# A PROJECT IN EUCLIDEAN GEOMETRY

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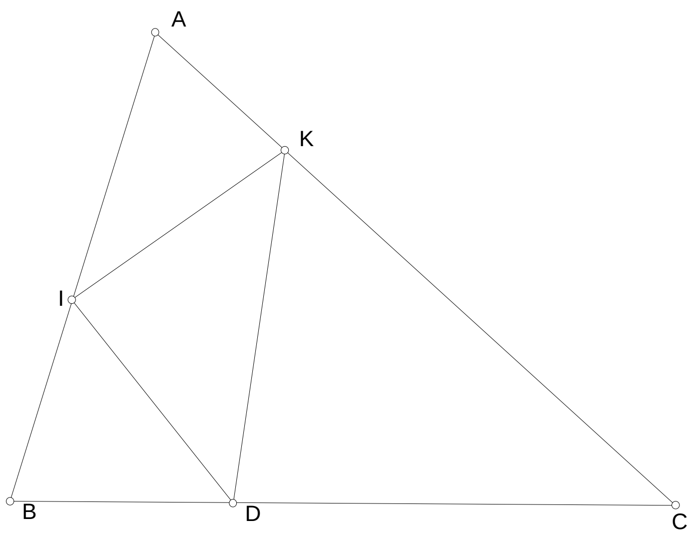
## ABSTRACT

One of the most effective instructional approaches in teaching Mathematics is project work, which, in an earlier paper, I connected with active learning, see Klaoudatos (1998). And this approach is going to be more interesting for the students if the project has been developed in collaboration with them. At the same time, these kinds of projects include 'dangers' for the teacher because of unexpected demands that might be found within. In this paper, I will describe such a project, which had been created in a problem solving class during the year 2000-2001 first semester.

Through successive generalizations of a simple geometric task, the students developed the following problem: *'In an ABC triangle, D is a point on BC from which we construct segments that form equal angles at the sides AB, AC, at the points I, K respectively. Which is the position of D so that the length of IK will be minimum?'*, see figure. The problem is expressed in terms of classical Euclidean geometry so that, at first glance, there is no evidence of the hidden difficulties.

The aim of the paper is twofold: first, to attract the attention of mathematics teachers and educators to the problem above, in order to be considered for project work, especially in teacher training courses. Second, to search for conditions under which the project could be effective.

The presentation will consist of four parts. In the introduction I will give some necessary information about the problem, while the second part consists of the solution of the problem. In the third part I will describe, shortly, the general theoretical framework in which the project took place. In the fourth part I will describe the way the project was conducted and some observations that emerged from the implementation phase. The same part contains tentative conclusions, as the research is still in process, that give hints on the factors that affected the successful accomplishment of the project. The research showed that the main factors were the time duration of the project, the previous experiences of the students in research projects and the belief systems of the students about mathematics.



# 1. Introduction

The project aroused from the following simple problem: In a right angle triangle ABC (A=90°) the point D is moving on BC. From D we construct segments perpendicular to the sides AB, AC at the points I, K respectively. Find the position of D in which IK has the minimum length.

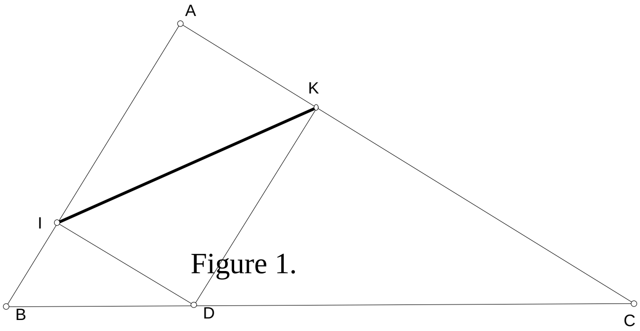



Figure 1.

