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On the connection theory of extensions of transitive Lie groupoids ☆

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Abstract

Due to a result by Mackenzie, extensions of transitive Lie groupoids are equivalent to certain Lie groupoids which admit an action of a Lie group. This paper is a treatment of the equivariant connection theory and holonomy of such groupoids, and shows that such connections give rise to the transition data necessary for the classification of their respective Lie algebroids.

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1. Introduction

A groupoid, in the categorical sense, is a category where every arrow is invertible. Lie groupoids are those ones whose object and arrow spaces are manifolds, and the structure maps (source s, target t, multiplication and inversion) are smooth, plus t and s are submersions. A Lie group can be considered as a Lie groupoid (with a single point as the object space), but in general Lie groupoids are inherently noncommutative, hence their extensive use in differential geometry. In the same fashion, a bundle of Lie groups is also a Lie groupoid, with its projection playing the role of both the source and target maps. In this paper we focus on the transitive case, namely those Lie groupoids $\Omega \rightrightarrows M$ such that the anchor map $(t,s): \Omega \to M \times M$ is a surjective submersion.

Given a transitive Lie groupoid $\Omega \rightrightarrows M$, an extension of this is a pair of Lie groupoid morphisms

$$F \overset{\iota}{\rightarrowtail} \Omega \overset{\phi}{\twoheadrightarrow} \Phi \tag{1}$$

where $\Phi \rightrightarrows M$ is a transitive Lie groupoid and $F \to M$ is a bundle of Lie groups. Much of the work of Mackenzie concerns such extensions, and for a general Lie groupoid there always exists a foliation whose restriction to every leaf is such an extension. When dealing with such an extension, it is quite difficult to keep track of the transition functions of all the groupoids involved. Mackenzie in [5], managed to reformulate such an extension to a single (transitive)

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Lie groupoid over the total space of a principal bundle, plus a group action. These are called PBG-groupoids, and an account of this correspondence is given in Section 1. It is therefore reasonable to expect that extensions of transitive Lie groupoids are classified by the transition functions of their corresponding PBG-groupoids, but it is necessary to keep track of the group action as well, and it is not certain that there exist transition functions which enjoy some kind of equivariance.

This is where connections are necessary. The transversals of an extension of transitive Lie groupoids correspond to connections in the respective PBG-groupoid which are suitably equivariant, and these are the connections whose infinitesimal theory and holonomy are given in this paper. The term *isometablic* is used for these connections, instead of equivariant, in order to highlight the non-standard nature of equivariance for the transition functions that they induce.

The classification of PBG-groupoids (extensions of transitive Lie groupoids) is a problem of a different order presented in [1]. In this paper we focus on the infinitesimal level, namely the classification of transitive PBG-algebroids. It is shown in [6] that every transitive Lie algebroid A (i.e., an extension of TM by a Lie algebra bundle L) locally can be written as $TU_i \oplus L_{U_i}$, and in [6] it is shown that such Lie algebroids are classified by pairs $(\chi, \alpha) = \{(\chi_{ij}, \alpha_{ij})\}_{i,j\in I}$, where $\alpha_{ij} : U_{ij} \to Aut(\mathfrak{h})$ are the transition functions of the sections of L and $\chi_{ij} : TU_{ij} \to U_{ij} \times \mathfrak{h}$ are certain differential 1-forms, and the two of them satisfy suitable compatibility conditions. Here $\mathfrak{h}$ is the fiber type of L and $\{U_i\}_{i\in I}$ is a simple open cover of M. This data arises from the fact that locally there exist flat connections, namely Lie algebroid morphisms $TU_i \to A_{U_i}$. In this paper we show that locally PBG-algebroids have flat isometablic connections, and this gives rise to transition data which is suitably equivariant, modulo a simple connectivity assumption. Namely, we prove that:

Theorem 1.1. *Let $P(M, G)$ be a principal bundle whose structure group G is simply connected and $\{P_i \equiv U_i \times G\}_{i\in I}$ a local trivialisation of this bundle. If A is a PBG-algebroid over this bundle then there exist transition data (χ, α) such that*

(i) $\chi_{ij}(Xg) = \chi_{ij}(X)g$ *and*
(ii) $\alpha_{ij}(ug) = \alpha_{ij}(u)g$

for all $X \in TP_{ij}$, $u \in P_{ij}$ and $g \in G$.

When A is integrable though, the simple connectivity assumption is no longer necessary. It will be shown that:

Theorem 1.2. *For any integrable PBG-algebroid A over a principal bundle $P(M, G)$ there exist transition data (χ, α) which satisfy* (i) *and* (ii) *of Theorem* 1.1.

Note that the in order to prove the above result, the equivariance of the sections of the PBG-groupoid which integrates A is investigated, and this is an important step to the classification of PBG-groupoids.

Finally, we give an account of the holonomy of isometablic connections which leads to the proof of an Ambrose–Singer theorem for isometablic connections. The treatment of holonomy presented here follows the fashion of [6], i.e. we use the equivariant version of deformable sections. This method provides more information for the relation of isometablic connections of PBG-Lie algebra bundles with their sections.

The structure of this paper is as follows: Sections 2 and 3 are an account of the correspondence between extensions and PBG structures, in both the groupoid and the algebroid level. A number of examples is included, most of which are used in the proofs later in the paper. Section 4 gives the connection theory of PBG-groupoids and PBG-Lie algebra bundles. In Section 5 we give the proofs of Theorems 1.1 and 1.2. Section 6 is an account of the holonomy of isometablic connections in the fashion of [6], and Section 7 gives the proof of the Ambrose–Singer theorem for isometablic connections.

2. PBG-groupoids

Let us fix some conventions first. All manifolds considered in the paper are C^∞-differentiable, Hausdorff, paracompact and have a countable basis for their topology. We consider the arrows in a groupoid “from right to left”, i.e.

the source $s\xi$ of an element ξ in a groupoid Ω is considered to be on the right and its target $t\xi$ on the left. The object inclusion map of a groupoid Ω over a manifold M is denoted by $1 : M \to \Omega$, namely $x \mapsto 1_x$. In [6] the expression 'Lie groupoid' stood for a locally trivial differentiable groupoid; here we use the expression 'transitive Lie groupoid' for clarity.

Definition 2.1. A *PBG-groupoid* is a Lie groupoid $\Upsilon \rightrightarrows P$ whose base is the total space of a principal bundle $P(M, G)$ together with a right action of G on the manifold Υ such that for all $(\xi, \eta) \in \Upsilon \times \Upsilon$ such that $s\xi = t\eta$ and $g \in G$ we have:

(i) $t(\xi \cdot g) = t(\xi) \cdot g$ and $s(\xi \cdot g) = s(\xi) \cdot g$,
(ii) $1_{u\cdot g} = 1_u \cdot g$,
(iii) $(\xi\eta) \cdot g = (\xi \cdot g)(\eta \cdot g)$,
(iv) $(\xi \cdot g)^{-1} = \xi^{-1} \cdot g$.

We denote a PBG-groupoid Υ over the principal bundle $P(M, G)$ by $\Upsilon \rightrightarrows P(M, G)$ and the right-translation in Υ coming from the G-action by $\widetilde{R}_g$ for any $g \in G$. The right-translation in P will be denoted by R_g. The properties in the previous definition show that the group G of the base principal bundle $P(M, G)$ acts on Υ by automorphisms, namely $\widetilde{R}_g$ is an automorphism of the Lie groupoid Υ over the diffeomorphism R_g for all $g \in G$. A morphism φ of Lie groupoids between two PBG-groupoids Υ and Υ' over the same principal bundle is called a morphism of PBG-groupoids, if $\varphi \circ \widetilde{R}_g = \widetilde{R}'_g \circ \varphi$ for all $g \in G$. In the same fashion, a *PBG-Lie group bundle* (PBG-LGB) is a Lie group bundle F over the total space P of a principal bundle $P(M, G)$ such that the group G acts on F by Lie group bundle automorphisms. We denote a PBG-LGB by $F \to P(M, G)$. It is easy to see that the gauge group bundle $I\Upsilon$ of a PBG-groupoid $\Upsilon \rightrightarrows P(M, G)$ is a PBG-LGB. Now let us describe the correspondence of transitive PBG-groupoids with extensions of Lie groupoids. For any given extension of Lie groupoids (1), choose a basepoint and take its corresponding extension of principal bundles

$$N \rightarrowtail Q(M, H) \overset{\pi(id,\pi)}{\twoheadrightarrow} P(M, G).$$

That is to say that $\pi(id, \pi)$ is a surjective morphism of principal bundles and H is an extension of the Lie group G by N. This gives rise to the principal bundle $Q(P, N, \pi)$, which Mackenzie in [5] called the transverse bundle of the previous extension. The Lie groupoid $\Upsilon = \frac{Q\times Q}{N}$ corresponding to the transverse bundle admits the following action of G:

$$\langle v_2, v_1\rangle g = \langle v_2 h, v_1 h\rangle,$$

for any $v_2, v_1 \in Q$ and $g \in G$, where $h \in H$ is any element with $\tilde{\pi}(h) = g$. This action makes $\Upsilon \rightrightarrows P(M, G)$ a PBG-groupoid. On the other hand, writing a transitive PBG-groupoid as an exact sequence $I\Upsilon \rightarrowtail \Upsilon \twoheadrightarrow P \times P$, the fact that G acts by Lie groupoid automorphisms allows us to quotient the sequence over G and obtain the extension of Lie groupoids

$$\frac{I\Upsilon}{G} \rightarrowtail \frac{\Upsilon}{G} \twoheadrightarrow \frac{P \times P}{G}. \tag{2}$$

The two processes are mutually inverse (for the details of the proof see [5]). This establishes the following result:

Proposition 2.2. *Any transverse PBG-groupoid $\Upsilon \rightrightarrows P(M, G)$ corresponds exactly to an extension of Lie groupoids $F \rightarrowtail \Omega \twoheadrightarrow \frac{P\times P}{G}$.*

Let us give now some examples of PBG-groupoids, which will also be useful in later sections.

Example 2.3. Consider a principal bundle $P(M, G)$ and a Lie group H. Suppose given an action by automorphisms of G on H, say $(g, h) \mapsto R_g(h)$ for all $g \in G$ and $h \in H$. That is to say that $R_g : H \to H$ is an automorphism of H for all $g \in G$. Form the trivial groupoid $P \times H \times P \rightrightarrows P(M, G)$. This is easily seen to be a transitive PBG groupoid. We will refer to it as the *trivial PBG-groupoid* corresponding to the given action.

Example 2.4. For this example, first let us recall the definition of an action groupoid. When a Lie group G acts on a manifold M the action groupoid $M \rtimes G$ is the product manifold $M \times G$ with the following groupoid structure: $s(x,g) = x$, $t(x,g) = xg$, $1_x = (x, e_G)$. The multiplication is defined by $(xg,h)\cdot(x,g) = (x,gh)$ and the inverse of (x,g) is (xg, g^{-1}). The action groupoid is transitive if and only if the action of G on M is transitive. Now consider a principal bundle $P(M,G)$. There is an action of G on $P \rtimes G$ defined by $\widetilde{R}_g(u,h) = (ug, g^{-1}hg)$. This action makes the action groupoid a PBG-groupoid $P \rtimes G \rightrightarrows P(M,G)$.

Example 2.5. A PBG vector bundle is a vector bundle E over the total space P of a principal bundle $P(M,G)$ such that the structure group G acts on E by vector bundle automorphisms. We denote a PBG vector bundle by $E \to P(M,G)$. The isomorphisms between the fibers of an arbitrary vector bundle define a transitive Lie groupoid called the frame groupoid. This is the groupoid corresponding to the frame principal bundle induced by the vector bundle under consideration. The frame groupoid $\Phi(E)$ of a PBG vector bundle $E \to P(M,G)$ has a canonical PBG structure over $P(M,G)$. Namely, for any $g \in G$, an isomorphism $\xi : E_u \to E_v$ between two fibers defines an isomorphism $\widetilde{R}_g(\xi) : E_{ug} \to E_{vg}$ by

$$\widetilde{R}_g(\xi)(V) = \xi(V \cdot g^{-1}) \cdot g$$

for all $V \in E_{ug}$. Moreover, there is a canonical morphism of PBG-groupoids $\varepsilon : P \rtimes G \to \Phi(E)$ over the principal bundle $P(M,G)$. Namely, $\varepsilon(u,g)$ is the isomorphism $\widehat{R}_g : E_u \to E_{ug}$ induced by the action of G on E. It is straightforward to show that ε is indeed a morphism of Lie groupoids. It also preserves the actions because:

$$\begin{aligned}\varepsilon\big((u,h)\cdot g\big) &= \varepsilon(u\cdot g, g^{-1}hg) = \big[(\widehat{R}_{g^{-1}hg})_{u\cdot g} : E_{u\cdot g} \to E_{u\cdot hg}\big] \\ &= \widetilde{R}_g\big[(\widehat{R}_h)_u : E_u \to E_{u\cdot h}\big] = \widetilde{R}_g\big(\varepsilon(u,h)\big).\end{aligned}$$

Let us now give an example of a PBG-groupoid that arises from an extension of principal bundles.

