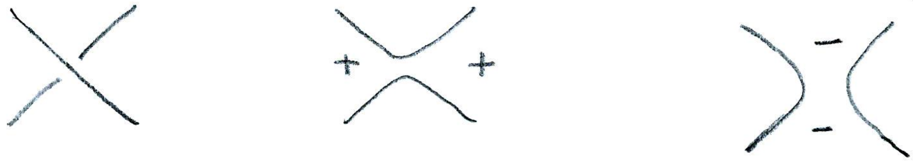
Application of Jones polynomial: Number of crossings of an alternating diagram.

Positive and negative regions.

left or +1 splitting

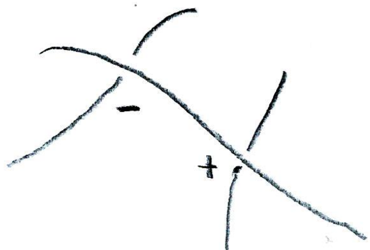
right or −1 splitting



We define the signs of the adjacent regions.

The signs are defined locally, near the crossing. But if the diagram is alternating, the sign is well defined for each region of $\mathbb{R}^2 \setminus D$.

alternating.



not alternating.

<u>Lemma</u> If D is a connected diagram with c crossings, then $\mathbb{R}^2 \setminus D$ has $c+2$ components.

<u>Proof</u> From Euler's formula, vertices + faces = edges + 2

But for a link diagram, the number of edges is $2 \times$ vertices, so faces = vertices + 2. //

<u>Proposition</u>. Let D be a connected, reduced, alternating link diagram, with c crossings, p positive regions and n negative regions.
Then the term in $\langle D \rangle$ of maximum degree is $(-1)^{n-1} A^{c+2n-2}$, and the term in $\langle D \rangle$ of minimum degree is $(-1)^{p-1} A^{-c-2p+2}$.

<u>Theorem</u> If a link L has a connected, reduced, alternating diagram D with c crossings then the difference Δ between the maximum exponent and the minimum exponent of the terms of $\langle D \rangle$ is $4c$.

Any two connected, reduced, alternating diagrams of the link L have the same number of crossings.

<u>Proof of Theorem</u> From the proposition

$$\Delta = c+2n-2 - (-c-2p+2)$$
$$= 2c + 2(n+p) - 4$$

But $n+p = c+2$, hence $\Delta = 2c + 2(c+2) - 4$
$$= 4c.$$

Δ is equal to the difference between the max and min exponent of the Kauffman polynomial, which is an isotopy invariant. Hence the number of crossings for any connected, reduced, alternating diagram representing L is the same, $\Delta/4$. //

Δ is called the <u>span</u> (εύρος) of the polynomial $\langle D \rangle$.

<u>Proof of Proposition.</u>

Let S be the state of D with all crossings $+1$. When we split all the crossings we end up with one component $\mathcal{U}$ around each negative region (see 66). So we end up with $\mathcal{U}_n$. The state S contributes to $\langle D \rangle$ an expression

$$A^c \langle \mathcal{U}_n \rangle = A^c \left((-1)^{n-1} (A^2 + A^{-2})^{n-1} \right)$$

where n is the number of components, and c is (number of splittings to the left) − (number of splittings to the right).

This expression contains the term with maximum exponent:

$$(-1)^{n-1} A^{c+2n-2}.$$

We must show all other states contribute terms with smaller exponent.

Assume S_1 is a state obtained from S by changing exactly one crossing from left to right. Let D_S be the diagram obtained after applying S, and D_{S_1} the diagram obtained after applying S_1.

Then D_S has n components, each component is the boundary of a negative region.

In D_{S_1} the splittings are the same, except at one crossing, where instead of joining positive to positive we join negative to negative. So in D_{S_1} two of the original negative regions are joined. Since the original diagram was reduced, and 4 different regions met at each crossing, two different negative regions are joined, and the number of negative regions becomes $n-1$. Hence $D_{S_1} = U_{n-1}$.