The position of D we are asking for can be found when AD becomes the height of the triangle from A. The same point is the solution of the generalization of the problem in every triangle, see Honsberger (1996, p. 43-45). At that moment I asked the students to form a new generalization, while not having any specific idea in my mind. Then, after a heated discussion, the students developed the following proposition: *'In an ABC triangle, D is a point on BC from which we construct segments that form equal angles, ω, at the sides AB, AC, at the points I, K respectively. Which is the position of D so that the length of IK will be minimum?'*

The problem, which I have not yet found in the bibliography, is stated in terms of classical Euclidean geometry, it is the result of the generalization of the two previous problems and for this reason is not easy to recognize the difficulties 'hidden' in it. Indeed, it demands very much experience in this kind of problem for someone to suspect that the solution requires most of the developments that signified the revival of modern Euclidean geometry after 1873, see Honsberger (1995, p.88). I focused my attention to this area only when I noticed in the computer screen that the segments IK, as D moves on BC, seem to move on a parabola, which, also, seems to be their envelope, see figure 2.

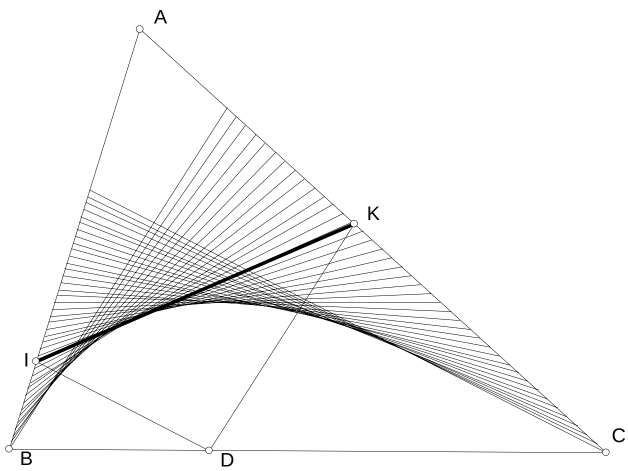



Figure 2.

One of the most powerful heuristics is 'search for relevant bibliography'. In F.G.-M (1952, vol.2, p.540, Greek edition) there is the following proposition: *'In a triangle ABC, every line IK which divides the sides AB, AC in segments inversely proportional from the vertex A, is tangent to a parabola, which is also tangent to the sides AB, AC at B and C'*. There is not any given proof, but there is reference in two papers, Brocard (1885) and Longchamps (1890), that I do not have up to now. On the other hand, Bullard (1935, 1937), gives the way in which we can construct the focus and directrix of this special parabola, see also Honsberger (1978, p.236-242).

According to this information, I developed the following plan:

1. I will prove that there is always a triangle in which IK segments divide the two sides in segments inversely proportional.
2. In this triangle, the IK segments have the parabola as an envelope.
3. I will search for the connection of these questions to that position of D in which IK has the minimum length.

The plan is divided into three cases, of which the first is the main one. From now on, ω will signify the angles which have been constructed from D to the sides AB, AC. Moreover, the solution is somehow condensed due to limited space, leaving the reader to clarify a few minor issues.

# 2. The solution of the problem

**2.1:** 1st case: ω=A.

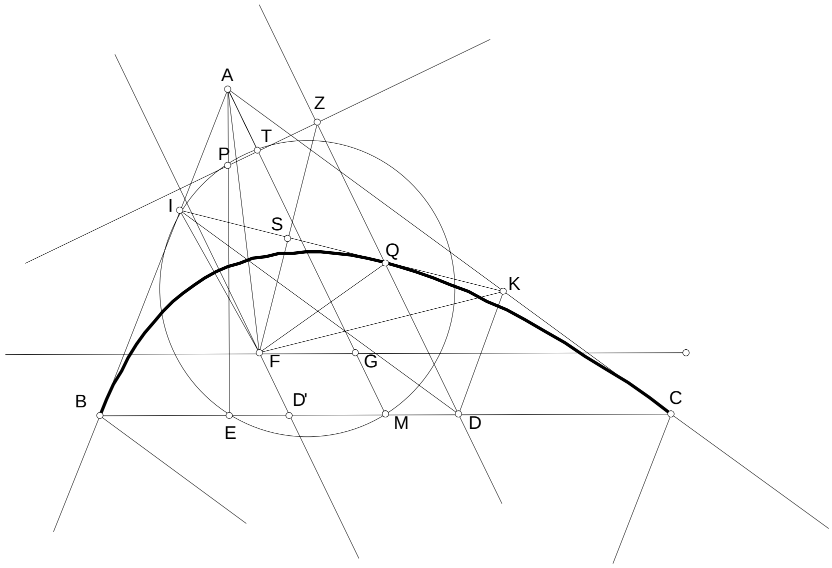



Figure 3.