Example 2.6. We consider the principal bundles $SU(2)(S^2, U(1), p)$ and $SO(3)(S^2, SO(2), p')$. For the first bundle, denote a typical element $\left[\begin{smallmatrix} s & t \\ -\bar{t} & \bar{s}\end{smallmatrix}\right]$ of $SU(2)$ such that $|s|^2 + |t|^2 = 1$ by (s,t). Regard $U(1)$ as a subgroup of $SU(2)$ by mapping every $z \in U(1)$ to $(z, 0)$ and let p be

$$(s,t) \mapsto \big(-2\,\mathrm{Re}(s\cdot t), -2\,\mathrm{Im}(s\cdot t), 1 - 2\cdot|t|^2\big).$$

For the second bundle, regard $SO(2)$ as a subgroup of $SO(3)$ by $A \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & 1\end{smallmatrix}\right]$ and let p' be $A \mapsto A \cdot e_3$ where (e_1, e_2, e_3) is the usual basis of $\mathbb{R}^3$. We define a morphism between these principal bundles making use of quaternions. Consider any $q = (s,t) \in SU(2)$ as a unit quaternion and every $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ as a vector quaternion. Then $q \cdot r \cdot q^{-1}$ is a vector quaternion, i.e. $q \cdot r \cdot q^{-1} \in \mathbb{R}^3$. For every $q = (s,t) \in SU(2)$ we define $A_q : \mathbb{R}^3 \to \mathbb{R}^3$ as $A_q(r) = q \cdot r \cdot q^{-1}$. This is an element in $SO(3)$. Let $\varphi : U(1) \to SO(2)$ be the restriction of A to $U(1)$. If we write a $z \in U(1)$ as $z = \mathrm{e}^{\imath\theta} = \cos\theta + \imath \sin\theta$. for some $\theta \in \mathbb{R}$, then it is easy to see that

$$\varphi(z) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta)\end{bmatrix} \in SO(2).$$

This shows both that φ is a surjective submersion and that its kernel is $\mathbb{Z}_2$. We therefore get the extension

$$\mathbb{Z}_2 \rightarrowtail SU(2)\big(S^2, U(1)\big) \overset{R(id_{S^2}, \phi)}{\twoheadrightarrow} SO(3)\big(S^2, SO(2)\big).$$

Its transverse bundle is $SU(2)(SO(3), \mathbb{Z}_2, A)$ and the PBG-groupoid it induces is

$$\frac{SU(2)\times SU(2)}{\mathbb{Z}_2} \rightrightarrows SO(3)\big(S^2, SO(2)\big),$$

where the action of $SO(2)$ on $\frac{SU(2)\times SU(2)}{\mathbb{Z}_2}$ is:

$$\big\langle (s_1, t_1), (s_2, t_2)\big\rangle \cdot z = \big\langle (s_1 z, t_1 \bar{z}), (s_2 z, t_2 \bar{z})\big\rangle.$$

It was shown in [6, II§7] that for any Lie group G and closed subgroup $H \leqslant G$ the gauge groupoid $\frac{G\times G}{H}$ is isomorphic to the action groupoid $\frac{G}{H} \rtimes G$, where G acts on $\frac{G}{H}$ via $(g, g'H) \mapsto (g \cdot g')H$. The isomorphism is

$$\langle g_1, g_2\rangle \mapsto \big(g_1 \cdot g_2^{-1}, g_2 H\big).$$

Therefore $\frac{SU(2)\times SU(2)}{\mathbb{Z}_2} \rightrightarrows SO(3)(S^2, SO(2))$ is isomorphic to the action PBG-groupoid $\frac{SU(2)}{\mathbb{Z}_2} \rtimes SU(2) \rightrightarrows SO(3)(S^2, SO(2))$. Note that $\frac{SU(2)}{\mathbb{Z}_2} \cong \mathbb{R}P^3$. The action of $SO(2)$ on $\mathbb{R}P^3 \rtimes SU(2)$ is

$$\big((w_1, w_2), (s, t) \cdot \mathbb{Z}_2\big) \cdot R_\theta = \big((w_1, w_2), (s \cdot z, t \cdot \bar{z}) \cdot \mathbb{Z}_2\big).$$

3. PBG-algebroids

This is an account of PBG structures on the algebroid level. We show that every PBG-Lie algebroid corresponds to an extension of Lie algebroids; the account in [5] gave only a partial result of this type. Some fundamental results are also proved, which will be useful in the study of the connection theory in later sections of this paper. Let us begin by describing the notion of a PBG structure on the algebroid level. We start with the special case of Lie algebra bundles.

Definition 3.1. A *PBG-Lie algebra bundle* (PBG-LAB) is a Lie algebra bundle L over the total space P of a principal bundle $P(M, G)$, together with an action of G on L such that each right-translation $\overline{R}_g : L \to L$ is a Lie algebra bundle automorphism over the right translation $R_g : P \to P$.

We denote a PBG-LAB by $L \to P(M, G)$. A morphism between two PBG-LABs over the same principal bundle is a morphism of LABs which preserves the group actions.

Example 3.2. Let $P(M, G)$ be a principal bundle and $\mathfrak{h}$ a Lie algebra which admits a right action $(V, g) \mapsto V \cdot g$ of G such that

$$[V, W] \cdot g = [V \cdot g, W \cdot g]$$

for all $V, W \in \mathfrak{h}$. Now the trivial LAB $P \times \mathfrak{h}$ admits the G-action $\overline{R}_g(u, V) = (ug, V \cdot g)$. This makes it a PBG-LAB, the *trivial PBG-LAB.*

Definition 3.3. A *Lie algebroid* is a vector bundle A on base M together with a vector bundle map $q : A \to TM$, called the *anchor* of A, and a bracket $[\,] : \Gamma A \times \Gamma A \to \Gamma A$ which is $\mathbb{R}$-bilinear, alternating, satisfies the Jacobi identity, and is such that

(i) $q([X, Y]) = [q(X), q(Y)]$,
(ii) $[X, uY] = u[X, Y] + q(X)(u)Y$

for all $X, Y \in \Gamma A$ and $u \in C^\infty(M)$.

Given a trivial PBG-LAB $P \times \mathfrak{h} \to P(M, G)$, recall that the Whitney sum vector bundle $TP \oplus (P \times \mathfrak{h})$ has a trivial Lie algebroid structure over P. The anchor map is the first projection and the Lie bracket on its sections is defined by the formula

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus \big\{X(W) - Y(V) + [V, W]\big\}$$

for all $X, Y \in \Gamma TP$ and smooth $\mathfrak{h}$-valued functions on V, W on P. The G-action on $P \times \mathfrak{h}$ defines a PBG structure on this Lie algebroid. Namely, with the notation used in the previous Example 3.2 define the following action of G on $TP \oplus (P \times \mathfrak{h})$:

$$\big(X \oplus (u, V), g\big) \mapsto \widehat{R}_g\big(X \oplus (u, V)\big) = T_u R_g(X) \oplus (ug, V \cdot g).$$

Denote by $\widehat{R}^\Gamma_g : \Gamma(TP \oplus (P \times \mathfrak{h})) \to \Gamma(TP \oplus (P \times \mathfrak{h}))$ the corresponding action on the sections of $TP \oplus (P \times \mathfrak{h})$. This is given by the formula

$$\widehat{R}^\Gamma_g (X \oplus V)_u = T_{ug^{-1}} R_g (X_{ug^{-1}}) \oplus (V_{ug^{-1}} \cdot g)$$

for all $X \in \Gamma TP$, $V \in C^\infty(P, \mathfrak{h})$ and $u \in P$. This action preserves the Lie bracket in $\Gamma(TP \oplus (P \times \mathfrak{h}))$ and is an action by automorphisms on the trivial Lie algebroid. This construction is an example of the notion of a PBG-algebroid.

Definition 3.4. A *PBG-algebroid* over the principal bundle $P(M, G)$ is a Lie algebroid A over P together with a right action of G on A denoted by $(X, g) \mapsto \widehat{R}_g(X)$ for all $X \in A$, $g \in G$ such that each $\widehat{R}_g : A \to A$ is a Lie algebroid automorphism over the right translation R_g in P.

We denote a PBG-algebroid A over $P(M, G)$ by $A \Rightarrow P(M, G)$. The action of G on A induces an action of G on ΓA, namely

$$X \cdot g = \widehat{R}_g \circ X \circ R_{g^{-1}}$$

for all $g \in G$ and $X \in \Gamma A$. The right-translation with respect to this action is denoted by $\widehat{R}^\Gamma_g : \Gamma A \to \Gamma A$ for all $g \in G$. With this notation Definition 3.4 implies that

$$\widehat{R}^\Gamma_g \big([X, Y]\big) = \big[\widehat{R}^\Gamma_g(X), \widehat{R}^\Gamma_g(Y)\big]$$

for all $X, Y \in \Gamma A$ and $g \in G$. The following result strengthens 3.2 of [5], where the quotient manifold was assumed to exist.

Proposition 3.5. *Let A be a transitive PBG-Lie algebroid on $P(M, G)$. Then the quotient manifold A/G exists and inherits a quotient structure of transitive Lie algebroid from A; further, it is an extension*

$$\frac{L}{G} \rightarrowtail \frac{A}{G} \twoheadrightarrow \frac{TP}{G}$$

of the Lie algebroid of the gauge groupoid of $P(M, G)$ by the quotient LAB L/G.

Proof. The main requirement is to prove that the manifold A/G exists. We apply the criterion of Godement [2]. Denote the projection $A \to P$ by p_A, and write

$$\Gamma' = \big\{(X, Xg) \mid X \in A,\ g \in G\big\};$$

we must show that Γ' is a closed submanifold of $A \times A$. Now $\Gamma' \subseteq (p_A \times p_A)^{-1}(\Gamma)$ where $\Gamma = \{(u, ug) \mid u \in P, g \in G\}$. Since Γ is a closed submanifold of $P \times P$, and $p_A \times p_A$ is a surjective submersion, it suffices to prove that Γ' is a closed submanifold of $(p_A \times p_A)^{-1}(\Gamma)$. Define

$$f : (p_A \times p_A)^{-1}(\Gamma) \to A, \qquad (X, Y) \mapsto Xg - Y,$$

where $p_A(Y) = p_A(X)g$. From the local triviality of A, it easily follows that f is a surjective submersion. The preimage of the zero section under p_A is Γ', and this shows that Γ' is a closed submanifold. Denote the quotient projection $A \to A/G$ by $\natural$.

The vector bundle structure of A quotients to A/G in a straightforward fashion. Since the anchor $q : A \to TP$ is G-equivariant, it quotients to a vector bundle morphism $A/G \to TP/G$ which is again a surjective submersion; denote this by π, and define $r = \tilde{q} \circ \pi$ where $\tilde{q}$ is the anchor of TP/G.

For the bracket structure of A/G, note first that $\Gamma(A/G)$ can be identified with the $C^\infty(M)$ module of G-equivariant sections of A as in the case of the Atiyah sequence of a principal bundle. Since the bracket on ΓA restricts to the G-equivariant sections by assumption, this bracket transfers to $\Gamma(A/G)$. It is now straightforward to check that this makes A/G a Lie algebroid on M with anchor r, and $\natural : A \to A/G$ a Lie algebroid morphism over p. That π is a Lie algebroid morphism with kernel L/G is easily checked. □

It is a straightforward exercise to verify that PBG-groupoids differentiate to PBG-algebroids. The next result, which appears in [5], gives a converse.

Theorem 3.6. *Let Υ be an s-simply connected locally trivial Lie groupoid with base the total space of a principal bundle $P(M,G)$. Suppose that for all $g \in G$ there is given a Lie algebroid automorphism $\widetilde{R}_g : A\Upsilon \to A\Upsilon$ which defines the structure of a PBG-algebroid on $A\Upsilon$. Then there is a natural structure of PBG-groupoid on Υ which induces on $A\Upsilon$ the given PBG-Lie algebroid structure.*

The examples which follow will be useful later on.

