TEACHING PERIODIC DECIMALS IN TERTIARY EDUCATION

Michael KOURKOULOS
Department of Education, University of Crete
74100 Rethymnon, Crete, Greece
e-mail: mkourk@edc.uoc.gr

ABSTRACT

Periodic decimals (pd) constitute a fundamental representation of rational numbers. Additionally, their study leads to the understanding of basic characteristics of the decimal system and to a more profound comprehension of the operation of division. Furthermore, the a-priori analysis of the properties of the pd indicates that this area is adequate for learning activities, in which students may work by combining the use of inductive and deductive methods. This claim is supported by the historical development of Number Theory.

However, only a very limited part of the Mathematics curriculum in elementary and secondary education is devoted to this area. Often, this is also true for the pre-service undergraduate studies of (elementary and secondary) schoolteachers of Mathematics in Mathematics and Education Departments.

In the first part of this work, we present results of an empirical study of the knowledge, which (elementary and secondary) schoolteachers of Mathematics and students of Departments of Education in Greece have on pd. Our results point out that the properties of pd are largely unknown both to the teachers and to the students who have been asked. For example, 96% of 207 students of Departments of Education don't know that in all non-terminating divisions, the quotient is a pd.

In the second part of this work, we give an outline of a teaching approach concerning pd, which has been applied in the Department of Education of the University of Crete. In the context of this teaching approach, students become able to discover the basic properties of pd and to prove many of them, by combining the use of inductive and deductive methods. Apart from a significant improvement of the students' knowledge concerning the pd and the decimal system, we have also observed an important improvement of their understanding of inductive methods and of the fruitfulness to combine such methods with deductive methods, in order to study problems in Arithmetic and elementary Number Theory.

1. Introduction

Apart from the classical representation of fractions as ratios of integers (a/b), students encounter them as decimals with a finite or infinite number of digits (finite or periodic decimals respectively). Students come in first contact with periodic decimals (**p.ds**) at the age of 10-11. For all the rest of their mathematical education, p.d. is one of the most frequently appearing mathematical objects in students' work. (e.g. It is possible to encounter them every time they work on a non-terminating division of entire or decimal numbers)

In compulsory education (10-15), the exploration of the properties of the p.ds can contribute to a better understanding of the decimal system, of the decimal development of fractions and of the division algorithm (see Kourkoulos M. 1999 ii). Nevertheless, as we will see in [2], not only students but also teachers know very little about p.d.

The a-priori analysis of the basic properties of p.d. (see Kourkoulos M. 1999 i, ch 4), as well as the experimental data presented in [3] indicate that p.d. is an adequate domain for the development of students' Mathematical culture (see also Tzanakis, Kourkoulos 1998)

This is because, the exploration of p.d.:

-Leads to fertile questions, which instigate students' interest.

- Gives the possibility to form conjectures and hypotheses

- Is suitable for the organization of experimental (inductive) exploration of the formulated conjectures. The degree of difficulty of the experimental exploration of the properties of p.ds varies. This offers important possibilities to the teacher to design activities in which an experimental research on mathematical conjectures is asked (This is an important but neglected issue in the current conditions of Mathematics education, see Polya 1954, Lakatos 1976).

It is worthwhile to note that the experimental investigation of the properties of the p.ds often can be facilitated by "the intelligent use of the calculator and the computer in teaching activities" (see also Bruillard, Vivet 1994).

-It creates the desire to look for the explanations and the justification of the properties, which are found empirically (Balacheff, 1982, Duval, 1993)

- With the use of appropriate teaching activities, already from the level of the compulsory education, students can combine experimental and deductive methods of work fruitfully. The students of this level can discover the elementary properties of the p.ds[1] and understand their explanation. (The results of an experimental teaching, which we realized with two classes of 13-14 years old pupils, on p.d. confirm the aforementioned concerning the abilities of Junior High School (J.H.S.) students, Kourkoulos 1999,ii).

A complete justification of other properties[2], can be taught at the High School (H.S.) level or higher (depending on how the curriculum is related to Number Theory). In parallel, this teaching can lead students to discover and/or understand better important properties of Number Theory (e.g. property 5 and Fermat's little theorem).

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[1] Such as: The quotient of every non-terminating division is a p.d (property1). Which are the terminating divisions (property2). The way that we transform a p.d. to the form of an ordinary fraction (property3). The length of the period (**l.p.**) of 1/a is equal to the l.p. of a/b, when a, b are primes between them (property4).