Now consider a state S' and a sequence of states

$$S = S_0 , S_1 , \dots , S_{m-1} , S_m = S'$$

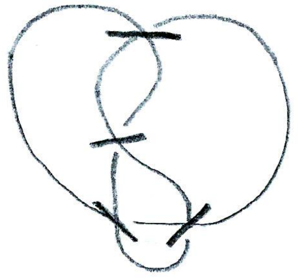
so that S_{i+1}, for $i = 0, \dots, m-1$, is obtained from S_i by changing exactly one crossing from $+1$ to -1. Let d_i be the difference between the numbers of positive and negative crossings in S_i. Then

$$d_{i+1} = d_i - 2 .$$

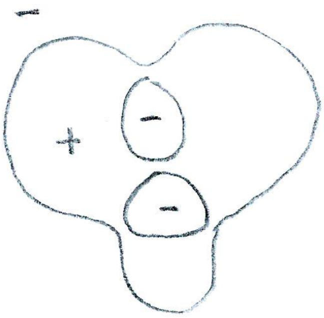
Next we look at how the number of components changes between D_{S_i} and $D_{S_{i+1}}$.

Example Figure 8 knot.

S_0



D_{S_0}

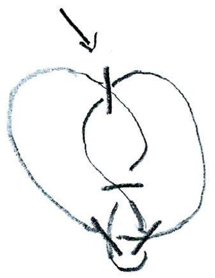


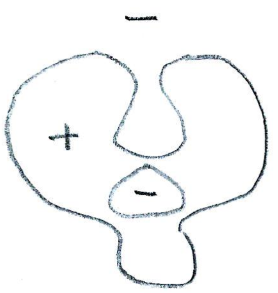
$p_0 = 1$

$n_0 = 3$ $\quad k_0 = 3$

$d_0 = 4$

S_1





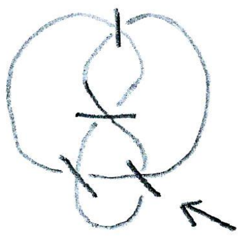
D_{S_1}

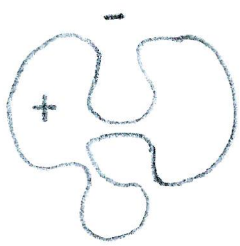
$p_1 = 1$

$n_1 = 2$ $\quad k_1 = 2$

$d_1 = 2$

S_2





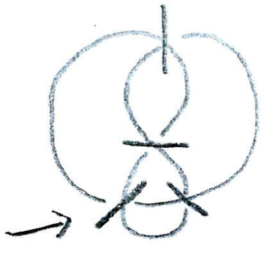
D_{S_2}

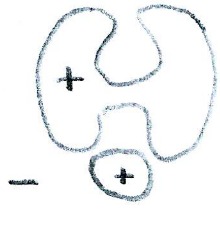
$p_2 = 1$

$n_2 = 1$ $\quad k_2 = 1$

$d_2 = 0$

S_3





D_{S_3}

$p_3 = 2$ $\quad k_3 = 2$

$n_3 = 1$

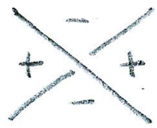
$d_3 = -2$

p_i : number of positive regions.

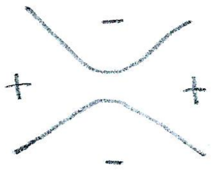
n_i : number of negative regions

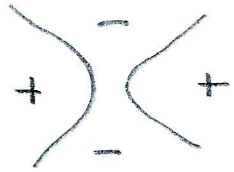
$k_i = |S_i|$: number of components.

d_i : (number of splittings to the left) − (number of splittings to the right) in D

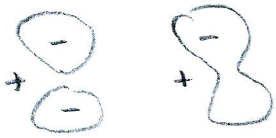


in D_{S_i} in $D_{S_{i+1}}$





If the two negative regions at the crossing are different, then changing the splitting from left to right, reduces the number of $\mathcal{U}$ components in the diagram by 1.