Example 3.7. Consider a transitive Lie algebroid A over a manifold M such that $\pi_1(M) \neq 0$. Let $\widetilde{M}$ be the covering space of M and $p : \widetilde{M} \to M$ the covering projection. Then we have the principal bundle $\widetilde{M}(M, \pi_1(M), p)$. Denote $p^{!!}A$ the pullback

$$\begin{tikzcd} p^{!!}A \arrow[r, "p^*"] \arrow[d, "q'"'] & p^*A \arrow[d, "p^*(q)"] \\ T\widetilde{M} \arrow[r, "T(p^*)"] & p^*TM \end{tikzcd}$$

Sections of $p^{!!}A$ are of the form $X' \oplus C$, where $X' \in \Gamma T\widetilde{M}$, $C \in \Gamma(p^*A)$ and

$$T(p^*)(X') = p^*(q)(C).$$

This construction is described in [4], but let us recall here the Lie algebroid structure of $p^{!!}A$. If we write $C = \sum u_i(X_i \circ p)$ with $u_i \in C(\widetilde{M})$ and $X_i \in \Gamma A$, then

$$p^*(q)(C) = \sum u_i\big(q(X_i \circ p)\big)$$

with $T(p) \circ X' = \sum u_i(q(X_i \circ p))$. We define a Lie algebroid structure on $p^{!!}A$ with q' as the anchor map and bracket given by

$$\begin{aligned} &\Big[X' \oplus \sum u_i(X_i \circ p), Y' \oplus \sum u_j(Y_j \circ p)\Big] \\ &\quad = [X', Y'] \oplus \Big\{\sum u_i u_j\big([X_i, Y_j] \circ p\big) + \sum X'(u_j)(Y_j \circ p) - \sum Y'(u_i)(X_i \circ p)\Big\}. \end{aligned}$$

Now the pullback Lie algebroid $p^{!!}A$ is a PBG-algebroid over $\widetilde{M}(M, \pi_1(M), p)$.

Example 3.8. Consider the action PBG-groupoid $P \rtimes G \rightrightarrows P(M,G)$ constructed in Example 2.4. This differentiates to the *action PBG-algebroid* $P \rtimes \mathfrak{g} \Rightarrow P(M,G)$. Here $P \rtimes \mathfrak{g}$ is the product manifold $P \times \mathfrak{g}$ and the anchor is the map that associates to every $V \in \mathfrak{g}$ the fundamental vector field $V^\dagger \in \Gamma TP$. The Lie bracket is given by the formula

$$[V, W] = V^\dagger(W) - W^\dagger(V) + [V, W]^\bullet, \tag{3}$$

where $[V, W]^\bullet$ stands for the point-wise bracket in $\mathfrak{g}$. The G-action that makes it a PBG-algebroid is of course

$$\widehat{R}_g(u, V) = \big(ug, \mathrm{Ad}_{g^{-1}}(V)\big).$$

Example 3.9. Given a PBG-LAB $L \to P(M,G)$, let $\Phi[L]$ denote the groupoid of Lie algebra isomorphisms between the fibers of L. This is a PBG-groupoid over $P(M,G)$ in the same way as $\Phi(L)$ (see Example 2.5). It was shown in [6] that $\Phi[L]$ differentiates to the Lie algebroid CDO$[L]$ over P. The notation CDO stands for “covariant differential operator”. It is a transitive Lie algebroid and its sections are those first or zeroth order differential operators $D : \Gamma L \to \Gamma L$ such that:

(i) For all $D \in \Gamma\,\mathrm{CDO}[L]$ there is a vector field $\sharp(D) \in \Gamma TP$ such that

$$D(f\mu) = fD(\mu) + \sharp(D)(f)\mu$$

for every $\mu \in \Gamma L$ and $f \in C^\infty(P)$.

(ii) The operators D act as derivations of the bracket, i.e.

$$D\big([\mu_1,\mu_2]\big)=\big[D(\mu_1),\mu_2\big]+\big[\mu_1,D(\mu_2)\big]$$

for all $\mu_1,\mu_2\in\Gamma L$.

The anchor map of CDO$[L]$ is exactly the map $\sharp$ established in (i). The adjoint bundle of this Lie algebroid is the LAB of endomorphisms of L which are derivations of the bracket. We therefore have the exact sequence

$$\mathrm{Der}(L)\rightarrowtail \mathrm{CDO}[L]\stackrel{\sharp}{\twoheadrightarrow} TP.$$

The action of G on $\Phi[L]$ differentiates to the action $(D,g)\mapsto R_g^{\mathrm{CDO}}(D)$ on the sections of CDO$[L]$ defined by

$$\big[R_g^{\mathrm{CDO}}(D)\big](\mu)=\overline{R}_g\circ D(\overline{R}_{g^{-1}}\circ\mu\circ R_g)\circ R_{g^{-1}}$$

for all $g\in G$, $D\in\Gamma\,\mathrm{CDO}[L]$ and $\mu\in\Gamma L$. (Recall that $\overline{R}_g$ denotes the right-translation on the sections of L.) This action makes CDO$[L]$ a PBG-algebroid over $P(M,G)$.

In general, CDO$[L]$ is the analogue of the automorphism group in the Lie algebroid framework. In the PBG setting, we have the following definition:

Definition 3.10. Let $A\Rightarrow P(M,G)$ be a (transitive) PBG-algebroid and $K\to P(M,G)$ a PBG-LAB. An *equivariant representation* of A on K is an equivariant Lie algebroid morphism $\rho: A\to\mathrm{CDO}[L]$.

Proposition 3.11. *Let $A\Rightarrow P(M,G)$ be a transitive PBG-algebroid. Then its adjoint bundle L is a PBG-LAB and* CDO$[L]$ *a PBG-algebroid, both over the principal bundle $P(M,G)$.*

Proof. Since A is transitive we have the exact sequence of Lie algebroids

$$L\stackrel{j}{\rightarrowtail} A\stackrel{\sharp}{\twoheadrightarrow} TP.$$

Let $(V,g)\mapsto\overline{R}_g(V)$ denote the action of G on ΓL defined by $j(\widehat{R}_g^{\Gamma}(V))=\widehat{R}_g^{\Gamma}(j(V))$. The result now follows from the fact that A is a PBG-algebroid. □

A Lie algebroid morphism $\varphi: A\to A'$ between two PBG-algebroids over the same principal bundle $P(M,G)$ is a *morphism of PBG-algebroids* if $\varphi\circ\widehat{R}_g=\widehat{R}_g\circ\varphi$ for all $g\in G$. It is easy to verify that morphisms of PBG-groupoids differentiate to PBG-algebroids.

Proposition 3.12. *Let Υ and Υ' be PBG-groupoids over the same principal bundle $P(M,G)$ such that Υ is s-simply connected. Then every morphism of PBG-algebroids $\varphi_*: A\Upsilon\to A\Upsilon'$ integrates to a unique morphism of PBG-groupoids $\varphi:\Upsilon\to\Upsilon'$.*

Proof. It was shown in [8] that the Lie algebroid morphism φ_* integrates uniquely to a morphism of Lie groupoids $\varphi:\Upsilon\to\Upsilon'$. It suffices to show that φ respects the G-actions. For every $g\in G$ let $\widetilde{R}_g$ and $\widetilde{R}'_g$ be the right-translations induced by the actions of G on Υ and Υ' respectively. These differentiate to the right-translations $\widehat{R}_g$ and $\widehat{R}'_g$ respectively on the Lie algebroid level. Since φ_* is a morphism of PBG-algebroids, for every $g\in G$ we have $\widehat{R}'_g\circ\varphi\circ\widehat{R}_{g^{-1}}=\varphi_*$. The morphism of Lie groupoids $\widehat{R}'_g\circ\varphi\circ\widehat{R}_{g^{-1}}$ differentiates to $\widehat{R}'_g\circ\varphi_*\circ\widehat{R}_g$. The uniqueness of φ yields $\widetilde{R}'_g\circ\varphi\circ\widetilde{R}_{g^{-1}}=\varphi$ for all $g\in G$. □

Example 3.13. The natural morphism of PBG-groupoids arising from a PBG-LAB $L\to P(M,G)$ differentiates to a morphism of PBG-algebroids $\varepsilon_*: P\succ\mathfrak{g}\to\mathrm{CDO}[L]$. Such morphisms are known as "derivative representations" in the context of Poisson geometry. We will refer to this one as the *natural derivative representation* induced by L.

A Lie subalgebroid A' of a PBG-algebroid A is a PBG-subalgebroid if the inclusion $A'\rightarrowtail A$ is a morphism of PBG-algebroids.

Proposition 3.14. *If $A \Rightarrow P(M,G)$ is a transitive PBG-algebroid with adjoint bundle L then* $\mathrm{ad}(A)$ *is a PBG-subalgebroid of* $\mathrm{CDO}[L]$.

Proof. The Lie algebroid $\mathrm{ad}(A)$ was shown to be a Lie subalgebroid of $\mathrm{CDO}[L]$ in [6]. It suffices to show that it is closed under the G-action defined in Example 3.9. Indeed, suppose $X \in \Gamma A$. Then:

$$\begin{aligned}\big[R_g^{\mathrm{CDO}}(\mathrm{ad}_X)\big](V) &= \overline{R}_g \circ \mathrm{ad}_X(\overline{R}_{g^{-1}} \circ V \circ R_g) \circ R_{g^{-1}} \\ &= \overline{R}_g \circ \big[X, \overline{R}_{g^{-1}} \circ V \circ R_g\big] \circ R_{g^{-1}} = \big[\overline{R}_g(X), V\big] = \mathrm{ad}_{\overline{R}_g(X)}(V)\end{aligned}$$

for all $V \in \Gamma L$. So, $R_g^{\mathrm{CDO}}(\mathrm{ad}_X) \in \mathrm{ad}(A)$. $\square$

The Lie algebroid $\mathrm{ad}(L)$ is integrable as a subalgebroid of the integrable Lie algebroid $\mathrm{CDO}[L]$. We denote the α-connected Lie groupoid it integrates to by $\mathrm{Int}(A)$, following the notation of [6]. The proof of the following statement is immediate.

Corollary 3.15. *If $A \Rightarrow P(M,G)$ is a transitive PBG-algebroid then* $\mathrm{Int}\,A \rightrightarrows P(M,G)$ *is a PBG-subalgebroid of* $\Phi[L]$.

We can also consider $\mathrm{ad}(A)$ as the image of the adjoint representation $\mathrm{ad}: A \to \mathrm{CDO}[L]$. In this sense, ad is a representation of PBG algebroids, i.e. a morphism of PBG-algebroids.

Definition 3.16. Let A be a PBG-algebroid and L a PBG-LAB, both over the same principal bundle $P(M,G)$. A representation $\rho: A \to \mathrm{CDO}[L]$ of A in L is a *representation of PBG-algebroids* if it is a morphism of PBG-algebroids.

In particular, for all $g \in G$, $X \in \Gamma A$ and $\mu \in \Gamma L$ it is required:

$$\big[R_g^{\mathrm{CDO}}\big(\rho(X)\big)\big](\mu) = \rho\big(\widehat{R}_g^{\Gamma}(X)\big)(\mu).$$

The developed form of this formula is:

$$\overline{R}_g \circ \rho(X)(\overline{R}_{g^{-1}} \circ \mu \circ R_g) = \rho\big(\widehat{R}_g^{\Gamma}(X)\big)(\mu) \circ R_g.$$

4. Infinitesimal connections for PBG structures

Lie groupoids and Lie algebroids provide a natural framework for the study of connection theory. For example, given a principal bundle $P(M,G,p)$, consider its corresponding Atiyah sequence, namely the exact sequence of vector bundles

$$\frac{P \times \mathfrak{g}}{G} \rightarrowtail \frac{TP}{G} \overset{p_*}{\twoheadrightarrow} TM$$

(where the action of G on $\mathfrak{g}$ implied is simply the adjoint), together with the bracket of sections of the vector bundle $\frac{TP}{G}$ which is obtained by identifying them with G-invariant vector fields on P. This bracket is preserved by the vector bundle morphism p_*. Thus the Atiyah sequence of a principal bundle is a transitive Lie algebroid. In this setting, infinitesimal connections of the principal bundle $P(M,G,p)$ correspond exactly to the right-splittings of the Atiyah sequence. The holonomy of such connections is studied using the global object corresponding to the Atiyah sequence and that is the transitive Lie groupoid corresponding to the principal bundle we started with, namely the quotient manifold $\frac{P \times P}{G}$ (over the diagonal action).