In figure 3, following Bullard, PT is the directrix of the parabola, where AE is the height, AM the median and the circle is the Euler circle, of the triangle ABC. Then the focus F is the intersection of the symmedian AF and the line FG parallel to BC so that MG=AT.

I constructed IK as perpendicular bisectors of the segments FZ, Z any point on the directrix, and it is easy to realize that every Z corresponds to a point D through the ZD perpendicular to PT, which is also parallel to AM. In figure 4, Q is the point of intersection of ZD and parabola, and it belongs to IK. This point is the only common point of IK and of the parabola, so that IK is tangent to it as the point Z is moving on PT or, as the point D is moving on BC. But the point D can also be determined as the intersection of the parallel lines to the sides of the triangle from I and K.

So, the following two propositions are equivalent:

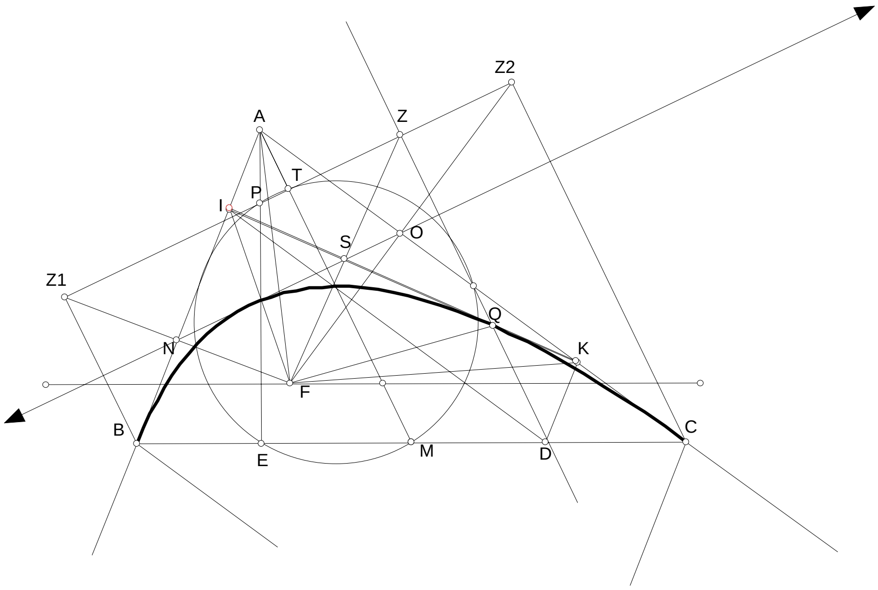



Figure 4.

1. The perpendicular bisectors IK of ZF, are tangents to parabola, which is the envelope of these segments.

2. The IK segments divide the sides AB, AC in segments inversely proportional, namely $\frac{BI}{IA}=\frac{AK}{KC}=\frac{BD}{DC}$.

Moreover, in figure 4, the points Z1, Z2, are the reflections of F through AB, AC respectively, so the points N, O are the midpoints of FZ1 and FZ2. Then, the segment NO is one of the positions of IK. On the other hand, NO is the Simson line of the triangle AIK and for this reason F is a point on the circumcircle of this triangle. As a result, the angles of IFK triangle are constant and then, as the Z moves, the triangle IFK remains similar to itself. From the similarity of the IFK and NFO triangles we have the proportion $\frac{IK}{NO}=\frac{IF}{FN}\geq 1$. So, the minimum length of IK happens when IF=FN, in other words, when D coincides to D′, which is the intersection of the axis of the parabola to BC, see figure 3. In the following I will regard D as ‘the position of minimum length’.

