TWO COMPONENTS IN LEARNING TO REASON USING DEFINITIONS

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ABSTRACT

This paper discusses the transition to the use of formal definitions in mathematics, using the example of convergent sequences in Real Analysis. The central argument is that where in everyday contexts humans categorize objects in flexible ways, the introduction of mathematical definitions imposes a much more rigid structure upon the sets so defined, and hence upon the acceptability of different types of argument. The result is that, in order to have their reasoning accepted in proof-based mathematics courses, students must do two things:

1. align their notion of what mathematical objects belong to a given set with the extension of the defined set, and
2. (more fundamentally) learn to express their reasoning about such sets exclusively in terms of the definitions or other results traceable to these.

The importance of these two components is illustrated using two examples. First, a student whose idea of what objects belong to the set of convergent sequences does not closely correspond with the definition, and whose reasoning is therefore insufficiently general. Second, a student whose set corresponds well to that given by the definition, and whose work is arguably more mathematically sophisticated, but who still does not "succeed" since he fails to reason using definitions in the required way.

Finally, pedagogical implications are discussed, with particular reference to tasks that require exploring the extension of defined sets. We consider the role of collaborative student work in promoting awareness of a broader range of examples within such sets. Further, we suggest that there is often a gap in the structure of the tasks that students are asked to complete; that many would benefit from tasks which begin with a term and require students to generate examples, in addition to the more usual task of beginning with an example and establishing its membership of a set.

Keywords: definitions, proof, advanced mathematics, imagery, Analysis

Human categorization and mathematically-defined sets

Human cultural categories are usually not "classical", in the sense that their extension is not determined by necessary and sufficient conditions for membership. Instead, many have "fuzzy" boundaries, such as the category described by the phrase *tall man*. They may also exhibit "prototype effects", as exemplified by the category *bird*, in which case there is general agreement that a robin is a "better example" than a penguin. Such effects may be attributed to considerable complexity in the internal structure of these categories (Rosch, 1978, Lakoff, 1987).

By contrast, mathematically defined "categories" or sets of objects do not have these attributes: the selection of a defining property precisely delimits a set, and does not distinguish any members as "better examples" than others[1]. This does not stop mathematicians regularly using certain examples in reasoning or explanation, and does not mean that it is necessarily easy to determine membership or otherwise in any particular case. However, in the logical structure of the subject, no special status is accorded to any particular examples, and this impacts upon accepted standards of argumentation in the subject: once a definition for a mathematical term is agreed, work that purports to establish results about the associated category must do so via arguments traceable to this definition (Tall, 1995).

This logical status of definitions should make some aspects of tasks set for students simple. Proof problems encountered at beginning university level generally either require showing that a particular object is a member of a mathematical category (e.g. "show that the sequence $(1/n)$ is convergent"), or showing that one category is a subset of another (e.g. "show that all convergent sequences are bounded"). The existence and status of definitions renders the "top level" (Leron, 1985) or "proof framework" (Selden & Selden, 1995) required in these cases very simple: one must either show that the object satisfies the definition, or show that one definition implies another. However, it is well recognized that students not only struggle with such tasks, but regularly employ alternative and less mathematically appropriate strategies such as generalization from an example or a "concept image" (Moore, 1994, Vinner, 1992, Harel & Sowder, 1998).

This paper examines the behaviour of such students, identifying two things they must accomplish in order to move from their existing reasoning habits, which are well adapted to everyday argumentation, to a mathematical approach to the use of definitions.

Research context

The students used as examples in the following took part in a research study in a top-ranking UK university. They were attending two pedagogically different first courses in Real Analysis, each of which covered work on sequences, completeness and series. The first of these courses was given in a traditional lecture format, the second was a new course in which students worked in groups in a smaller classroom, attempting to answer a structured sequence of questions which led to them proving the majority of the major results for themselves[2] (Alcock & Simpson, 2001). A number of students from each course attended biweekly

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1 The word "category" will be used rather than "set" from now on in order to highlight the fact that student behaviours would often be appropriate when handling everyday categories: it does not refer to categories in the sense of Category Theory.