Consider a transitive PBG-algebroid

$$L \rightarrowtail A \twoheadrightarrow TP \tag{4}$$

over the principal bundle $P(M,G)$ and form the associated extension of Lie algebroids

$$\frac{L}{G} \rightarrowtail \frac{A}{G} \twoheadrightarrow \frac{TP}{G}. \tag{5}$$

We will write $\Phi = \frac{P \times P}{G}$ and $B = \frac{A}{G}$, $K = \frac{L}{G}$. As vector bundles $L \cong p^* K$, $A \cong p^* B$ and $TP \cong p^*(A\Phi)$. These are actually isomorphisms of Lie algebroids with respect to the action Lie algebroid structures on the pullback bundles; see [5] for this. A connection (or an infinitesimal connection if emphasis is required) in A is a right-inverse $\gamma : TP \to A$ to (4). If γ is equivariant with respect to the G actions, that is, if

$$\gamma \circ TR_g = \widehat{R}_g \circ \gamma \tag{6}$$

for all $g \in G$, then γ quotients to a linear map $\gamma^G : A\Phi \to B$ which is right-inverse to (5). Conversely, given a right-inverse $\sigma : A\Phi \to B$ to (5), the pullback map $p^*\sigma$ is a connection in A which satisfies (6). Connections in A satisfying (6) are not equivariant in the standard sense of, for example, [3], and in order to distinguish from the standard notion we call these connections *isometablic*. This is an adaptation of the Greek translation of the Latin word "equivariant".

Definition 4.1. Let $A \Rightarrow P(M, G, p)$ be a transitive PBG-algebroid. A connection $\gamma : TP \to A$ is called *isometablic*, if it satisfies (6).

Writing a transitive PBG-algebroid as an extension of PBG-algebroids $L \overset{\iota}{\rightarrowtail} A \overset{\natural}{\twoheadrightarrow} TP$, the curvature of any connection γ in A is the 2-form $\Omega_\gamma : TP \times TP \to L$ defined by

$$\Omega_\gamma(X, Y) = \gamma\big([X, Y]\big) - \big[\gamma(X), \gamma(Y)\big].$$

If γ is isometablic then its curvature also preserves the actions, namely for all $g \in G$ we have

$$\Omega_\gamma \circ (TR_g \times TR_g) = \overline{R}_g \circ \Omega_\gamma.$$

An isometablic *back connection* of $A \Rightarrow P(M, G)$ is a morphism of vector bundles $\omega : A \to L$ such that $\omega \circ \iota = id_L$ and $\iota \circ \omega \circ \widehat{R}_g = \overline{R}_g \circ \iota \circ \omega$ for all $g \in G$. Isometablic connections correspond to isometablic back connections via the formula

$$\iota \circ \omega + \gamma \circ \natural = id_A.$$

Example 4.2. Consider the PBG-algebroid $TP \oplus (P \times \mathfrak{h})$ over a principal bundle $P(M, G)$ constructed in Section 3. The connection $\gamma^0 : TP \to TP \oplus (P \times \mathfrak{h})$ defined by $X \mapsto X \oplus 0$ for all $X \in TP$ is isometablic and flat. This is the *standard flat connection*.

If $\varphi : A \to A'$ is a morphism of PBG-algebroids over the same principal bundle and γ is an isometablic connection in A then $\varphi \circ \gamma$ is also an isometablic connection in A'. Any connection in a transitive PBG-algebroid $A \Rightarrow P(M, G)$ gives rise to a Koszul connection ∇ of the adjoint bundle L. Namely, define $\nabla^\gamma : \Gamma TP \times \Gamma L \to \Gamma L$ by

$$\nabla^\gamma_X(V) = \big[\gamma(X), \iota(V)\big]$$

for all $X \in \Gamma TP$ and $V \in \Gamma L$. This is the *adjoint connection* induced by γ. If γ is an isometablic connection, then its adjoint connection satisfies

$$\begin{aligned} \overline{R}_g\big(\nabla^\gamma_X(V)\big) &= \overline{R}_g\big(\big[\gamma(X), \iota(V)\big]\big) = \big[\overline{R}_g\big(\gamma(X)\big), \overline{R}_g\big(\iota(V)\big)\big] \\ &= \big[\gamma\big(TR_g(X)\big), \iota\big(\overline{R}_g(V)\big)\big] = \nabla^\gamma_{TR_g(X)}\big(\overline{R}_g(V)\big) \end{aligned}$$

for all $X \in \Gamma TP$ and $V \in \Gamma L$. The curvature of γ satisfies:

$$\odot\big\{\nabla^\gamma_X\big(\Omega_\gamma(Y, Z)\big) - \Omega_\gamma\big([X, Y], Z\big)\big\} = 0$$

for all $X, Y, Z \in \Gamma TP$. Here $\odot$ denotes the sum over all the permutations of X, Y and Z. This is the (second) *Bianchi identity*.

Proposition 4.3. *Suppose given a transitive PBG-algebroid $A \Rightarrow P(M, G, p)$ and consider its corresponding extension of Lie algebroids over M. The connections of the* (*transitive*) *Lie algebroid $\frac{A}{G} \to M$ are equivalent to the isometablic connections of A which vanish on the kernel T^pP of $Tp : TP \to TM$.*

Proof. Consider an isometablic connection $\gamma : TP \to A$ such that $\gamma(X) = 0$ if $X \in T^pP$. This quotients to a splitting $\gamma^{/G} : \frac{TP}{G} \to \frac{A}{G}$. Given a connection $\delta : TM \to \frac{TP}{G}$ of the principal bundle $P(M,G)$, define

$$\widetilde{\gamma} = \gamma^{/G} \circ \delta : TM \to \frac{A}{G}.$$

The assumption that γ vanishes on the kernel of Tp makes the definition of $\widetilde{\gamma}$ independent from the choice of δ. It follows immediately from the assumption that δ is a connection of $P(M,G)$ and $\gamma^{/G}$ is a splitting of (5) that this is a connection of the Lie algebroid $\frac{A}{G}$.

Conversely, given a connection $\theta : TM \to \frac{A}{G}$ of the Lie algebroid $\frac{A}{G}$, compose it with the anchor map $p^* : \frac{TP}{G} \twoheadrightarrow TM$ of the Atiyah sequence corresponding to the bundle $P(M,G,p)$ to the vector bundle morphism

$$\bar{\theta} = \theta \circ p^* : \frac{TP}{G} \to \frac{A}{G}.$$

Denote $\natural : TP \to \frac{TP}{G}$ and $\natural^A : A \to \frac{A}{G}$ the natural projections. Since $\natural^A$ is a pullback over $p : P \to M$, there is a unique vector bundle morphism $\gamma : TP \to A$ such that

$$\natural^A \circ \gamma = \bar{\theta} \circ \natural.$$

Due to the G-invariance of $\natural$ and $\natural^A$the morphism of vector bundles $\widehat{R}_{g^{-1}} \circ \gamma \circ TR_g$ also satisfies the previous equation for every $g \in G$, therefore it follows from the uniqueness argument that γ is isometablic. It is an immediate consequence of the previous equation that γ vanishes at T^pP.

To see that it is indeed a connection of A, let us recall the fact that θ is a connection of $\frac{A}{G}$. This gives $p^* \circ \sharp^{/G} \circ \theta = id_{TM}$. Now $\sharp^{/G} = \natural \circ \sharp$ and by definition we have $p^* \circ \natural = Tp$, therefore $Tp \circ \sharp \circ \theta = id_{TM}$. Now take an element $X \in TP$. Then $Tp(X) \in TM$, and it follows from this equation that there exists an element $g \in G$ such that

$$(\sharp \circ \theta)\big(Tp(X)\big) = X \cdot g.$$

Multiplying this by g^{-1} and using the G-invariance of Tp we get

$$\sharp \circ (\theta \circ Tp) = id_{TP}.$$

Finally, from the properties of the pullback, it follows immediately that γ is the map $(\pi, \bar{\theta} \circ \natural)$, where $\pi : TP \to P$ is the natural projection of the tangent bundle. It is straightforward to check that this reformulates to $(\pi, \theta \circ Tp)$, and this proves that γ is a connection. □

Corollary 4.4. *Let $A \Rightarrow P(M,G)$ be a transitive PBG-algebroid. A flat connection of the Lie algebroid $\frac{A}{G} \to M$ gives rise to a unique flat connection of A which vanishes on the kernel of Tp.*

Definition 4.5. An *isometablic Koszul connection* of the PBG-LAB $L \Rightarrow P(M,G)$ is a vector bundle morphism $\nabla : TP \to \mathrm{CDO}[L]$ such that

$$\nabla_{TR_g(X)}\big(\overline{R}_g(V)\big) = \overline{R}_g\big(\nabla_X(V)\big)$$

for all $g \in G$, $X \in \Gamma TP$ and $V \in \Gamma L$.

Now let us give a few examples of such connections.

Example 4.6. Suppose $P \times \mathfrak{h} \Rightarrow P(M,G)$ is a trivial PBG-LAB (in the sense of Example 3.2). Then the *Koszul connection* $\nabla^0 : \Gamma TP \to \Gamma\, \mathrm{CDO}[P \times \mathfrak{h}]$ defined by $\nabla^0_X(V) = X(V)$ for every $X \in \Gamma TP$, $V \in C^\infty(P, \mathfrak{h})$ is isometablic and flat. To verify this, note that it is the adjoint connection of the standard flat connection γ^0 introduced in Example 4.2.

Further examples arise when we consider the Hom functor. Now we present some of them, which are necessary for the proof of the Ambrose–Singer theorem in Section 7.

Suppose $L \to P(M,G)$ and $L' \to P(M,G)$ are PBG-Lie algebra bundles. The vector bundle $\mathrm{Hom}^n(L,L') \to P$ is a Lie algebra bundle with Lie bracket

$$[\varphi_1,\varphi_2](\mu_1,\dots,\mu_n) = \big[\varphi_1(\mu_1,\dots,\mu_n),\varphi_2(\mu_1,\dots,\mu_n)\big]$$

for all $\varphi_1,\varphi_2 \in \mathrm{Hom}^n(L,L')$ and $\mu_1,\dots,\mu_n \in \Gamma L$. It also admits the following action of G:

$$\big[R_g^{Hom}(\varphi)\big](\mu_1,\dots,\mu_n) = \varphi\big(R_g(\mu_1),\dots,R_g(\mu_n)\big)$$

for all $g \in G$, $\varphi \in \mathrm{Hom}(L,L')$ and $\mu_1,\dots,\mu_n \in \Gamma L$. With this bracket and this action, $\mathrm{Hom}(L,L')$ becomes a PBG-LAB. At this point, let us recall the definition of an equivariant represention of a PBG-groupoid on a PBG-vector bundle (for the definition of a PBG-vector bundle see Example 2.5).

Definition 4.7. Let $\Upsilon \rightrightarrows P(M,G)$ be a transitive PBG-groupoid and $E \xrightarrow{\pi} P(M,G)$ be a PBG-vector bundle (both over the same principal bundle $P(M,G)$). An *equivariant representation of* Υ *on* E is a smooth map $\Upsilon * E \to E$ such that:

(i) $\pi(\xi\cdot v)) = t(\xi)$ for all $(\xi,v) \in \Upsilon * E$.
(ii) $(\xi_1\xi_2)\cdot v = \xi_1\cdot(\xi_2,v)$ for all $(\xi_1,\xi_2) \in \Upsilon * \Upsilon$, $(\xi_2,v) \in \Upsilon * E$.
(iii) $1_{\pi(v)}\cdot v = v$ for all $v \in E$.
(iv) $(\xi g)\cdot(vg) = (\xi\cdot v)\cdot g$.

The set $\Upsilon * E$ consists of the pairs $(\xi,v) \in \Upsilon \times E$ such that $\alpha(\xi) = \pi(v)$.