[2] Such as: The l.p. of 1/p divide p-1, when p prime (property5). The l.p. of 1/(a*b) is equal to the LCM of the l.p. of 1/a and of the l.p. of 1/b, when a, b primes between them (property6). The l.p. of $1/p^n$ is equal to $m*p^{n-k}$, when p prime, m the l.p. of 1/p, $(1/p)^k$ the bigger power of 1/p for which the l.p. is equal m and $n \geq k$ (property7).

The assertion that the domain of p.ds is appropriate for the combination of experimental (inductive) and theoretical (deductive) treatment is corroborated by the historical development of Number Theory (see Dicson 1971).

From the end of the 18^{th} century until the end of the 19^{th} century there is a vigorous interest and activity of the mathematical community on the subject of p.ds (However, we can find research works on this subject dated from the end of the 17^{th} century (e.g. G.W. Leibnitz 1677, J. Wallis 1685) until the beginning of the 20^{th} century (e.g. Weixer 1916, Hoppe 1917), see Dicson L.E., 1971).

When we consider the path that research has followed on this subject, we observe that there often appears the following scheme: formulation of conjectures, experimental research, possible modification of the conjectures, proof.

A characteristic example is the path which led to the discovery of the properties 6 and 7: Wallis in 1685 claimed, without proof, that the l.p. of 1/(m.n.) is equal to the LCM of the l.p. of 1/m and of 1/n, if m and n have prime factors different from 2 and 5 and gives as example the 1/(3*7) (This example constitutes also a counter-example to an anterior assertion of Leibnitz).

In 1771 J. Bernoulli published a table with the periods of 1/p, where p is an odd prime smaller than 200 and a table with the periods of $1/(d_1*d_2)$, where p_1, p_2 are odd primes, smaller than 25. From this table he confirmed Wallis' claim when $p_1 \neq p_2$ but he rejected it for the case that $p_1 = p_2$. He also remarked that if p>3 the l.p. of $1/p^2$ is m*p where m is the l.p. of 1/p (This proposition is not valid in all cases, e.g. it is not valid for p=487, but it is correct when the l.p. of 1/p and the l.p. of $1/p^2$ are not equal. A more general answer to the problem of the l.p. of $1/d^n$ is given by property 7).

Finally, in 1843, Thibauld formulated the properties 6 and 7 correctly in the same work. In this work only property 6 is proven. Property 7 is proven some time later (1846) by E. Prouhet.

2. The Teachers

Although, the p.ds constitute an important aspect of fundamental notions such as the rational numbers and the decimal system and despite of the fact that the investigation of their properties can have very positive effects on pupils' mathematical culture, the curricula in use in primary and secondary education, in Greece, reserve a very limited place to the study of p.ds.

In primary education the official instructions for the curriculum indicate that it must simply mentioned the existence of p.d. along with the existence of non-terminating divisions that have as quotient a p.d. There is no mention that further explanations on the subject, or any other properties of p.d. should be taught in primary education. According to the official instructions, in the J.H.S. level , only properties 1, 2, 3 should be taught (their teaching is placed to the 2^{nd} year of J.H.S.). At the H.S. level the official instructions of the curriculum don't mention any property of p.d. that should by taught (or any other activities on p.d. that should by realized).

Furthermore, the data presented in Appendix 1 indicate that, concerning the fruitful educational use of the subject, certain additional difficulties can come the teachers: In the investigated samples teachers appears to know very few things about p.d.

After answering to the questionnaire, all secondary education's teachers were individually interviewed. In these interviews the teachers of J.H.S. (18 out of the 32) stated, that p.d. is one of those subjects that are taught briefly or not at all ("it is taught briefly", "I don't teach it when time presses", "it is usually exempt of the exams",…). Only 2 out of the 14 H.S. teachers explain the transformation of p.d. to fraction by using Geometrical Progression (2nd year of H.S.). Also, no-one don't teaches topics concerning p.d. in the course of Number Theory (course taught only to students with a Scientific or Technological orientation).

The aforementioned point out that, moreover the fact that the teachers asked know a few elements[3] concerning the p.d., they have not consecrate some systematic work on the teaching of the subject.

3. Experimental Teaching

3.1. In order to investigate the possibilities and the difficulties that students encounter concerning the p.d. we have realize an experimental course with 24 students of 3rd and 4th years of the Department of Education of the University of Crete. Below we present the outline of this course and some significant elements of its realization.

The course was optional and lasted for 11 weeks (one meeting of 3hours per week).