**2.2**: 2nd case: $\frac{A}{2}\leq\omega\leq A$

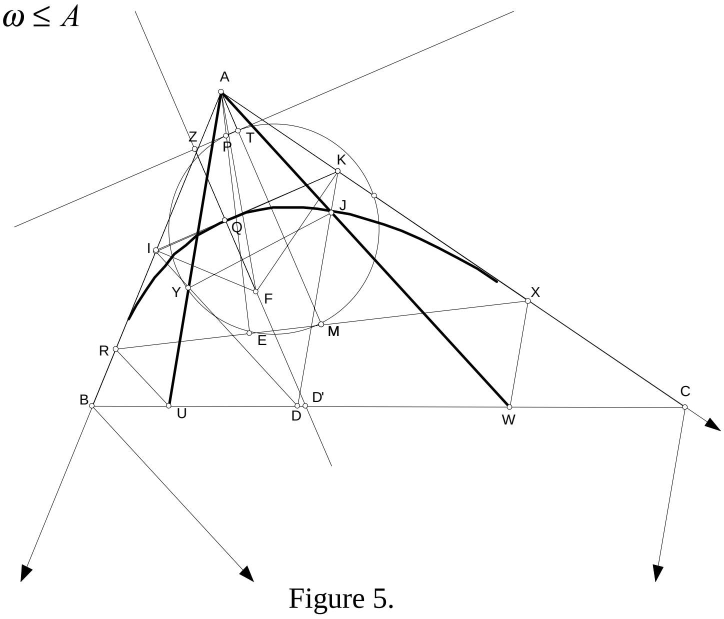



Figure 5.

In figure 5, angle BAW = angle UAC = ω.Then the proportion $\frac{UY}{YA}=\frac{AJ}{JW}$ is transferred to $\frac{RI}{IA}=\frac{AK}{KX}$. Thinking by analogy to the first case, in triangle ARX, the position of minimum length is D≠D′, which is determined as the point of intersection of the parallels to AW, AU from I, K.

**2.3:** 3rd case: ω> A.

Working in the same way it is easy to recognize that again D≠D′, as can be sown in figure 6, where the triangles UAW and RAX are constructed exactly in the same way as in the previous case.

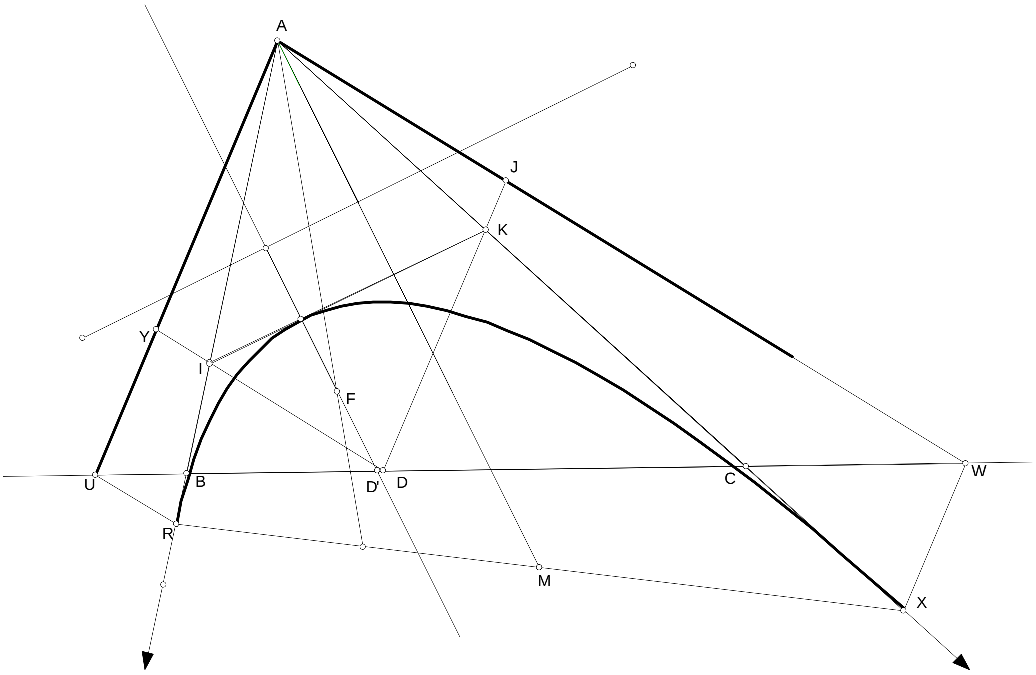



Figure 6.