Consider now the following representations of Lie groupoids:

(i) $\Phi[L] * \mathrm{Hom}^n(L, P\times\mathbb{R}) \to \mathrm{Hom}^n(L,P\times\mathbb{R})$ defined by

$$(\xi,\varphi) \mapsto \varphi\circ(\xi^{-1})^n$$

(ii) $\Phi[L] * \mathrm{Hom}^n(L,L) \to \mathrm{Hom}^n(L,L)$ defined by

$$\big((\xi,\xi'),\varphi\big) \mapsto \xi'\circ\varphi\circ(\xi^{-1})^n$$

(iii) $(\Phi[L]\times_{P\times P}\Phi[L']) * \mathrm{Hom}^n(L,L') \to \mathrm{Hom}^n(L,L')$ defined by

$$\big((\xi,\xi'),\varphi\big) \mapsto \xi'\circ\varphi\circ(\xi^{-1})^n.$$

Note that the elements of the Lie groupoid $(\Phi[L]\times_{P\times P}\Phi[L]) \rightrightarrows P(M,G)$ are of the form

$$(\xi : E_u \to E_v, \xi' : E_u \to E_v)$$

for $u,v \in P$. The Lie group G acts on it by

$$(\xi,\xi')\cdot g = \big(\widehat{R}_g\circ\xi\circ\widehat{R}_{g^{-1}}, \widehat{R}_g\circ\xi'\circ\widehat{R}_{g^{-1}}\big)$$

for all $g \in G$, and under this action it is a PBG-groupoid. In [6] it is proved that these representations are smooth and they induce the following representations of Lie algebroids respectively:

(i) $\mathrm{CDO}[L] \to \mathrm{CDO}[\mathrm{Hom}^n(L,P\times\mathbb{R})]$ defined by

$$X(\varphi)(\mu_1,\dots,\mu_n) = q(X)\big(\varphi(\mu_1,\dots,\mu_n)\big) - \sum_{i=1}^n \varphi\big(\mu_1,\dots,X(\mu_i),\dots,\mu_n\big).$$

(ii) $\mathrm{CDO}[L] \to \mathrm{CDO}[\mathrm{Hom}(L,L)]$ defined by

$$X(\varphi)(\mu_1,\dots,\mu_n) = X\big(\varphi(\mu_1,\dots,\mu_n)\big) - \sum_{i=1}^n \varphi\big(\mu_1,\dots,X(\mu_i),\dots,\mu_n\big).$$

(iii) $\mathrm{CDO}[L] \oplus \mathrm{CDO}[L'] \to \mathrm{CDO}[\mathrm{Hom}^n(L, L')]$ defined by

$$\big[(X \oplus X')(\varphi)\big](\mu) = X'\big(\varphi(\mu)\big) - \varphi\big(X(\mu)\big).$$

The definition of the above representations and the fact that the standard flat connection is isometablic show that these representations are equivariant. Therefore if ∇, ∇' are isometablic Koszul connections on L, L' respectively then through the previous representations one gets isometablic Koszul connections on $\mathrm{Hom}^n(L, P \times \mathbb{R})$, $\mathrm{Hom}^n(L, L)$ and $\mathrm{Hom}(L, L')$.

5. Equivariant transition data

This section gives the proof of Theorems 1.1 and 1.2.

Proposition 5.1. *Let $L \to P(M, G)$ be a PBG-LAB. If $P(M, G)$ has a flat connection and M is simply connected, then $L \Rightarrow P(M, G)$ has a flat isometablic Koszul connection.*

Proof. If $P(M, G)$ has a flat connection and M is simply connected, then it is isomorphic to the trivial bundle $M \times G(M, G, pr_1)$. So it suffices to prove that every PBG-LAB over a trivial principal bundle has a flat isometablic Koszul connection.

The action $\delta : (M \times G) \times G \to M \times G$ is $((x, g), h) \mapsto (x, gh)$ for all $x \in M$ and $g, h \in G$. In other words, G only acts on itself by right translations. For every $(x, g) \in M \times G$, the partial map $\delta_{(x,g)} : G \to M \times G$ is $\delta_{(x,g)} = (\mathrm{const}_x, \ell_g)$, where $\ell_g : G \to G$ denotes the left translation in G. Now form the action groupoid $(M \times G) \succ G \rightrightarrows M \times G$. This differentiates to the Lie algebroid $(M \times G) \times \mathfrak{g}$ with Lie bracket the one given by (3). Here the fundamental vector field $V^\dagger$ corresponding to a smooth map $V : M \times G \to \mathfrak{g}$ is

$$\begin{aligned} V^\dagger_{(x,g)} &= T_{(x,e)}\delta_{(x,g)}(V_{(x,g)}) = T_{(x,e)}(\mathrm{const}_x, \ell_g)(V_{(x,g)}) \\ &= 0 \oplus T_e\ell_g(V_{(x,g)}) = T_e\ell_g(V_{(x,g)}) \end{aligned}$$

for all $(x, g) \in M \times G$. Since the action of G on $M \times G$ leaves M unaffected, let us concentrate on the action of G on itself by left translations. Form the action groupoid $G \succ G \rightrightarrows G$. The map $\varphi : G \succ G \to G \times G$ defined by $(g, h) \overset{\varphi}{\mapsto} (g, gh)$ for all $g, h \in G$ makes the action groupoid isomorphic to the pair groupoid $G \times G \rightrightarrows G$. On the Lie algebroid level this differentiates to the isomorphism of Lie algebroids $\varphi_* : G \succ \mathfrak{g} \to TG$ given by

$$(g, V) \xmapsto{\varphi_*} T_e\ell_g(V)$$

for all $g \in G$ and $V : G \to \mathfrak{g}$. Since it is an isomorphism of Lie algebroids, its inverse maps the (usual) Lie bracket of TG to the Lie bracket in $G \times \mathfrak{g}$ defined by (3).

Now consider the derivative representation $\varepsilon_* : (M \times G) \succ \mathfrak{g} \to \mathrm{CDO}[L]$ introduced in Example 3.13. Since the action of G on $M \times G$ does not affect M, we can consider the Lie algebroid of the action groupoid to be $M \times (G \succ \mathfrak{g})$. Finally, define $\nabla : \Gamma TM \times \Gamma TG \to \Gamma\,\mathrm{CDO}[L]$ to be

$$\nabla_X(V) = \varepsilon_*\big(X\big(\varphi_*^{-1}(V)\big)\big).$$

This is a morphism of Lie algebroids because both φ_*^{-1} and ε_* are. Therefore it is a flat Koszul connection. Its isometablicity is immediate. □

Proposition 5.2. *If the PBG-LAB $L \to P(M, G)$ has a flat isometablic Koszul connection and P is simply connected, then it is isomorphic to a trivial PBG-LAB* (*as it was introduced in Example* 3.2).

Proof. Suppose $\nabla : TP \to \mathrm{CDO}[L]$ is a flat isometablic Koszul connection. Then it is a morphism of PBG-algebroids, and since P is simply connected it can be integrated to a morphism of PBG-groupoids $\varphi : P \times P \to \Phi[L]$ (see Proposition 3.12). Choose a $u_0 \in P$, define $\mathfrak{h} = L_{u_0}$ and consider the following representation of G on $\mathfrak{h}$:

$$\rho_*(g)(V) = \varphi(u_0, u_0 \cdot g^{-1})\big(\overline{R}_{g^{-1}}(V)\big),$$

for all $V \in \mathfrak{h}$ and $g \in G$. This map is easily shown to be a representation, to preserve the Lie bracket of $\mathfrak{h}$ because $\overline{R}_g([V,W]) = [\overline{R}_g(V), \overline{R}_g(V)]$ for all $V, W \in \mathfrak{h}$, and $\varphi(u_0, u_0 \cdot g^{-1}) \in \Phi[L]$. Now, the product $P \times \mathfrak{h}$ becomes a PBG-LAB with action

$$(u, V) \cdot g = \big(u \cdot g, \rho_*(g^{-1})(V)\big).$$

Consider the map $\Psi : P \times \mathfrak{h} \to L$ defined as $\Psi(u, V) = \varphi(u, u_0)(V)$. This is clearly an isomorphism of Lie algebra bundles. Moreover,

it preserves the action because:

$$\begin{aligned}\Psi\big((u,V)\cdot g\big) &= \varphi(u\cdot g, u_0)\big(\varphi(u_0, u_0\cdot g)\big(\overline{R}_g(V)\big)\big) = \varphi(u\cdot g, u_0\cdot g)\big(\overline{R}_g(V)\big)\\ &= \big[\overline{R}_g \circ \varphi(u, u_0)\circ \overline{R}_{g^{-1}} \circ \overline{R}_g\big](V) = \varphi(u, u_0)(V)\cdot g = \overline{R}_g\big(\Psi(u, V)\big)\end{aligned}$$

for all $(u, V) \in P \times \mathfrak{h}$ and $g \in G$. $\square$

The previous two propositions prove the existence of a special trivialization for a certain class of PBG-LABs which takes into account the group action. Namely, suppose $L \to P(M, G)$ is a PBG-LAB such that the Lie group G is simply connected. Choose a cover $\{P_i\}_{i\in I}$ by principal bundle charts. That is, for every $i \in I$ there is an open subset $U_i \subseteq M$ such that $P_i \cong U_i \times G$. Without harm to the generality we may consider U_i to be contractible, therefore P_i will be simply connected. Then, Proposition 5.1 shows that the PBG-LAB $L_{P_i} \to P_i(U_i, G)$ has a flat isometablic Koszul connection. Now Proposition 5.2 shows that there are Lie algebras $\mathfrak{h}_i$ acted upon by G and isomorphisms of PBG-LABs $\psi_i : P_i \times \mathfrak{h}_i \to L_{P_i}$. If we choose one of these Lie algebras $\mathfrak{h}$ and consider the composition of ψ_i with a chosen isomorphism $\mathfrak{h} \cong \mathfrak{h}_i$, we obtain a trivialization for L which respects the group action.

Proposition 5.3. *Suppose $L \to P(M, G)$ is a PBG-LAB such that the Lie group G is simply connected and $\{U_i\}_{i\in I}$ a simple open cover of M. Then, for any section-atlas $\{P_{U_i} \cong U_i \times G\}_{i\in I}$ there exists a section-atlas $\{\Psi_i : P_i \times \mathfrak{h} \to L_{P_i}\}_{i\in I}$ of the vector bundle L such that*

$$\Psi_i(ug, V \cdot g) = \Psi_i(u, V) \cdot g$$

for all $i \in I$, $V \in L_{U_i}$ and $g \in G$.

Let $A \Rightarrow P(M, G)$ be a PBG-algebroid. We recall from [6] the construction of the transition data (χ, α). It follows from [6, IV§4] that locally $\frac{A}{G}$ has flat connections which, due to Corollary 4.4 give rise to local flat isometablic connections $\theta_i^* : TP_i \to A_{P_i}$. The transition data of A is defined as:

$$\chi_{ij} : TP_{ij} \to P_{ij} \times \mathfrak{h}, \qquad \chi_{ij} = \Psi_i^{-1}(\theta_i^* - \theta_j^*)$$

and

$$\alpha_{ij} : P_{ij} \to Aut(\mathfrak{h}), \qquad \alpha_{ij}(u) = \Psi_{i,u} \circ \Psi_{j,u}^{-1}.$$

Moreover, they satisfy the following, where Δ stands for the Darboux derivative:

(i) $d\chi_{ij} + [\chi_{ij}, \chi_{ij}] = 0$, i.e. each χ_{ij} is a Maurer–Cartan form,
(ii) $\chi_{ij} = \chi_{ik} + \alpha_{ij}(\chi_{jk})$ whenever $P_{ijk} \neq \emptyset$,
(iii) $\Delta(\alpha_{ij}) = \mathrm{ad} \circ \chi_{ij}$ for all i, j.

Now Theorem 1.1 follows immediately from the definition of the transition data and Proposition 5.3.

For the proof of Theorem 1.2 first let us recall first the following result from [8]:

Theorem 5.4. *Let Ω, Ξ be Lie groupoids over the same manifold M and $\mu : A\Omega \to A\Xi$ a Lie algebroid morphism. If Ω is s-simply connected, then there exists a unique morphism of Lie groupoids $\varphi : \Omega \to \Xi$ which differentiates to μ, i.e. $\varphi^* = \mu$.*

Consider a PBG-groupoid $\Upsilon \rightrightarrows P(M,G)$ and its corresponding Lie algebroid $A\Upsilon \Rightarrow P(M,G)$ with adjoint bundle $L\Upsilon$. The extension of Lie algebroids corresponding to that is

$$\frac{L\Upsilon}{G} \rightarrowtail \frac{A\Upsilon}{G} \twoheadrightarrow \frac{TP}{G}.$$

It follows from [6, IV§4] that the Lie algebroid $\frac{A\Upsilon}{G}$ (over M) has local flat connections $\tilde{\theta}_i^*: TU_i \to (\frac{A\Upsilon}{G})_{U_i}$. Due to Corollary 4.4 these give rise to flat isometablic connections $\theta_i^*: TP_i \to A\Upsilon_{P_i}$. Since the connections $\tilde{\theta}_i^*$ are flat, they can be regarded as morphisms of Lie algebroids. With the assumption that every U_i is contractible, and by force of the previous theorem, it follows that the $\tilde{\theta}_i^*$s integrate uniquely to morphisms of Lie groupoids $\tilde{\theta}_i: U_i \times U_i \to \frac{\Upsilon}{G}{}_{U_i}^{U_i}$. It was shown in the proof of Proposition 4.3 that the isometablic flat connections θ_i^* corresponding to the $\tilde{\theta}_i^*$s are in essence the maps $\tilde{\theta}_i^* \circ Tp$, therefore they also integrate uniquely to morphisms of Lie groupoids

$$\theta_i: P_i \times P_i \to \Upsilon_{P_i}^{P_i}.$$

The uniqueness argument of Theorem 5.4 yields that the θ_is are morphisms of PBG-groupoids. That is because for every $g \in G$ the map $\theta_i^g(u,v) = \theta_i(ug, vg)g^{-1}$ is also a morphism of Lie groupoids and differentiates to θ_i^*. It therefore follows from the uniqueness of θ_i that $\theta_i^g = \theta_i$ for all $g \in G$, consequently θ_i is equivariant.