The course was not in the classic form of a series of lectures. Instead, students worked in groups and the emphasis was given on their research work. In such a course, the learning of properties and procedures and the production (or reproduction) of well-done proofs are not the only elements considered as valuables. Experimental research, the formulation of conjectures and questions are also considered as important elements of doing mathematics. In traditional teaching these elements are disregarded, because the teaching focuses mostly in the learning of properties and algorithms or methods and little attention is given to the procedures leading to these properties and algorithms or methods. One, in order to appreciate the importance of the aforementioned elements and to begin to understand their role in doing mathematics, it is necessary to consecrate a relatively long period to the research of the properties of a mathematical area. This is important especially for prospective teachers' mathematics education. However, our students had never done that, since they had followed a traditional mathematical education. This fact as well as the elements presented in [1] led to use the aforementioned non-conventional form of course.

3.2. Before the beginning of the course, a questionnaire was given to the 24 students in order to evaluate their knowledge related to our subject. Some significant results of this questionnaire are presented in Appendix 2.

Taking into consideration the knowledge of the students, as it appears in the questionnaire, and the amount of inductive and deductive work that they had to do, we concluded that their work would have been more efficient if they had worked in groups of four. So, in the first meeting six groups of four students were formed. The students determined the formation of groups. However, the teacher interfered in the formation of 2 groups, in order to avoid the formation of groups in which basic knowledge and/or skills would be completely absent.

The rest 2 hours of this meeting were dedicated to the reviewing of necessary knowledge (algorithm of primes factors analysis of integers, algorithms of LCM and GCD, property "if a,b,c integers, GCD(a,b)=1 and a divides b*c then a divides c").

<u>In the 2nd meeting</u>, students initially made certain divisions (11/4, 5/7, 6/11, 3/17) and found some digits of their quotient. After this, the teacher told them that, as it is apparent, for some divisions the quotient is a finite decimal [**f.d**.], for others the quotient is a p.d. and maybe (maybe not) there are divisions whose quotient is a decimal non-finite and non-periodic [**n-f.n-p.d**.], so

[3] It is interesting to note that the only properties known by the majority of secondary education's teachers who have been asked, are the properties 1,2 and 3, which, are , also, the only properties contained in the schoolbooks of mathematics (Mathematics' Schoolbook of 2nd year of J.H.S., O.E.D.B. 1995, pp 58-62)

the purpose of this course is to explore this subject and to research related questions. The teacher also gave some indicative questions:

i) Which divisions have as quotient a t.d. and which a p.d. ?

ii) Are there divisions for which the quotient is a n-f.n-p.d.?

iii) What can we know for the period of a p.d. before we made the division (such as the length of the period)? Can we find methods permitting the quick calculation of the period (or a part of it), in all or in some cases?

The first question that students chose to investigate, concerned the divisions which have as quotient a f.d., as they knew some examples of this kind. The experimental investigation on this question permitted them to find that all tested divisions with divisors2, 4 and 8 have as quotient a f.d. and the same holds for 5. They also saw that all divisions by powers of 10 are terminated, the explication was easy for the students because of the particular algorithm of the division in this case. These results leaded some of them to conjecture that the same holds for the other powers of 2.

Some students, considering treated examples (such as 6/11, 2/3, 5/14, 7/12), introduced a new distinction: p.d. whose period starts immediately after the decimal point (**i.s.p.d.**) and p.d. whose period starts later (**l.s.p.d.**)

As every group of students had to continue alone their research until the next meeting, many of the students were anxious about the way of choosing a sample of examples which would permit them to obtain interesting results, especially concerning the questions for which specific conjectures had not yet emerged (such as i,ii above)[4]. So the problem was discussed in class. Two students proposed to take all the divisions with Dividend and divisor between 1 and 20. Some students proposed to link Dividend and divisor with a simple relation (they proposed D=d+1, D=d+2 and D=d-1), apparently because they found it difficult to vary in a systematic way two independent variables. Others objected that, in this way, they will have only one example for every Dividend and divisor. After some discussion, they concluded that it was better to keep first constant either the divisor or the Dividend and to vary the other, and then, after having finished with one divisor (or Dividend), to proceed in the same way to the next one. Concerning the kind of numbers to be tested, they proposed even and odd numbers, numbers bigger or smaller than 20 and only four proposed to test samples of prime numbers and samples of composite numbers.

<u>In the 3nd meeting</u> Students had found that in all tested divisions with divisor of the form 2^n ($1 \leq n \leq 9$) the result was a p.d. , and the same holds for the powers of 5 (tested until 5^5).