# 3. Some theoretical notions

One of the most effective ways to teach Mathematics, in my view, are problem solving, see Klaoudatos (1994), Klaoudatos and Papastavridis (2002). Through this kind of teaching the students could develop the feeling of personal contribution to their learning and at the same time they could acquire the ‘inner freedom to act’, Sierpinska (1995, p.8), which are both components of active learning, see Anthony (1996).

A necessary condition for problem solving to be successful as a way of teaching is ‘to get involved in problems’, and for this reason I consider project work as an effective teaching strategy, Klaoudatos (1998). But how can we form an appropriate classroom environment to promote such strategies? Actually, this question refers to the ‘situation context’ which together with the ‘task context’ are the two kinds of context that Wedege (1999) has discriminated.

More specifically, I developed in a problem-solving classroom, an experimental teaching environment through which the students, working in groups, could develop a general approach for problem solving together with the understanding of specific mathematical ideas. The development of such an environment was influenced by the ideas of Polya, which I have condensed to the following steps, Polya (1973, see in particular the first three chapters):

1. Experimentation and observation
2. Pattern recognition

3. The development of a guess
4. Testing the guess

It is expected that students, working through the steps above, will be led to the acts of understanding, see Sierpinska (1990, 1994 p.56). The acts of understanding describe the crucial moments in the process of concept formation and are the following four: *identification, discrimination, generalization and synthesis*. And, although Sierpinska connects these acts with the notion of epistemological obstacle, in the case of the specific problem of the paper I use them only to give a short description of the students' thinking and reasoning. This theoretical framework is interesting because it is in close connection to problem solving and offers some tools for the evaluation of this process.

The problem of the paper was developed in the first semester of 2000, in a problem solving class of undergraduate students, where I had proposed it as a project, not knowing the solution and not having any idea of the difficulties hidden within. The students accepted it at once and were enthusiastic because of their contribution in its development. Thus, I considered it as an ideal teaching situation in which the teacher would act 'as learner', as constructivists claim. But soon I realized that, working in the broader scheme of 'teaching mathematics through project work', I could not help the students because I did not know what mathematics were involved and which ideas I had to organize and put forward. The result of the endeavor was the abandonment of the project after three weeks of students' work, partly due to students'frustration and partly because of the limited available time and the students' other obligations. The situation reminded me of Brousseau's observation that is, in words of Sierpinska (2000), *'if the teacher finds it useful to act as if she did not know how to solve the problem, this should only be good acting and not the actual state of the teacher's mind'*, Brousseau (1997, p.45-47).

# 4. Observations and tentative conclusions of the implementation phase

Only when I solved the problem I felt confident to propose it again as project work. The aim of the project was to search for the conditions under which the project could be effective. Indeed, in the first semester of the next year 2001, I proposed it to the post-graduate students of my problem solving class of Mathematics Education Section. Roughly speaking, the 17 students of the class could be divided into two groups: those who had the required knowledge or, at least, part of it, who were experienced secondary teachers, who I will call teachers, and those who had just finished their graduate studies and did not have that knowledge, who I will call young students.

The time duration of the project was one month, in which the students formed four groups that had to submit a weekly progress report. These reports were discussed in the classroom every week, where all the students made comments and discriminated between potential and non-potential ideas and strategies. I gave the minimum relevant bibliography only after the first week, allowing them to work on their own ideas for a week. By minimum bibliography, I mean, the F.G-M proposition and Bullard's two papers to start their research. At the same time I warned the students that they had to learn some new material, in sources that they had to find by themselves. It was agreed that this project would be included in the evaluation of the course. Finally, before I proposed the problem, the students had already solved the two previous problems, from which it was developed.

I proposed the problem, making a short presentation of it in the class, where I gave some information and used the figure 2, where there is an unusual combination of a triangle and a curve that seemed like a parabola.

The required knowledge consisted of the geometric properties of parabola, the Euler circle, the Simson line, the notions of symmedian and envelope and the triangles of Brocard. All the teachers were familiar with the parabola, the Euler circle and the Simson line, while some of them knew all the require knowledge. So, the main part of their job was to synthesize this knowledge in order to manage the project. But the young students encountered special difficulties because, not only did they have to find new sources and learn new material, but also they had to synthesize it.