If the two negative regions at the crossing are the same, then changing the splitting from left to right creates a new positive region, and increases the number of $\mathcal{U}$ components in the diagram by 1.

Assume $D_{S_i} = \mathcal{U}_{k_i}$. Then the term contributed by D_{S_i} to $\langle D \rangle$ is

$$A^{d_i} \langle \mathcal{U}_{k_i} \rangle = A^{d_i} (-1)^{k_i - 1} (A^2 + A^{-2})^{k_i - 1}$$

For $D_{S_{i+1}}$ this term is

$$A^{d_{i+1}} \langle \mathcal{U}_{k_{i+1}} \rangle = A^{d_{i+1}} (-1)^{k_{i+1} - 1} (A^2 + A^{-2})^{k_{i+1} - 1}$$

We have seen that $d_{i+1} = d_i - 2$

and $k_{i+1} = k_i \pm 1$.

It follows that

$$\deg\left(A^{d_{i+1}} \langle U_{k_{i+1}} \rangle \right) \leq \deg\left(A^{d_i} \langle U_{k_i} \rangle \right).$$

Since $k_1 = n-1$, $d_1 = c-2$, the degree of the terms contributed by state S' are

$$\leq \deg\left(A^{c-2} (-1)^{n-2} (A^2 + A^{-2})^{n-2} \right)$$

$$= c - 2 + 2(n-2) < c + 2n - 2.$$

Similarly we show that the term in $\langle D \rangle$ of minimum exponent is the term corresponding to the state $\hat{S}$ with all splittings to the right, which is

$$(-1)^{p-1} A^{-c-2p+2}.$$

//.

We have shown that all connected, reduced, alternating diagrams of a link have $c = \Delta/4$ crossings. To complete the proof of the Tait conjecture we must show that any other connected diagram of the link has $\geq \Delta/4$ crossings.

<u>Theorem</u> If a link L possesses a connected diagram D with c crossings, then the span of the bracket polynomial $\langle D \rangle \leqslant 4c$.

If S is a state of a connected diagram, the <u>dual state</u> $-S$ is obtained by changing the sign at all crossings.

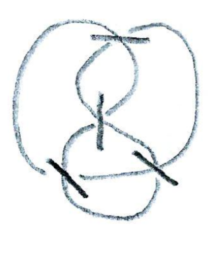
<u>Dual State Lemma</u>

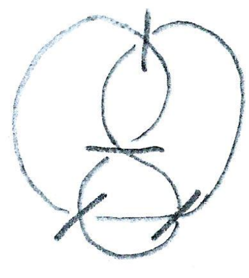
Let S be a state of a connected diagram D, and $-S$ be the dual state. Let r be the number of regions for D, and $|S|$ the number of components of D_S. Then

$$|S| + |-S| \leq r.$$

<u>Example</u> $r = 6$.

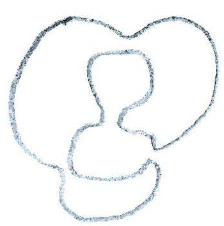
S

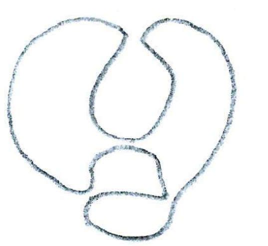




$-S$

D_S

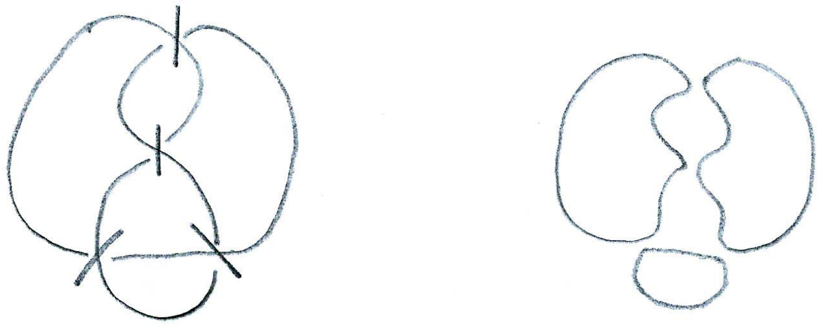




S_{-S}

$|-S| = 1$.