Just like the non-integrable case, the aim is to show that there exists an equivariant section-atlas of the PBG-LAB $L\Upsilon$. Then, the transition data (χ, α) it defines has to be equivariant. To this end, we will show that on global level there exist equivariant section-atlases for the PBG-LGB $I\Upsilon$, which differentiate to the desired atlases of $L\Upsilon$.

First, let us give some notation. For a Lie groupoid $\Omega \to M$ we denote $\Omega_x = s^{-1}(\{x\})$, $\Omega^x = t^{-1}(\{x\})$ and Ω_x^x the Lie group $\Omega_x \cap \Omega^x$.

Now fix a basepoint u_0 in P and for every $i \in I$ choose a $u_i \in P_i$ and an arrow $\xi_i \in \Upsilon_{u_0}^{u_i}$. Define $\sigma_i : P_i \to \Upsilon_{u_0}$ by

$$\sigma_i(u) = \theta_i(u, u_i) \cdot \xi_i.$$

This is a section of the principal bundle $\Upsilon_{u_0}(P, \Upsilon_{u_0}^{u_0})$. Consider the Lie group $H = \Upsilon_{u_0}^{u_0}$ and define a (left) G-action $\rho_i : G \times H \to H$ by

$$\rho_i(g^{-1})(h) = \sigma_i(u_i g)^{-1} \cdot (\xi_i g) \cdot (hg) \cdot (\xi_i g)^{-1} \cdot \sigma_i(u_i g).$$

Last, consider the sections $\psi_i : P_i \times H \to I\Omega_{P_i}$ of $I\Omega$ defined by

$$\psi_i(u,h) = \sigma_i(u) \cdot h \cdot \sigma_i(u)^{-1}.$$

The proof of the following proposition is an immediate calculation.

Proposition 5.5. *The sections ψ_i are equivariant in the sense that*

$$\psi_i\big(ug, \rho_i(g^{-1})(h)\big) = \psi_i(u,h) \cdot g.$$

Their transition functions $\tilde{\alpha}_{ij} : P_{ij} \to Aut(H)$ are equivariant in the sense

$$\tilde{\alpha}_{ij}(ug)\big(\rho_{jj}(g^{-1})(h)\big) = \rho_i(g^{-1})\big(\tilde{\alpha}_{ij}(u)(h)\big).$$

These sections clearly differentiate to sections $\Psi : P_i \times \mathfrak{h} \to L\Upsilon_{P_i}$ of $L\Upsilon$. To show that these Ψ_is, as well as their transition functions $\tilde{\alpha}_{ij}$ are equivariant in the sense of Proposition 5.3 (thus they give rise to equivariant transition data for $A\Upsilon \Rightarrow P(M,G)$), we need to show that the G-actions ρ_i are local expressions of the canonical G-action on the PBG-LGB $I\Upsilon$.

For every $i \in I$, consider the action groupoid $P_i \rtimes G \rightrightarrows P_i(U_i, g)$ (recall Example 2.4) and define a map $\tilde{\rho}_i : P_i \rtimes G * I\Upsilon_{P_i} \to I\Upsilon_{P_i}$ by

$$\tilde{\rho}_i\big((u,g), \eta \in \Upsilon_u^u\big) = \psi_i\big(ug, \rho_i(g^{-1})\big(\psi_{i,u}^{-1}(\eta)\big)\big).$$

Obviously, $\pi(\tilde{\rho}_i((u,g),\eta)) = ug = t(u,g)$ and $\tilde{\rho}_i((u,e_G),\eta) = \eta$. It is a straightforward exercise to verify that

$$\tilde{\rho}_i\big((ug_1, g_2) \cdot (u, g_1), \eta\big) = \tilde{\rho}_i\big((ug_1, g_2), \tilde{\rho}_i(u, g_1), \eta\big).$$

Also, each $\tilde{\rho}_i(u,g)$ is an automorphism of Υ_u^u, therefore it is a representation of the Lie groupoid $P_i \succ G$ on the Lie group bundle $I\Upsilon_{P_i}$, in the sense of [7]. The following proposition allows us to "glue" the $\tilde{\rho}_i$s together to a global map.

Proposition 5.6. *For all $i, j \in I$ such that $P_{ij} \neq \emptyset$, $u \in P_{ij}$, $g \in G$ and $\eta \in \Omega_u^u$ we have*

$$\tilde{\rho}_i\big((u,g),\eta\big) = \tilde{\rho}_j\big((u,g),\eta\big).$$

Proof. The equivariance of the α_{ij}'s gives:

$$\begin{aligned}\tilde{\rho}_i\big((u,g),\eta\big) &= \psi_i\big(ug, \rho_i(g^{-1})\big(\psi_{i,u}^{-1}(\eta)\big)\big) = \psi_i\big(ug, \rho_i(g^{-1})\big(\tilde{\alpha}_{ij}(u)\big(\psi_{j,u}^{-1}(\eta)\big)\big)\big)\\ &= \psi_i\big(ug, \tilde{\alpha}_{ij}(ug)\big(\rho_j(g^{-1})\big(\psi_{i,u}^{-1}(\eta)\big)\big)\big) = \psi_j\big(ug, \rho_j(g^{-1})\big(\psi_{i,u}^{-1}(\eta)\big)\big) = \tilde{\rho}_j\big((u,g),\eta\big). \qquad \square\end{aligned}$$

Now define $\rho : (P \succ G) * I\Omega \to I\Omega$ by $\rho((u,g), \eta \in \Omega_u^u) = \tilde{\rho}_i((u,g),\eta)$, if $u \in P_i$. This is a representation because each $\tilde{\rho}_i$ is. As a matter of fact, ρ is a lot simpler than it seems. Since the sections $\{\psi_i\}_{i\in I}$ are equivariant we have:

$$\rho\big((u,g),\eta\big) = \psi_i\big(ug, \rho_i(g^{-1})\big(\psi_{i,u}^{-1}(\eta)\big)\big) = \psi_i\big(u, \psi_{i,u}^{-1}(\eta)\big)\cdot g = \eta \cdot g.$$

So ρ is, in fact, just the PBG structure of $I\Upsilon$.

Conversely, it is possible to retrieve the local representations $\{\rho_i\}_{i\in I}$ from the PBG structure of $I\Upsilon$. Suppose $\{\sigma_i : P_i \to \Upsilon_{u_0}\}_{i\in I}$ is a family of sections of Υ. Consider the charts $\psi_i : P_i \times H \to I\Upsilon_{P_i}$ defined as $\psi_{i,u}(h) = I_{\sigma_i(u)}(h)$ and define $\tilde{\rho}_i : P_i \succ G \to Aut(H)$ by

$$\tilde{\rho}_i(u,g)(h) = \psi_{i,ug}^{-1}\big(\psi_{i,u}(h)\cdot g\big)$$

for all $g \in G$, $h \in H$ and $u \in P_i$. This is a morphism of Lie groupoids over $P_i \to \cdot$. For every $i \in I$ choose $u_i \in P_i$ and define

$$\rho_i(g^{-1})(h) = \tilde{\rho}_i(u_i,g)(h) = \psi_{i,u_i g}^{-1}\big(\psi_{i,u}(h)\cdot g\big).$$

Then,

$$\rho_i(g^{-1})(h) = I_{\sigma_i(u_i g)^{-1}}\big(I_{\sigma_i(u_i)}(h)\cdot g\big) = \sigma_i(u_i g)^{-1}\cdot\big(\sigma_i(u_i)g\big)\cdot(hg)\cdot\big(\sigma_i(u_i)^{-1}g\big)\cdot\sigma_i(u_i g).$$

The latter is exactly the original definition of the ρ_i's. Finally, these considerations prove Theorem 1.2.

6. Holonomy

In this section we introduce isometablic path connections for PBG-groupoids and prove that they correspond to isometablic infinitesimal connections in the transitive case. Then we use isometablic path connections to study the holonomy of transitive PBG-groupoids. In the following we restrict to transitive PBG-groupoids $\Upsilon \rightrightarrows P(M,G)$ over a principal bundle $P(M,G)$. Let us fix some notation first. We denote $C(I,P)$ the set of continuous and piecewise smooth paths in P. Moreover, $P_0^s(\Upsilon)$ denotes the set of continuous and piecewise smooth paths $\delta : I \to \Upsilon$ which commence at an identity of Υ and $s\circ\delta : I \to M$ is constant. These sets obviously admit right actions from G.

Definition 6.1. An *isometablic C^∞ path connection* in Υ is a map $\Gamma : C(I,P) \to P_0^s(\Upsilon)$, usually written $c \mapsto \bar{c}$, such that:

(i) $\bar{c}(0) = 1_{c(0)}$ and $\bar{c}(t) \in \Upsilon_{c(0)}^{c(t)}$ for all $t \in I$.
(ii) If $\varphi : [0,1] \to [\alpha,\beta] \subseteq [0,1]$ is a diffeomorphism then $\overline{c\circ\varphi}(t) = \bar{c}(\varphi(t))\cdot[\bar{c}(\varphi(0))]^{-1}$.
(iii) If $c \in C(I,P)$ is differentiable at $t_0 \in I$ then $\bar{c}$ is also differentiable at t_0.
(iv) If $c_1, c_2 \in C(I,P)$ have $\frac{dc_1}{dt}(t_0) = \frac{dc_2}{dt}(t_0)$ then $\frac{d\bar{c}_1}{dt}(t_0) = \frac{d\bar{c}_2}{dt}(t_0)$.
(v) If $c_1, c_2, c_3 \in C(I,P)$ are such that $\frac{dc_1}{dt}(t_0) + \frac{dc_2}{dt}(t_0) = \frac{dc_3}{dt}(t_0)$ then $\frac{d\bar{c}_1}{dt}(t_0) + \frac{d\bar{c}_2}{dt}(t_0) = \frac{d\bar{c}_3}{dt}(t_0)$.
(vi) For all $g \in G$, $\overline{R_g\circ c} = \widetilde{R}_g \circ \bar{c}$.

The first five properties in the above definition constitute the standard definition of a path connection in a Lie groupoid, as it was postulated in [6, II§7]. The isometablicity is expressed by the sixth property. This definition has the following consequences which were proved in [6, II§7].

Proposition 6.2.

(i) $\bar{\kappa}_u = \kappa_{1_u}$ *for all* $u \in P$ *(where* κ_u *denotes the path constant at* u*).*
(ii) $\overline{c^{\leftarrow}}(t) = \bar{c}^{\leftarrow}(t) \cdot [\bar{c}(t)]^{-1}$ *(where* $c^{\leftarrow}$ *denotes the inverse path of* c*).*
(iii) $\overline{c' \cdot c} = (R_{c(1)} \circ \overline{c'}) \cdot \bar{c}$ *for all composable paths* $c, c' \in C(I, P)$*.*

For every $c \in C(I, P)$ denote $\hat{c} \equiv c(1)$.

Corollary 6.3.

(i) $\hat{\kappa}_u = 1_u$.
(ii) $\widehat{c^{\leftarrow}} = (\hat{c})^{-1}$.
(iii) $\widehat{c' \cdot c} = \widehat{c'} \cdot \hat{c}$.