2 The course was based on Burn, 1992.

interviews in pairs. The interviews were semi-structured and comprised an introductory discussion of recent material and the students' experience of the course, a task-based section in which the pair worked largely without intervention from the interviewer, and a final section in which they reviewed their work on this task as well as responding to questions about their more views of proof and definitions in general.

One task that generated particularly rich data was the following, which was set in week 7 of the course:

Consider a sequence (a_n). Which of the following is true?
a) (a_n) is bounded $\Rightarrow$ (a_n) is convergent,
b) (a_n) is convergent $\Rightarrow$ (a_n) is bounded,
c) (a_n) is convergent $\Leftrightarrow$ (a_n) is bounded,
d) none of the above.
Justify your answer.

The interview excerpts presented in the following two sections show students who have decided upon the correct answer to this question, and are now attempting to produce justifications.

Generalization from a "prototype": Wendy

In everyday argumentation it is often acceptable to make statements about entire categories of objects based on generalization either from a specific example or from a more generalized "prototype" representing what is considered typical of the category in question. This is sensible in everyday life, where categories are not delimited by definitions, but is often inappropriate in advanced mathematics, at least in contexts such as beginning university courses where the student is required to learn about mathematical concepts as they are currently understood by the community. We can see what happens when students try to apply this strategy it in the following interview excerpts, in which Wendy's justification for her answer to the question involves a generalization from an image of a monotonic convergent sequence.

W: Well if it converges, you get closer and closer…
Pause (drawing).
W: Is that enough to like, justify it…a little diagram, what have you?

Prompted for a proof, she does not do much more than describe her picture:

W: [*Draws a monotonic increasing convergent sequence*] It's convergent… yes so if it's convergent it's always…or…say it could be the other way round it could be…going down this way [*draws a monotonic decreasing convergent sequence*]. It converges, so it's always above that limit.

In the context of the material she is supposed to be learning, Wendy's argument is inadequate in two ways. First, it is based on inviting the listener to agree with the generalization, without further explication of properties of convergent sequences from which one can deduce the conclusion. Second, her reasoning seems to indicate that she is only considering monotonic sequences, and hence is not properly arguing about the whole category. Such problems are well recognized in studies of students' use of visual imagery, in which it is noted that focus on a particular image can lead to a fixation with irrelevant details or even the introduction of false data (in this case, the assumption that all the sequences concerned are monotonic) (Presmeg, 1986).

In this case it is not clear whether Wendy thinks that all convergent sequences are monotonic, or whether she simply considers this subcategory more important in some way than other kinds of example (this would not be unreasonable, given that a great many of the sequences she has encountered so far will have been monotonic). It may also be argued that this is preliminary reasoning, much like any mathematician would perform, and that Wendy can be expected to refine her argument. Unfortunately this is not the case. It proves difficult to dislodge Wendy's fixation with monotonic sequences: despite repeated prompts from the interviewer to consider other types of example, she keeps returning to reasoning depending upon this property. Essentially, she *acts* as though she is unaware of the extension of the category of convergent sequences as delimited by the definition, and the result is that her reasoning is insufficiently general.

Abstraction of properties from a prototype: Cary

In everyday argumentation, if a generalization is questioned, we may provide extra justification by citing some properties of objects in the category in order to clarify why our conclusion must hold; saying, in effect, "I am correct *because*…". Depending upon the parties present, these properties are likely to be chosen spontaneously in order to draw on mutual experience.

We see this in Cary's attempt at the same problem. He begins in a way similar to Wendy, by making sketches in order to reach a first hypothesis:

> C: I've drawn…er…convergent sequences, such that…I don't know, we have er…curves… er…approaching a limit but never quite reaching it, from above and below, and oscillating either side.

However, he is not content to assume a generalization. Instead he first performs a mental check for any possible counterexamples, postponing his conclusion until he has completed this to his own satisfaction:

> C: I was trying to think if there's a sequence…which converges yet is unbounded both sides. But there isn't one. Because that would be…because then it wouldn't converge. Erm…so I'll say b) is true.

