# Topological manifold

A **topological manifold of dimension** $n$ is a set $M$ together with a set of subsets $\{U_i \subseteq M : i \in \mathcal{A}\}$ such that

1. the $U_i$ cover $M$, that is $M = \bigcup_{i\in\mathcal{A}} U_i$,
2. for every $i \in \mathcal{A}$, there is a bijection $\varphi_i : U_i \longrightarrow B^n$ from $U_i$ to the unit ball in $\mathbb{R}^n$, $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$.
3. for every $i, j \in \mathcal{A}$ with $U_i \cap U_j \neq \varnothing$, the mapping
$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$
is continuous as a mapping between subsets of $\mathbb{R}^n$.

The mapping $\varphi_i : U_i \longrightarrow B^n$ is a **coordinate chart** for the topological manifold $M$. The set $\{U_i \subseteq M : i \in \mathcal{A}\}$ together with the set of coordinate charts $\{\varphi_i : U_i \longrightarrow B^n\,, i \in \mathcal{A}\}$ is an **atlas** for the topological manifold $M$. The mappings $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$ are the **transition functions** of the atlas.

**Example 1** $\mathbb{R}^n$ itself is a topological manifold of dimension $n$. For every point $a \in \mathbb{R}^n$, we define $U_a = \{x \in \mathbb{R}^n : |x - a| < 1\}$, and the coordinate chart $\varphi_a : x \longmapsto x - a$. The set $\{U_a : a \in \mathbb{R}^n\}$ is clearly a covering of $\mathbb{R}^n$, and the transition functions are continuous, $\varphi_a \circ \varphi_b^{-1}(y) = y + b - a$.

**Example 2** The circle $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is a topological manifold of dimension 1. We can define an atlas with two coordinate charts.

Let $g : (-1, 1) \longrightarrow S^1$ be the mapping $t \longmapsto (\cos \pi t, \sin \pi t)$. The image of $g$ is the set $U_1 = S^1 \setminus \{(-1, 0)\}$ and $g$ is an injection. So there is an inverse $g^{-1} : U_1 \longrightarrow (-1, 1)$. We define the coordinate chart $\varphi_1(x) = g^{-1}(x)$.

Similarly, we define $g : (-1, 1) \longrightarrow S^1$ to be the mapping $t \longmapsto (-\cos \pi t, \sin \pi t)$. The image of $h$ is the set $U_2 = S^1 \setminus \{(1, 0)\}$ and $h$ is an injection. So there is an inverse $h^{-1} : U_2 \longrightarrow (-1, 1)$. We define the coordinate chart $\varphi_2(x) = h^{-1}(x)$.

Check that the transition functions are continuous.

**Example 3** We shall define the structure of a topological manifold of dimension 2 on the set $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$: We define an atlas consisting of a covering by 6 subsets, $U_u$, $U_d$, $U_l$, $U_r$, $U_f$, $U_b$ (where the indices stand for *up, down, left, right, front* and *back*),

$$\begin{aligned} U_u &= \{(x, y, z) \in S^2 : z > 0\}\,, & \varphi_u(x, y, z) &= (x, y) \in \mathbb{R}^2\,,\\ U_d &= \{(x, y, z) \in S^2 : z < 0\}\,, & \varphi_d(x, y, z) &= (x, y) \in \mathbb{R}^2\,,\\ U_l &= \{(x, y, z) \in S^2 : y < 0\}\,, & \varphi_l(x, y, z) &= (x, z) \in \mathbb{R}^2\,,\end{aligned}$$

$$\begin{aligned}
U_r &= \{(x,\, y,\, z) \in S^2 : y > 0\}\,, & \varphi_r(x,\, y,\, z) &= (x,\, z) \in \mathbb{R}^2\,,\\
U_f &= \{(x,\, y,\, z) \in S^2 : x > 0\}\,, & \varphi_f(x,\, y,\, z) &= (y,\, z) \in \mathbb{R}^2\,,\\
U_b &= \{(x,\, y,\, z) \in S^2 : x < 0\}\,, & \varphi_b(x,\, y,\, z) &= (y,\, z) \in \mathbb{R}^2\,.
\end{aligned}$$

Check that for $(s,\, t) \in B^2$, $\varphi_u^{-1}(s,\, t) = (s,\, t,\, \sqrt{1-s^2-t^2})$, and that if $t < 0$, $\varphi_u^{-1}(s,\, t) \in U_u \cap U_l$ and the transition function is $\varphi_l \circ \varphi_u^{-1}(s,\, t) = (s, \sqrt{1-s^2-t^2})$, which is clearly a continuous mapping between subsets of $\mathbb{R}^2$.