Proposition 6.4. *If* $c_1, c_2 \in C(I, P)$ *are such that* $\frac{dc_1}{dt}(t_0) = \lambda \frac{dc_2}{dt}(t_0)$ *for some* $t_0 \in I$*, then* $\frac{d\bar{c}_1}{dt}(t_0) = \lambda \cdot \frac{d\bar{c}_2}{dt}(t_0)$*.*

Theorem 6.5. *If* $\varphi : P_U \times (-\varepsilon, \varepsilon) \to P$ *is a local* 1*-parameter group of local transformations for* P*, where* $P_U \cong U \times G$ *is the image of a chart of the principal bundle* $P(M, G)$*, then the map* $\bar{\varphi} : \Upsilon^{P_U} \times (-\varepsilon, \varepsilon) \to \Upsilon$ *constructed in* [6, II§7] *is a local* 1*-parameter group of local transformations on* Υ *and*

(i) $t \circ \bar{\varphi}_t = \varphi_t \circ t$ *for all* $t \in (-\varepsilon, \varepsilon)$*.*
(ii) *If* $\varphi_{v \cdot g} = R_g \circ \varphi_v$ *for all* $g \in G$*,* $v \in P_u$ *then* $\bar{\varphi}_{\xi \cdot g} = \widetilde{R}_g \circ \bar{\varphi}_\xi$ *for all* $\xi \in \Upsilon^{P_U}$ *and* $g \in G$*.*

Proof. The first assertion was proved in [6, II§7]. For the second one we have:

$$(\widetilde{R}_g \circ \bar{\varphi}_\xi)(t) = \bar{\varphi}_\xi(t) \cdot g = \overline{R_g \circ \varphi_\xi}(t) = \bar{\varphi}_{\xi \cdot g}(t). \qquad \square$$

Theorem 6.6. *There is a bijective correspondence between isometablic* C^∞ *path connections* $\Gamma : c \mapsto \bar{c}$ *in* $\Upsilon \rightrightarrows P(M, G)$ *and isometablic infinitesimal connections* $\gamma : TP \to A\Upsilon$ *such that a corresponding* Γ *and* γ *are related by*

$$\frac{d}{dt}\bar{c}(t_0) = TR_{\bar{c}(t_0)}\left[\gamma\left(\frac{d}{dt}c(t_0)\right)\right] \tag{7}$$

for all $c \in C(I, P)$ *and* $t_0 \in I$*.*

Proof. Suppose given an isometablic C^∞ path connection Γ. For $v \in P$ and $X \in T_v P$ take any $c \in C(I, P)$ with $c(t_0) = v$ and $\frac{dc}{dt}(t_0) = X$ for some $t_0 \in I$ and define $\gamma(X) = TR_{(\bar{c}(t_0))^{-1}}[\frac{d}{dt}(\bar{c}(t_0))]$. The smoothness and $\mathbb{R}$-linearity of this connection are proven exactly as in [6, III§7]. We now prove its isometablicity.

$$\begin{aligned}\gamma\big(TR_g(X)\big) &= TR_{(\overline{R_g \circ c}(t_0))^{-1}}\left[\frac{d}{dt}\big(\overline{R_g \circ c}(t_0)\big)\right] = TR_{\widetilde{R}_g \circ (\bar{c}(t_0))^{-1}}\left[\frac{d}{dt}\big(\widetilde{R}_g \circ \bar{c}(t_0)\big)\right] \\ &= (TR_{(\bar{c}(t_0))^{-1} \cdot g} \circ T\widetilde{R}_g)\left[\frac{d}{dt}\big(\bar{c}(t_0)\big)\right] = T(R_{(\bar{c}(t_0))^{-1} \cdot g} \circ \widetilde{R}_g)\left[\frac{d}{dt}\big(\bar{c}(t_0)\big)\right].\end{aligned}$$

For all $\xi \in \Upsilon$ we have

$$R_{(\bar{c}(t_0))^{-1} \cdot g} \circ \widetilde{R}_g(\xi) = (\xi \cdot g) \cdot \big(\big(\bar{c}(t_0)\big)^{-1} \cdot g\big) = \big(\xi \cdot \big(\bar{c}(t_0)\big)^{-1}\big) \cdot g = \widetilde{R}_g \circ R_{(\bar{c}(t_0))^{-1}}(\xi).$$

Therefore,

$$T(R_{(\bar{c}(t_0))^{-1} \cdot g} \circ \widetilde{R}_g)\left[\frac{d}{dt}\big(\bar{c}(t_0)\big)\right] = T\widetilde{R}_g(TR_{(c(t_0))^{-1}})\left[\frac{d}{dt}\big(\bar{c}(t_0)\big)\right] = T\widetilde{R}_g\big(\gamma(X)\big).$$

Conversely, suppose given an isometablic infinitesimal connection $\gamma : TP \to A\Upsilon$. It was shown in [6, III§7] that given a path $c \in C(I, P)$ such that $c(0) = u_0$ the re is a unique path $\bar{c} \in P_0^s(\Upsilon)$ which satisfies (7). For any $g \in G$ the path $R_g \circ c$ has initial data $(R_g \circ c)(0) = u_0 \cdot g$. Thus there is a unique path $\overline{R_g \circ c} \in P_0^s(\Upsilon)$ satisfying the differential equation

$$\frac{d}{dt}(\overline{R_g \circ c})(t_0) = TR_{\overline{R_g \circ c}(t_0)}\left[\gamma\left(\frac{d}{dt}(R_g \circ c)(t_0)\right)\right]$$

with initial data $\overline{R_g \circ c}(0) = 1_{u_0 \cdot g}$. The path $\widetilde{R}_g \circ \bar{c}$ also has initial data $\widetilde{R}_g \circ \bar{c}(0) = \widetilde{R}_g(1_{u_0}) = 1_{u_0} \cdot g = 1_{u_0 \cdot g}$. Therefore it suffices to show that it also satisfies the differential equation (7). Indeed:

$$\frac{d}{dt}(\widetilde{R}_g \circ \bar{c})(t_0) = T\widetilde{R}_g\left(\frac{d}{dt}\bar{c}(t_0)\right) = (T\widetilde{R}_g \circ TR_{\bar{c}(t_0)})\left[\gamma\left(\frac{d}{dt}c(t_0)\right)\right] = T(\widetilde{R}_g \circ R_{\bar{c}(t_0)})\left[\gamma\left(\frac{d}{dt}c(t_0)\right)\right].$$

It is easy to see that $\widetilde{R}_g \circ R_{\bar{c}(t_0)} = R_{(\widetilde{R}_g \circ \bar{c})(t_0)}$, therefore

$$\begin{aligned} T(\widetilde{R}_g \circ R_{\bar{c}(t_0)})\left[\gamma\left(\frac{d}{dt}c(t_0)\right)\right] &= TR_{(\widetilde{R}_g \circ \bar{c})(t_0)}\left[(T\widetilde{R}_g \circ \gamma)\left(\frac{d}{dt}c(t_0)\right)\right] = TR_{(\widetilde{R}_g \circ c)(t_0)}\left[(\gamma \circ TR_g)\left(\frac{d}{dt}c(t_0)\right)\right] \\ &= TR_{(\widetilde{R}_g \circ c)(t_0)}\left[\gamma\left(\frac{d}{dt}(R_g \circ c)(t_0)\right)\right]. \qquad \square \end{aligned}$$

Corollary 6.7 (Of the proof). *Let $\gamma : TP \to A\Upsilon$ be an isometablic infinitesimal connection in Υ and Γ the corresponding isometablic path connection. Then*:

(i) *For all $X \in \Gamma TP$, $\mathrm{Exp}(t \cdot \gamma(X))(v) = \Gamma(\varphi, v)(t)$ where φ_t is the local flow of X near v and $\Gamma(\varphi, v) : \mathbb{R} \to \Upsilon_v$ is the lift of $t \mapsto \varphi_t(v)$.*
(ii) *The restriction of the exponential map on the image of γ is equivariant.*
(iii) *The restriction of the exponential map on $\gamma(\Gamma^G TP)$ is equivariant.*

Proof. The first was proved in [6, III§7]. For the second assertion consider $\{\varphi_t\}$ a local flow of $X \in \Gamma TP$ near $v \in P$. Then $\{\psi_t = R_g \circ \varphi_t \circ R_{g^{-1}}\}$ is a local flow of $TR_g(X_v)$. Therefore,

$$\begin{aligned} \mathrm{Exp}\big(t \cdot T\widetilde{R}_g\big(\gamma(X)\big)\big)(v) &= \mathrm{Exp}\big(t \cdot \gamma\big(TR_g(X)\big)\big)(v) = \Gamma(\psi, v \cdot g)(t) \\ &= \Gamma(R_g \circ \varphi_t \circ R_{g^{-1}}, v \cdot g)(t) = \big[\widetilde{R}_g^{\Gamma} \circ \Gamma(\varphi, v)\big](t) = \widetilde{R}_g^{\Gamma} \circ \mathrm{Exp}\big(t \cdot \gamma(X)\big)(v). \end{aligned}$$

The third assertion is immediate because if $X \in \Gamma^G TP$ then both $\{\varphi_t\}$ and $\{\psi_t\}$ are flows of X at v and $v \cdot g$, respectively. $\square$

Definition 6.8. Let Γ be an isometablic C^∞ path connection in Υ. Then $\Psi = \Psi(\Gamma) = \{\bar{c}(1){:}\ c \in C(I, P)\} \subseteq \Upsilon$ is called the *holonomy subgroupoid* of Γ. The vertex Ψ_v^v is the *holonomy group* of Γ at v.

We know from [6] that Ψ is in general an s-connected Lie subgroupoid of $\Upsilon \rightrightarrows P(M, G)$. We will finish this section by showing that in case Γ is isometablic then it is in fact a PBG-subgroupoid of Υ.

Proposition 6.9. *Let Υ, Υ' be PBG-groupoids over $P(M, G)$ and $\varphi : \Upsilon \to \Upsilon'$ an (equivariant) morphism of PBG-groupoids over $P(M, G)$. Let γ be an isometablic infinitesimal connection in Υ and Γ its corresponding path connection. The isometablic path connection Γ' corresponding to the produced connection $\gamma' = \varphi_* \circ \gamma$ is $\Gamma' = \varphi \circ \Gamma$ and its holonomy subgroupoid is $\Psi' = \varphi(\Psi)$.*

Proof. Take $c \in C(I, P)$. Then, since $\bar{c}$ satisfies Eq. (7) for γ it immediately follows that $\varphi \circ \bar{c}$ satisfies Eq. (7) for $\varphi_* \circ \gamma$. Thus $\Gamma' = \varphi \circ \Gamma$. $\square$

Definition 6.10. Let Υ, Υ' be PBG-groupoids over the principal bundle $P(M, G)$. The PBG-groupoid Υ' is called a *PBG reduction* of Υ if there is a morphism of Lie groupoids $\varphi : \Upsilon' \to \Upsilon$ which is equivariant and an injective immersion.

Corollary 6.11. *If Υ' is a PBG reduction of Υ and the isometablic connection $\gamma : TP \to A\Upsilon$ takes values in $A\Upsilon'$ then $\Psi \leqslant \Upsilon'$.*

Theorem 6.12. *Let Γ be an isometablic path connection in Υ. Then the holonomy subgroupoid Ψ of Γ is a PBG-subgroupoid of Υ.*

Proof. The fact that Ψ is a Lie subgroupoid of Υ is proved in [6, III§7]. We will only show that the action of G on Υ can be restricted to Ψ. Let $\xi = \bar{c}(1)$ for some $c \in C(I, P)$. Then $\xi \cdot g = \widetilde{R_g} \circ \bar{c}(1) = \overline{R_g \circ c}(1) \in \Psi$. □

Corollary 6.13. *For each $X \in \Gamma TP$ and all t sufficiently near* 0*, $\gamma(X) \in \Gamma A\Psi$ and* $\mathrm{Exp}(t \cdot \gamma(X)) \in \Psi$.

Proof. These are reformulations of Corollary 6.7. □

7. Deformable sections

The aim of this section is to prove an Ambrose–Singer theorem for transitive PBG-groupoids. The result passes through the study of deformable sections for representations of vector bundles on PBG-groupoids (see Definition 4.7). Again, all groupoids regarded in this section are considered to be transitive.

Definition 7.1. Let $\Upsilon \rightrightarrows M$ be a (transitive) Lie groupoid, $E \to M$ a vector bundle and $\rho : \Upsilon * E \to E$ a smooth representation of Υ on E. A section $\mu \in \Gamma E$ is called Υ*-deformable* if for every $x, y \in M$ there is a $\xi \in \Upsilon_x^y$ such that $\rho(\xi, \mu(x)) = \mu(y)$.

If $\mu \in \Gamma E$ is Υ-deformable then the *isotropy groupoid* of Υ at μ is $\Phi(\mu) = \{\xi \in \Upsilon: \ \xi(\mu(s(\xi))) = \mu(t(\xi))\}$. A section μ is Υ-deformable iff its value lies in a single orbit. The condition ensures that the isotropy groupoid is transitive.