One group (the 2nd) claimed that they had found the explanation of this property and they presented it with the following example:

$7\mathbf{:}64=\frac{7}{16}=7\times\frac{1}{16}=7\text{x}\frac{1}{2}\text{x}\frac{1}{2}\text{x}\frac{1}{2}\text{x}\frac{1}{2}$ =7x0,5x0,5x0,5x0,5 , they remarked that the initial division gives the same result as a series of multiplications of f.ds, so the result is necessarily a f.d. (in the example 0,4375). They remarked, also, that this procedure will function in the same way with any other power of 2 as divisor and any other Dividend.

Other students said that the procedure could also be applied for the powers of 5 and some time later they remarked that it could be applied for the divisors of the form $2^n\text{x}5^k$.

The teacher remarked that other interesting results could be found from this procedure; for example results concerning the decimal part or other characteristics of the quotient. After these

[4] How to choose the sample of cases which will be examined, is an important problem of experimental research and the students who have received a traditional education have important difficulties on this subject even in simpler cases, such as the test of a precisely stated conjecture

instigation three students (not of the 2nd group) found that the procedure points out that, if a fraction $a/2^n$ (or $a/5^n$) is irreducible, the decimal part of the quotient of the corresponding division have n digits, and its digits, regardless of the decimal point, form the number $ax5^n$ (or $ax2^n$). [5]

The procedure conceived by the 2nd group uses the consideration of a division as a fraction, the transformation of a fraction to a multiplication (and to a series of multiplications) and the transformation of a fraction to a decimal. These transformations and changes of point of view and their inversions are basic tools in the research of the explanations for the properties of p.d. The majority of students realized this fact and began to use systematically these transformations in their research after the 5th-6th meeting. [6]

Another element creating difficulties in students' work was that they focused their attention in what they considered as "principal elements" of a division (Dividend, divisor and quotient) and they paid less attention to the remainder and even smaller to the sequence of partial remainders obtained during a division. This behavior changed slowly. This fact was the reason for which they significantly delayed to find the proof of the property that all non-terminating divisions give a p.d. (at the 9th meeting)[7].

From the 3rd until the 6th meeting they posed the question "how one can directly perform the four operations with p.ds.?" and they arrived to produce well organized algorithms, for the addition the subtraction of p.ds and the multiplication of a p.d. with a f.d. The patterns of these algorithms pointed out that if the quotient of 1/a is an **i.s.p.d.** and l.p.(1/a)=n then there is periodic repetition of the n first decimal digits of the quotient of b/a. After this, the remaining question: "In which cases the l.p.(a/b)< l.p.(1/a) ?" led them to conceive couples of examples of the type $1/(p_1xp_2)$, $p_1/(p_1xp_2)$ and of the type $1/p^2$, p/p^2 (where p, p_1,p_2 are primes). On the one hand these examples led them to discover property 4 and on the other hand, these examples and their extensions, conducted them to the discovery of the properties 6 and 7 (later they achieved to prove property 4 and 6 but not property 7)

During the construction of the aforementioned algorithms, they had often used p.ds, which they constructed directly by repetition of an arbitrary period. So they posed the question if these p.ds come from a division and how one can find it. The 5th group found the first part of the answer, which was presented in the 6th meeting with the following example:

0,353535… = 0,35 +0,0035+0,000035[8]….=

=357/100+357/10000+357/1000000…=357x(1/100+1/10000+1/1000000….)=

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[5] The students of the group which conceived the procedure, had not found these elements not because their discovery is difficult but because they regarded the procedure only as an answer to a specific conjecture ("When the divisor is 2^n, is the quotient a f.d.?"). They didn't examine if elements of the answer can enlighten other relative questions. This restricted way to look at answers and questions is characteristic of students who have received a traditional mathematical education. In the beginning of the course the students presented this attitude very frequently but progressively their behavior changed and after the 8th-9th meeting the majority of them, having an answer, looked for elements of this answer, which could help answering other questions.

[6] The issue about terminating divisions was not settled until the 7th meeting, when students arrived to pove that the only irreducible fractions transformable to f.d. are those of the form $a/(2^n x5^k)$. At this moment, the experimental investigations performed so far had convinced students that very probably these divisions are the only ones, which terminate. This conviction led them to reasoning in which was used the reductio at absurdum. This and the aforementioned transformations led them to the proof of the property.