Following this he begins trying to formulate properties that will hold for all the objects he wishes to consider, and that can be used to demonstrate that his conclusion is correct:

> C: If it converges…that has to be…well I don't suppose you can say bounded. It doesn't have to be monotonic…. Erm…Yes, I'm trying to think if there's like…if you can say the first term is like the highest or

lowest bound but it's not. Because then you could just make a sequence which happens to go…to do a loop up, or something like that.

Not surprisingly, finding an appropriate property proves difficult, and Cary rejects several possibilities (that the sequences must be monotonic, that the first term would serve as a bound). Note, however, that not only are his attempts at argumentation more sophisticated than Wendy's, but that he also appears to have a better awareness of what kinds of object are classed as convergent sequences. He is aware that such need not be monotonic or even necessarily have the property that each term is nearer to the limit than its predecessor.

A teacher would recognize that the property Cary needs is the definition of convergence, but this appears not to occur to him. When eventually prompted for the definition by the interviewer, he writes down an incomplete version, and then returns to his previous attempts to abstract properties from his prototypical images. Eventually however, he is persuaded to complete his definition, at which point he realizes that this is useful, and is able to quickly construct the essence of an appropriate argument:

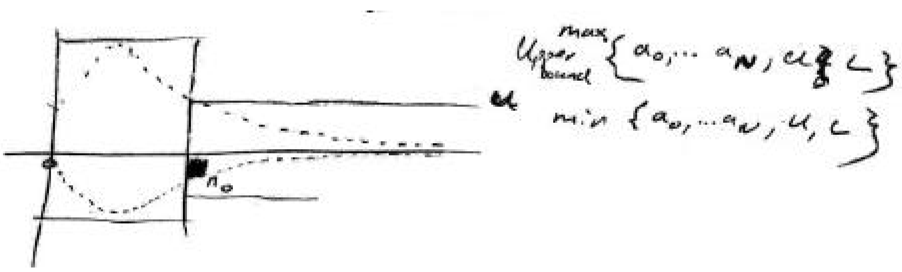

Figure 1: Cary's diagram to illustrate his definition-based argument that all convergent sequences must be bounded

C: …Yes, your n_0…that could just be called your n_0 instead, so going back to your definition up there, there exists this point here, such that after that point, i.e. when n is greater than n_0, the sequence…that statement there won't be less than any epsilon which you just happen to pick…. And so it's…and so the upper bound – so because there's finitely many terms before n_0, then er…your upper bound will either be plus or minus epsilon, or it'll be the maximum of those finite terms beforehand.

Notice that his diagram at this stage is drawn so as to illustrate some of the possible forms of non-monotonic sequences.

Pedagogical implications

Wendy's case is reminiscent of much that is seen in the literature on inappropriate uses of generalization from examples when a deductive proof is required (Chazan, 1993, Harel & Sowder, 1998). Here we would like to emphasize that the situation might improve for Wendy if she had a better awareness of what objects belong to the category of convergent sequences as this is defined in the course. Such an awareness should make her less likely to overgeneralize from a restricted range of cases, and more likely to recognize the potential pitfalls of relying on relatively fixed images.

Of course this may not be enough. Cary's case makes it clear that an idea of the category that corresponds closely to its defined counterpart, even when combined with a relatively mature approach to mathematical argumentation, is neither sufficient nor efficient in learning to produce the type of argument that is expected at this level.

In attempting to remedy these problems, we could simply attempt to enforce or at least heavily encourage the use of definitions. However, it might be argued that this is what lecturers already think they are doing, and that while some students do take this advice on board and become competent in using definitions, students like Wendy and Cary are far from atypical. A more student-centered approach would be to capitalize on the strategies already in use: after all, Cary is employing good mathematical thinking, and it would be desirable from a pedagogical perspective to capitalize on his existing strengths. This should not be impossible, as in the same interview it becomes apparent that on a philosophical level he already understands the role of definitions remarkably well:

> I: Do you feel that you now see maths in a different way?
> C: Not maths, but arguments.
> I: Right…can you explain how?
> C: We had this…I walked into the kitchen. I thought, I'll have an early night, I was going to make a cup of tea,
> I: Mm,
> C: And there was two people around the table, arguing about whether or not law came from morals?
> I: Right.
> C: And erm…so I was listening to them, and I thought, they're getting this all wrong. So I started joining in, and…and I found myself, *defining* stuff, and I was like, I cannot argue with you unless I have it defined, exactly what I'm supposed to be arguing about…

It appears that, with encouragement, it should not be too great a step for Cary to enact this understanding in his mathematical work. For others, the step to be made is greater, as indicated by this short continuation of the earlier extract from the interview with Wendy:

> W: Is that enough to like, justify it…a little diagram, what have you?
> I: Well, I'd like you to prove it, if you can.
> W: Oh dear! (*laughs*) Oh right, well, if a to the n…

This indicates that Wendy does not consider proof to be a natural extension of her existing efforts at justification. This is not uncommon among students who regularly employ visual imagery in their work, and is epitomized in a remark by Fred:

> F: Well it's not really scientifically proven. Because I think…I think I'm right, but it's not… it's not…if we've got to prove it then that's a different kettle of fish altogether.

However, even Wendy clearly has some idea of what is meant by convergent sequence, and we do not want to create a situation in which "proof" for her becomes any more removed from her intuitive ideas than it already is. It should be possible to help her move on from her present position without asking her to completely change her thinking: the fact that Cary's eventual answer is closely linked to his diagram indicates that building on this type of imagery can lead to the type of argument we would like to see. Indeed, the study as a whole indicates that students who reach the strongest understanding are often those who have access to both formal and imagery-based representations and who move flexibly between these. Adam is such a student, and he explicitly remarks upon this link:

> A: It's not usually enough to stick the definition down, you have to stick it down and then remind yourself of what it means.

So how can we promote such an awareness of the link between the definition and the objects that a student thinks of as belonging to a given category? One approach supported by the results of this study is the use of collaborative student work. It was found that those students in the new (problems-based) course showed more inclination to be critical of their initial conclusions, and to test these by attempting to check for counterexamples, than their peers on the lecture course. For example, Kate's initial thinking about the question described above is similar to Wendy's:

> K: …it would be bounded wouldn't it, by its first term…
> J: We don't know if it's increasing or…
> K: And its last term.
> J: Depends if it's increasing or decreasing doesn't it?
> K: Well it would be bounded, either below – if it was decreasing it would be bounded above…

However, she and Jenny go on to question their conclusion, attempting to think of examples for which their argument will not work.

> K: But it's just, this one.
> J: Is there such a sequence that we don't know…
> K: Yes that's what I mean, is it true?
> *Pause.*
> K: Can you think of one?…Because I can't.

This behaviour does not necessarily reflect a mature awareness of the philosophy of advanced mathematics; it often appeared as an unexamined reaction to repeated experiences of being proved wrong. However it does appear that regular feedback and challenge from teachers and peers led to students developing the habit of subjecting their thinking to more rigorous checks, the effect of which is that their work reflected a better correspondence between their views of what objects belong to central categories and the formal versions of these.

A further suggestion is generated by noting a gap in the types of task required of beginning university students: we often ask students to show that some specific object is a member of some mathematical category (beginning with an object and concluding with a category), or to show that some category is a subset of another (beginning and concluding with categories). We also set tasks demanding that manipulations be performed on one specific object in order to obtain another, for instance (as an initial task) finding an N such that $|1/n| < \frac{1}{10}$ whenever n is greater than N (beginning with an object and concluding with another). Far less common are tasks that begin with a mathematical category and ask the student to provide examples (beginning with a category and requiring objects). Dalhberg and Housman suggest that example generation in response to a new definition is a feature of the thinking of better-performing students (Dahlberg & Housman, 1997). Hence it seems that, if well designed, tasks that start with a definition and ask for a range of examples might create a sense of the link between a definition and the objects included in the associated category. They could therefore help students like Wendy to bring their idea of what is in a given category into line with that determined by the agreed definition, and help students like Cary to think more readily of the definition as a natural basis for constructing arguments about mathematical categories.

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