Theorem 7.2. *Let $\Upsilon \rightrightarrows P(M, G)$ be a (transitive) PBG-groupoid and $E \to P(M, G)$ a vector bundle on which G acts by isomorphisms. Let $\rho : \Upsilon * E \to E$ be an equivariant representation and $\mu \in \Gamma^G E$. Then the following propositions are equivalent*:

(i) *The section μ is Υ-deformable.*
(ii) *The isotropy groupoid $\Phi(\mu)$ is a PBG-subgroupoid of Υ.*
(iii) *The PBG-groupoid Υ has a section atlas $\{\sigma_i : P_i \to \Upsilon_{u_0}\}_{i \in I}$ such that $\rho(\sigma_i(u)^{-1})(\mu(u))$ is a constant map $P_i \to E_{u_0}$.*
(iv) *The PBG-groupoid Υ possesses an isometablic infinitesimal connection γ such that*

$$(\rho_* \circ \gamma)(\mu) = 0.$$

Proof. (i) ⇒ (ii) It was shown in [6, III§7] that $\Phi(\mu)$ is a closed and embedded Lie subgroupoid of Υ. It remains to show that if $\xi \cdot g \in \Phi(\mu)$ for all $\xi \in \Phi(\mu)$ and $g \in G$. Indeed,

$$\rho\big(\xi \cdot g, \mu\big(s(\xi \cdot g)\big)\big) = \rho\big(\xi \cdot g, \mu\big(s(\xi)\big) \cdot g\big) = \rho\big(\xi, \mu\big(s(\xi)\big)\big) \cdot g = \mu\big(t(\xi)\big) \cdot g = \mu\big(t(\xi \cdot g)\big).$$

(ii) ⇒ (iii) Since $\Phi(\mu)$ is a Lie subgroupoid of Υ, the principal bundle $(\Phi(\mu))_{u_0}(P, (\Phi(\mu))_{u_0}^{u_0}, t)$ is a reduction of $\Upsilon_{u_0}(P, \Upsilon_{u_0}^{u_0}, t)$. Therefore there is a section atlas $\{\sigma_i : P_{U_i} \to \Upsilon_{u_0}\}_{i \in I}$ of Υ such that $\sigma_i(u) \in (\Phi(\mu))_{u_0}$ for all $u \in P_{U_i}$. So $\sigma_i(u)^{-1} \in (\Phi(\mu))_{u_0}$ for all $u \in P_{U_i}$. Equivalently, $\rho(\sigma_i(u)^{-1}, \mu(u)) = \mu((u_0))$ for all $u \in P_i$.

(iv) ⇒ (i). Suppose $\Psi \leqslant \Upsilon$ is the holonomy PBG-subgroupoid of γ and $\Psi' \leqslant \Phi(E)$ the holonomy PBG-subgroupoid of $\rho_* \circ \gamma$. Then $\Psi' = \rho(\Psi)$ and from [6, III§7] we have:

$$\begin{aligned}(\rho_* \circ \gamma) = 0 &\Rightarrow \xi \cdot \mu\big(s(\xi)\big) = \mu\big(t(\xi)\big) \ \forall \xi \in \Phi(E)\\ &\Rightarrow \rho\big(\eta, \mu\big(s(\eta)\big)\big) = \mu\big(t(\eta)\big) \ \forall \eta \in \Upsilon.\end{aligned}$$

Therefore μ is Υ-deformable.

(ii) $\Rightarrow$ (iv) The isotropy subgroupoid is a PBG-groupoid, therefore it has an isometablic connection $\gamma : TP \to A\Phi(\mu)$. This is also a connection of Υ. From [6, III§4] we have that $\Gamma A\Phi(\mu) = \{X \in \Gamma A\Upsilon:\ \rho_*(X)(\mu) = 0\}$. Therefore $\rho_*(\gamma(X))(\mu) = 0$ for all $X \in \Gamma TP$. □

Proposition 7.3. *Let L be a vector bundle over $P(M,G)$ on which G acts by isomorphisms and* [ ] *a section of the vector bundle* $\mathrm{Alt}^2(L;L)$. *Then the following three conditions are equivalent*:

(i) *The fibers of L are pairwise isomorphic as Lie algebras.*
(ii) *L admits an isometablic connection ∇ such that*

$$\nabla_X[V,W] = \big[\nabla_X(V), W\big] + \big[V, \nabla_X(W)\big]$$

for all $X \in \Gamma TP$ and $V, W \in \Gamma L$.
(iii) *L is a PBG-Lie algebra bundle.*

Proof. Let $\rho : \Phi(L) * \mathrm{Alt}^2(L,L) \to \mathrm{Alt}^2(L,L)$ denote the representation defined by

$$\rho(\xi,\varphi) = \xi \circ \varphi \circ (\xi^{-1} \times \xi^{-1})$$

for all $\xi \in \Phi(L)$ and $\varphi \in \mathrm{Alt}^2(L,L)$. We have already discussed why this is an equivariant representation. Now, (i) is the condition that [ ] is $\Phi[L]$-deformable and (iii) is the condition that $\Phi[L]$ admits a section atlas $\{\sigma_i\}_{i\in I}$ such that the corresponding charts for $\mathrm{Alt}^2(L,L)$ via ρ map $[\ ] \in \Gamma\,\mathrm{Alt}^2(L,L)$ to constant maps $P_i \to \mathrm{Alt}^2(L_{u_0}, L_{u_0})$. So, (i) and (iii) are equivalent by the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 7.2.

We also have that $\rho_* : \mathrm{CDO}(L) \to \mathrm{CDO}(\mathrm{Alt}^2(L,L))$ is

$$\rho_*(D)(\varphi)(V,W) = D\big(\varphi(V,W)\big) - \varphi\big(D(V),W\big) - \varphi\big(V,D(W)\big).$$

Therefore (ii) is the condition that L admits an isometablic connection ∇ such that $(\rho_* \circ \nabla)([\ ]) = 0$. Hence (i) $\Leftrightarrow$ (ii) follows from the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 7.2. □

Proposition 7.4. *Let E and E' be vector bundles over the principal bundle $P(M,G)$ on which G acts by isomorphisms and let $\varphi : E \to E'$ be an equivariant morphism over P. Then the following conditions are equivalent*:

(i) *The map $P \ni u \mapsto \mathrm{rk}(\varphi_u) \in \mathbb{Z}$ is constant.*
(ii) *There exist equivariant connections ∇ and ∇' of E and E' respectively such that $\nabla'_X(\varphi(\mu)) = \varphi(\nabla_X(\mu))$ for all $\mu \in \Gamma E$ and $X \in \Gamma TP$.*
(iii) *There exist atlases $\{\psi_i : P_i \times V \to E_{P_i}\}_{i\in I}$ and $\{\psi'_i : P_i \times V' \to E'_{P_i}\}_{i\in I}$ for E and E' respectively such that each $\varphi : E_{U_i} \to E'_{U_i}$ is of the form $\varphi_i(x,v) = (x, f_i(v))$ where $f_i : V \to V'$ is a linear map depending only on i.*

Proof. This is also an application of Theorem 7.2. Consider the equivariant representation $(\Phi[E] \times_{P\times P} \Phi[E]) * \mathrm{Hom}(E,E') \to \mathrm{Hom}(E,E')$ we discussed in Section 3. The equivariant morphism $\varphi : E \to E'$ can be regarded as an equivariant section of the vector bundle $\mathrm{Hom}(E,E')$, namely assigning to every $u \in P$ the linear map $\varphi_u : E_u \to E'_u$. The following lemma shows that (i) is the condition that φ is $(\Phi(E) \times_{P\times P} \Phi(E'))$-deformable:

Lemma. *Let $\varphi_1 : V \to V'$ and $\varphi_2 : W \to W'$ be morphisms of vector spaces such that* $\dim V = \dim W$, $\dim V' = \dim W'$ *and* $\mathrm{rk}(\varphi_1) = \mathrm{rk}(\varphi_2)$. *Then there are isomorphisms $s : V \to W$ and $s' : V' \to W'$ such that $s' \circ \varphi_1 = \varphi_2 \circ s$.*

We also discussed in Section 3 that the induced representation $\rho_* : \mathrm{CDO}(E) \oplus \mathrm{CDO}(E') \to \mathrm{CDO}(\mathrm{Hom}(E,E'))$ is $(X \oplus X')(\varphi)(\mu) = X'(\varphi(\mu)) - \varphi(X(\mu))$. Therefore (ii) is exactly the condition that $\Phi(E) \times_{P\times P} \Phi(E')$ possesses an isometablic infinitesimal connection γ such that $(\rho_* \circ \gamma)(\mu) = 0$. Finally, condition (iii) is clearly condition (iii) of Theorem 7.2. □

Definition 7.5. Let E, E' be two vector bundles over the principal bundle $P(M,G)$ on both of which G acts by automorphisms. Let $\varphi : E \to E'$ be an equivariant morphism of vector bundles over M. Then, φ is

(i) *of locally constant rank* if $u \mapsto \mathrm{rk}(\varphi_u): P \to \mathbb{Z}$ is locally constant;
(ii) *a locally constant morphism* if it satisfies condition (iii) of Proposition 7.4.

The proof of the following proposition is analogous to the one of Proposition 7.4.

Proposition 7.6. *Let L, L' be PBG-LABs on $P(M,G)$ and let $\varphi: L \to L'$ be a morphism of PBG-LABs. Then the following conditions are equivalent*:

(i) *For each $u, v \in P$ there are Lie algebra isomorphisms $\alpha: L_u \to L_v$ and $\alpha': L'_u \to L'_v$ such that $\varphi_v \circ \alpha = \alpha' \circ L_u$.*
(ii) *L and L' posses isometablic Lie connections ∇ and ∇' respectively such that $\varphi(\nabla_X(V)) = \nabla'_X(\varphi(V))$ for all $V \in \Gamma L$ and $X \in \Gamma TP$.*
(iii) *There exist Lie algebra bundle atlases $\{\psi_i: P_i \times \mathfrak{g} \to L_{P_i}\}_{i\in I}$ and $\{\psi'_i: P_i \times \mathfrak{g}' \to L'_{P_i}\}_{i\in I}$ for L and L' respectively such that each $\varphi: L_{P_i} \to L'_{P_i}$ is of the form $\varphi(u,W) = (u, f_i(W))$ where $f_i: \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra morphism depending on i.*

If P is connected then f_i may be chosen to be independent of i as well.

Proposition 7.7. *Let L be a PBG-Lie algebra bundle on $P(M,G)$ and let L^1 and L^2 be PBG-subLABs of L. Then $L^1 \cap L^2$ is a PBG-subLAB of L if there is a Lie connection ∇ of L such that $\nabla(\Gamma L^1) \subseteq \Gamma L^1$ and $\nabla(\Gamma L^2) \subseteq \Gamma L^2$.*

Proof. Without the PBG condition, this was proved in [6, I II§7]. All we need to show is that the intersection of two PBG-subLABs is PBG, i.e. the action of G on L can be restricted to $L^1 \cap L^2$. This is immediate. □

We are now ready to proceed to the proof of the Ambrose–Singer theorem for isometablic connections. Consider a (transitive) PBG-groupoid $\Upsilon \rightrightarrows P(M,G)$ and an isometablic infinitesimal connection γ on Υ.

Proposition 7.8. *Let L' be a PBG-subLAB of $L\Upsilon$ such that*

(i) *$\overline{R}_\gamma(X,Y) \in L'$ for all $X, Y \in TP$.*
(ii) *$\nabla^\gamma(\Gamma L') \subseteq \Gamma L'$.*

Then there is a PBG reduction $A' \leqslant A\Upsilon$ defined by

$$\Gamma A' = \{X \in \Gamma A\Upsilon:\ X - \gamma(q(X)) \in \Gamma L\}$$

which has L' as adjoint bundle and is such that $\gamma(X) \in A'$ for all $X \in TP$.

Proof. Again, without the PBG condition this is was proved in [6, III§7]. We only need to show that A' is a PBG-algebroid. Indeed, if $X \in \Gamma A'$ then $\widehat{R}^\Gamma_g(X) \in \Gamma A'$ because:

$$\widehat{R}^\Gamma_g(X) - \gamma(q(\widehat{R}^\Gamma_g(X))) = \widehat{R}^\Gamma_g(X)(X - \gamma(q(X))) \in \Gamma L'. \qquad \square$$

Proposition 7.9. *There is a least PBG-subLAB denoted $(L\Upsilon)^\gamma$ of $L\Upsilon$ which has the properties* (i) *and* (ii) *of Proposition 7.8.*

Proof. It suffices to prove that if L^1 and L^2 both satisfy (i) and (ii) of Proposition 7.8 then $L^1 \cap L^2$ does also. The only point that is not clear is that $L^1 \cap L^2$ is a PBG-subLAB, and since ∇^γ is a Lie connection this is established by Proposition 7.7. □

Theorem 7.10. *Let $\Upsilon \rightrightarrows P(M,G)$ be a PBG-groupoid and $\gamma: TP \to A\Upsilon$ an isometablic infinitesimal connection. Let Γ be its corresponding C^∞ path connection and Ψ its holonomy groupoid. Then $A\Psi = (A\Upsilon)^\gamma$.*

Proof. We showed in Corollary 6.13 that γ takes values in $A\Psi$. Hence $L\Psi$ satisfies the conditions of Proposition 7.8 and therefore $L\Psi \geqslant (L\Upsilon)^\gamma$ and $A\Psi \geqslant (A\Upsilon)^\gamma$. On the other hand, γ takes values in $(A\Upsilon)^\gamma$ and $A\Psi \leqslant (A\Upsilon)^\gamma$. □

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