$|S| = 1$



$|-S_0| = 3$

$$|S_0| + |-S_0| = r$$

<u>Proof of Theorem</u> (assuming the lemma).

Consider the state S_0. We have shown that it contributes the term of maximum degree $c + 2|S| - 2$, and no other state contributes a term of higher degree.

Similarly, the state $-S_0$ contributes a term of minimum degree $-c - 2|-S| + 2$.

$$\begin{aligned} \text{So} \quad \Delta &= c + 2|S| - 2 - (-c - 2|-S| + 2) \\ &= 2c + 2(|S| + |-S|) - 4. \\ &\leq 2c + 2r - 4. \end{aligned}$$

But $r = c + 2$, so

$$\Delta \leq 4c.$$

//

From: W.B.R. Lickorish,
An Introduction to Knot Theory.

(71)

TABLE 3.1. Jones Polynomial Table

3_1	-1	1	0	1	**0**						
4_1	1	-1	**1**	-1	1						
5_1	-1	1	-1	1	0	1	0	**0**			
5_2	-1	1	-1	2	-1	1	**0**				
6_1	1	-1	1	-2	**2**	-1	1				
6_2	1	-2	2	-2	2	**-1**	1				
6_3	-1	2	-2	**3**	-2	2	-1				
7_1	-1	1	-1	1	-1	1	0	1	0	0	**0**
7_2	-1	1	-1	2	-2	2	-1	1	**0**		
7_3	**0**	0	1	-1	2	-2	3	-2	1	-1	
7_4	**0**	1	-2	3	-2	3	-2	1	-1		
7_5	-1	2	-3	3	-3	3	-1	1	0	**0**	
7_6	-1	2	-3	4	-3	3	**-2**	1			
7_7	-1	3	-3	**4**	-4	3	-2	1			
8_1	1	-1	1	-2	2	-2	**2**	-1	1		
8_2	1	-2	2	-3	3	-2	2	-1	**1**		
8_3	1	-1	2	-3	**3**	-3	2	-1	1		
8_4	1	-2	3	-3	3	**-3**	2	-1	1		
8_5	**1**	-1	3	-3	3	-4	3	-2	1		
8_6	1	-2	3	-4	4	-4	3	**-1**	1		
8_7	-1	2	**-2**	4	-4	4	-3	2	-1		
8_8	-1	2	-3	**5**	-4	4	-3	2	-1		
8_9	1	-2	3	-4	**5**	-4	3	-2	1		
8_{10}	-1	2	**-3**	5	-4	5	-4	2	-1		
8_{11}	1	-2	3	-5	5	-4	4	**-2**	1		
8_{12}	1	-2	4	-5	**5**	-5	4	-2	1		
8_{13}	-1	2	-3	5	-5	**5**	-4	3	-1		
8_{14}	1	-3	4	-5	6	-5	4	**-2**	1		
8_{15}	1	-3	4	-6	6	-5	5	-2	1	0	**0**
8_{16}	-1	3	-5	6	-6	6	**-4**	3	-1		
8_{17}	1	-3	5	-6	**7**	-6	5	-3	1		
8_{18}	1	-4	6	-7	**9**	-7	6	-4	1		
8_{19}	**0**	0	0	1	0	1	0	0	-1		
8_{20}	-1	1	-1	2	-1	**2**	-1				
8_{21}	1	-2	2	-3	3	-2	2	**0**			

The coefficients of the Jones polynomial.
A bold entry is a coefficient of t^0.
For example,

$$V_{6_1}(t) = t^{-4} - t^{-3} + t^{-2} - 2t^{-1} + 2 - t + t^2.$$