In general, I can say that all the teachers, with the exception of two who did not proceeded in the fourth step of the Polya's scheme, solved the problem working in groups by using the Geometer's Sketchpad software in the first three steps. On the other hand all students had to overcome many difficulties throughout the task. In the following, I will concentrate on the young students.

Six of the seventeen students of the class were young students who participated in two groups, group A, which consisted of two teachers and two young students, and group B, which consisted of four young students and one teacher. The young students of group B showed an unexpected behavior: After the second week, they ignored the bibliography given to them and accepted a proposition made by the teacher of the group, to follow a different path towards the solution, which remained unfinished. A similar behavior showed by the other two young students of group A, who, although they solved the problem based mainly on the work of the other two members, they submitted their own 'solution', avoiding the given bibliography, which, also, remained unfinished.

The discussions in the classroom, the observations during the project work and the interviews that followed the end of the task, led to three main factors: The time duration of the project work, the previous experiences in research projects and the belief systems about mathematics. More specifically:

1.The time duration of the project affected the cognitive and meta-cognitive processes as well as the affective domain. The 'distance' that the young students had to cover in new knowledge seemed too long in a month. A student described the problem as *'A well without a bottom' because by the time we had completed a piece of new knowledge, a new area was opened*. For those who, typically, covered this distance, referring to the two young students of group A, there was another obstacle. They had to transfer the new knowledge to the actual problem. But the time for them was too short to make the *synthesis,* namely to connect appropriately the various pieces of new knowledge to the problem. So, while I could recognize the acts of understantig in the various steps of problem solving, in the final step the synthesis was absent. On the other hand, the heuristics did not work for them. A student said that *I tried to follow the heuristic: if you cannot solve the problem, try a simpler one. But soon I realized that I had already solved all the simpler problems before the project.* As a result, as time passed, the students' positive attitudes transformed into anxiety and then frustration.

2. The interviews uncover a rather unexpected fact. All the seventeen students insisted that it was the first time they faced a project with a 'sense' of genuine research work. Especially the young students had difficulties in finding the relevant bibliography and the way they could use it. Most of them had never heard of the F.G.-M book. A student said that *up to that time, I believed that all Euclidean geometry was included in the school textbook and there was nothing beyond that.*

3.The beliefs system about mathematics of the young students had an impact on their failure. One of the most striking points was the presence of the parabola in the triangle. A student said *I could not accept the parabola together with the triangle, because the parabola does not belong to Euclidean geometry, it belongs to analytic geometry.* I believe that this was one of the reasons, the young students of group B immediately accepted the proposition for another path towards the solution without using the parabola. On the other hand, their experiences in project work up to that time, created the belief that the solution of every problem should be based on familiar and specific knowledge and, in the case of given bibliography, the bibliography should contain all the necessary information for the problem. And, as a student said, *in every case we knew that, eventually, we would solve the problem within a few hours' work. But, then, after some time, we began to realize that, perhaps, we could not solve it.*

According to the above points, we can conclude that the young students were unprepared to handle this research project. This conclusion poses an important question: How can the teacher create an experimental environment in his classroom without having any experience in it? On the other hand, I stressed time as a decisive factor and not the process of the development of the required knowledge, because this knowledge is of elementary nature and I believe that it will be learned in a matter of time. It is another question, not of this paper, about the 'type of knowledge' that can be developed through this kind of work, as Sierpinska (1998, p. 58) proposes. At the same time, I recognized many of the results stated in international bibliography about the decisive role of the beliefs and affects in problem solving, which survive in tertiary education. See, for example, the review of the domain in Barkatsas and Hunting (1996), the compartmentalization perspective of mathematics and the dichotomy between 'theory and exercises', Schoenfeld (1992, p. 342), the presence of the parabola that acted as an epistemological obstacle, Sierpinska (1994, p. 125). In my opinion, the teacher has a limited influence on the factors above, except, perhaps, time. They mainly depend on the view of mathematics that an education system has adopted and the education praxis that arises from it. The research is still in progress and for this reason I have described the conclusions as tentative ones.

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