[7] From the beginning of the course until the 9th meeting, from time to time, students proposed fractions that they considered as probable **n-f.n-p.d**. The experimental investigation, which some times was long, had always showed that they were p.ds

[8] this analysis of a p.d. was often used during the construction of the algorithms

357x(0,01+0,0001+0,000001)= 357x0,010101… here they claimed that the remaining problem was to find a division (fraction) having as quotient the decimal 0,010101… They also explained that this procedure can be applied when the period is longer or different. Other students had the idea to apply the same procedure on 1/11, which they also knew as 0,0909…, so they obtained 1/11 =9x0,0101…and from this 1/99=0,0101…After this, they made directly the division 1/99 and some others of the form 1/9..9 and they understood that they give the prescribed quotients. Following this they found easily the rule to transform an **i.s.p.d.** to a fraction (the case of **l.s.p.d.** was treated in the 7^{th} meeting)

During the last meetings, students discovered the property 5 and the property of complementarity between the first and the second half of the period of a/p (where p is prime and l.p.(a/p)=2xn). They also searched extensions of this property and other ways to find the period and its length faster, because, at this stage of the course, these problems were considered as major ones by the students.

Furthermore, some groups considered as an intriguing problem the question "Are there other primes than 3 for which the l.p.(1/p)=l.p.($1/p^2$)?" and they realized, using Excel, extended empirical researches on this question. They arrived to found p=487 but having examined the primes until 1000000 they didn't arrive to find another prime of this kind. Students had understood that properties 4, 6 and 7 permit to reduce the problem of finding the l.p. of a/b to the problem of finding the l.p of the prime factors of b. This conception had reinforced their interest in the aforementioned problem because it was considered as the last element in order to complete this reduction.

3.4. At the end of the course students' knowledge concerning the p.d. and the operation of division has been considerably improved (see Appendix 2).

They also have a more profound understanding of the decimal system (e.g. they understand its limitations concerning the representation of rational numbers).

Furthermore, their ability to select a sample of cases in order to perform an experimental investigation in arithmetic as well as the way that they consider the treated examples have significantly evolved.

At the domain of p.d. most of them have formulated questions and conjectures.

Moreover, the majority of students have began to express evaluations on problems and properties, characterizing them as more or less important or interesting. These evaluations depend on the links that they conceive between the characterized problem (or property) and the other properties and problems known to them.

Concerning algorithms, they appreciate other elements besides correctness; especially they take under consideration the "cost" and the efficiency of an algorithm and this consideration can push them to further research even in cases that an algorithmic answer already exists.

Finally, some of them have spent considerable time working on problems just because they found them intriguing, which is an indication that they begin to find some fun in mathematics.

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Appendix 1

A questionnaire was given to 110 Primary Education Teachers[9] (PET), to 32 Secondary Education Mathematics Teachers[10] (SET) and to 207 students[11] (S) of 1st and 2nd year of the Department of Education of the Univ. of Crete. The following three questions were included in the questionnaire:

Q1) When we have an irreducible fraction a/b<1(a,b, are integers) and we want to transform it into a decimal number by dividing a with b, in which cases the division terminate [12]?

Q2) The a/b and c/b are irreducible fraction (a,b,c,d are integers) . The length of the period of a/b is n. What can we say for the length of the period of c/b?

Q3) Which others properties of periodic decimals do you know?

The results are the following:

Q1	Correct Answer	Partial answers[13]	Wrong Answer	Answer that they don't know[14]	No answer
PET	20% (22)	18% (20)	24%(27)	34% (37)	4% (4)
SET	56% (18)	9% (3)	0	35% (11)	0
S	2% (4)	23% (47)	30% (62)	42% (86)	3% (6)

Q2	Correct Answer	Answer with a conjecture[15]	Wrong Answer	Answer that they don't know	No answer
PET	14%(15)	3% (3)	6% (7)	72% (79)	5% (6)
SET	25% (8)	0	0	75% (24)	0
S	1,5% (3)	2% (4)	16%(33)	67,5% (140)	13%(27)

Q3	Give property1	Give prop.3	Give no properties	Report property 4, 5,6 or other
PET	23% (25)	14% (15)	68% (75)	No-one
SET	88% (28)	78% (25)	13% (4)	No-one
S	3,5% (7)	1,5% (3)	95% (197)	No-one

The questionnaire to the students was given after we have looked at teachers' answers. Therefore, a more elementary question were added:

Q4i) The division 157:47 A) terminates B) it doesn't terminate but the decimal digits of the quotient are repeated periodically C) it don't terminate and the decimal digits of the quotient are not repeated periodically D) I don't know. Q4ii) The same question for the division 453:67

[9] The in service carrier of these teachers was 2-11 years (average 6,3 years).
[10] The in service carrier of these teachers was 10-29 years (average 18,5 years).
[11] Pre-service school teachers of primary education
[12] The term "the division terminate" was explain orally, for the case that some one don't clearly understand it. The same hold for the term "length of the period" of the 2nd question
[13] They present particular cases in which the division is terminated such as: "when we divide by 2 or 5", "when the divisor is 2, 4, 5 and 10", "when the divisor is 10,100, 1000 etc "
[14] We asked them to check a corresponding "don't know" box in case that they didn't know.
[15] "Probably they have the same number of digits…", "Maybe, they have the same number of digits"

Q4	Answer B in both divisions	Ans. A in both divisions	Ans. C in both divisions [16]	Ans. C in one division and B in the other[17]	Ans. C in one division and A in the other[18]	Ans. C in one division and no answer for the other	Ans. E in both divisions	No answer
S	4% (8)	1,5%(3)	63%(131)	10% (20)	2% (4)	7% (14)	5%(10)	8%(17)

Appendix 2

Some results of the initial and the final questionnaire of the experimental teaching

Q1[19]	Correct Answer	Partial answers	Wrong Ans.	Answer that they don't know	No ans.
Initial	0	17% (4)	38% (9)	45% (11)	0
Final	79% (18)	8% (3)	8% (2)	13% (2)	0

Q2	Correct Answer	Answer with a conjecture	Wrong Ans.	Answer that they don't know	No ans.
Initial	0	0	13% (3)	79% (19)	8% (2)
Final	84%(20)	0	8% (2)	8% (2)	0

Q4	B in both divisions	A in both divisions	C in both divisions	C in one division and B in the other	C in one division and A in the other	C in one division and no answer for the other	E in both divisions
Initial	0	0	71%(17)	8% (2)	4% (1)	0	17% (4)
Final	84%(20)	0	4% (1)	4% (1)	0	0	8% (2)

Comment In the initial questionnaire, the percentage of correct answers of these students is, in all common questions, a little smaller than the percentage of correct answers of their younger colleagues presented in the previous page [20]

Q5) Analyze an integer in primes factors. Q6) Find the LCM of two integers Q7) Find the GCD of two integers. For each one of Q5, Q6, Q7 two examples were asked (e.g. Analyze in primes factors 5940 and 13260). Success (S) is considered the correct answer in both. [21]

Q8) Determine the remainder of a division when the quotient found have a decimal part

[16] All students of this category found some digits of the quotient and as they didn't find periodicity or termination of the divisions they concluded that C is the correct answer.

[17] The students of this category acted as the students of the precedent category for one division. For the other division, errors in the execution of the division or conceptual errors (such as the misinterpretation of the repetition of one digit) led them to select B

[18] Errors in the execution of one division led them to select A for this division.

[19] The questions Q1,Q2,Q4 are the same as Q1,Q2,Q4 in Appendix 1

[20] This slight, but systematic, difference is probably due to the reform of the curriculum of mathematics in H.S., realized between 1997 and 2000. Because of the reform the younger students coming from the "Theoretical orientation" of the H.S., have received a significantly longer and reinforced mathematical education, during their H.S. studies, than the older ones. (More than 80% of the students in the Departments of education in Greece come from the "Theoretical orientation" of the H.S.)

[21] These algorithms were taught to the students on the elementary and JHS level and they were re-taught in the University, as part of compulsory courses

Q9)Make correctly the verification of a division when the quotient found have a decimal part.

	Q5 S	Q5 F	Q6 S	Q6 F	Q7 S	Q7 F	Q8 S	Q8 F	Q9 S	Q9 F
Initial	11	13	9	15	7	17	1	23[22]	3	21
Final	19	5	18	6	17	7	19	5	20	4

Q10, Q11) Find the l.p. of $13/(3x13^2x11^2)$ and of $28/(37x3x11^3x7^3)$.[23] (The l.p of 1/3, 1/13, 1/11, 1/7, 1/13 were given.)

	Q10 S	Q10 F	Q11 S	Q11 F
Final	17	7	16	8

[22] At the initial questionnaire 21 out of the 24 students believe that the remainder of a division is in all cases an integer. The failure also in Q9 is related to this misconception.

[23] To answer this question is necessary to combine properties 4,6 and 7.