1} - t^{-2} + t^{-3}.\end{aligned}$$

We know $V_H(t) = f[H]\left(t^{-1/4}\right) = -t^{-5/2} - t^{-1/2}$

We verify: $-\left(t^{1/2} - t^{-1/2}\right)\left(t^{-5/2} + t^{-1/2}\right) = -t^{-2} - 1 + t^{-3} + t^{-1}$. ✓.