TEACHING FOR UNDERSTANDING FOR TEACHING: ADDRESSING THE CHALLENGE

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ABSTRACT

I present a constructivist approach that is used in some TE mathematics classes at MSU. This approach employs exclusively a model of a 'mathematical situation', a set of physical operations and a physical language to reason about **al**l students' mathematical doings while a unique system of reinforcements, grading and assessment methods, support the learning experience.

Introduction

Elementary school pre-service teachers tend to ignore the need to learn mathematics, as 'we don't need it', 'we already know it', where the 'it' refers to what they conceptualize as the mathematics that is taught in k-5.

By and large, students are puzzled when they are confronted with a 'weird' question like 'what is…' which very few of them would answer and even fewer would be able to explain "why…" by means of a concrete example or by means of abstract reasoning. All the more when one gets to fractions; which usually involves intense emotions. The Math-Educator reader is probably familiar with the Mantra 'but I know HOW to do it…' which one could frequently hear in my office especially during the period of learning of fractions. Though the student stated that it is extremely important for her/him as a future teacher to be able to explain WHY she/he 'flips' the ½ and WHAT does the 6 stand for in $3 \div \frac{1}{2}$ still when 'forced' to try and construct some kind of explanation she/he would cling to the Mantra 'but I know how…'.

More often than not it feels as if the students are trying as hard as they can to 'protect' their 'fragile' mathematical assets, to keep it intact and away from me. Thus in the beginning of the course it seems that the students perceive me as the 'destroyer' of their knowledge rather than the one who is supposed to help them to construct it. This is supported by a student's comment in a first-day-of-class Attitude-Questionnaire: 'I'm a little unsure about fractions so I'm nervous that we'll spend so much time on them…' referring to the Syllabus that shows that a large fraction of the course will be dedicated to fractions.

Therefore I've employed a somewhat 'aggressive' constructivist approach in order to get the students to unpack their fragile mathematical assets and to re-construct a more flexible and a deeper understanding of the different basic mathematical concepts. The main goals of this approach are to 'force' prospective teachers to understand their math-doings and to develop a reasoning attitude towards the learning and the teaching of mathematics.

Though at this time I do not have any formal research results, I will provide findings that could indicate that this approach is effective in achieving its goals. Also, I'll bring findings that could indicate that it is also the students that find this approach helpful.

However, I would like to emphasize that this approach was developed for pre-service elementary school teachers and not for elementary school students. It was meant to help the future elementary school students to make sense of their mathematical knowledge and doings, and not to teach them the basic concepts. At the same time this approach provided them with an efficient tool to analyze and to understand their future students' doings of mathematics. Nevertheless, I believe that there are some aspects of this approach that could be adjusted to help young students in their learning of mathematics.

The Approach

Much of Math-Education is about understanding students' difficulties in grasping the Abstract-Formal Mathematical concepts and algorithms. Thus our proposed approach retreats to natural-physical doings in trying to promote a "natural" or an intuitive understanding of the basic mathematical concepts as a well-established, firm base for the understanding of the more abstract, formal concepts.

Therefore our approach is based almost entirely on natural operations that our mind can perform without any formal learning ("Join", "Take Away", instead of Addition and Subtraction). Furthermore we have used a "natural" physical language, assuming that we "don't know" any formal mathematics and so we cannot use words that mathematicians "invented" such as addition, subtraction, division etc.

Thus the main principles of our approach are:

a Constructing a Natural-Intuitive understanding of the mathematical concepts:
  - a.1 Using only "Natural" operations, which require no formal learning of "how", such as 'to Join'.
  - a.2 Using "Physical" language and avoiding Formal mathematical language: JOIN but not ADD, CUT INTO but not DIVIDE etc.
  - a.3 Using Visualization tools such as drawings and 'role acting': 'I'm the first set and you are the second, How many of 'you' can be made out of me' or acting out the 'joining'/ 'taking-away' of the sets/elements etc.

b Building on a deep understanding of simple Whole Number situations as a basis for all further learning:
  - b.1 Using a "Whole Number" Language: 'we have HALF groups …' and not 'we have half a group'.
  - b.2 . Fractions are 'just' Numbers: ½ or 1½ is as good of a number as 1, 2 or 10…
  - b.3 Using Whole ("Natural") Number models to deal with "new" non-natural kinds of numbers: $\frac{2}{3} \div \frac{5}{7} \approx 6 \div 5$ so 'DO THE SAME'.
  - b.4 Sequencing a continuum from whole number situations to fractions: Something like $4 \div 1, 4 \div 2, 4 \div 4, 4 \div 3, 1 \div 4, ½ \div 1, ½ \div 2, \ldots 1 \div ½, 1½ \div ½, \ldots 1¾ \div 15⁄17 \ldots$.

c Using an abstract model, which conceptualize all basic mathematical situations as the same, to channel and to mold the reasoning.

d Promoting a Reasoning attitude:
  - d.1 Any statement must be accompanied by reasons: Why is it true/false, Why is it important/not important, Why is interesting/not interesting?
  - d.2 Making as many statements as possible about any given situation: Declarative statements: 'we have 3 sets', descriptive statements: 'there are 3, 2 and 3 elements in the sets', or relational statements: 'these 2 sets are of equal size' or 'this is larger than that' etc.
  - d.3 Exploring each situation thoroughly rather than employing a "task-oriented" exploration: All the Why's, and How's questions as well as 'What this means', 'What if we change this…", "Which units are used" etc. .
  - d.4 Concentrating on a few carefully chosen examples rather than investigating many situations.
  - d.5 Massive Reasoning tasks (using our physical language and our model exclusively): Why $\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$.
  - d.6 Massive investigations (trying to reason) of other student's understandings and solutions.

e Developing awareness of the meta-cognitive processes:
  - e.1 Using the model to compare the same/different situations, reasoning and doings.
  - e.2 Using the same language/doings in comparable situations.
  - e.3 Reasoning about the Thinking: 'Why this procedure and not another', 'what made me think of this…' 'Does it remind me of anything…' etc.

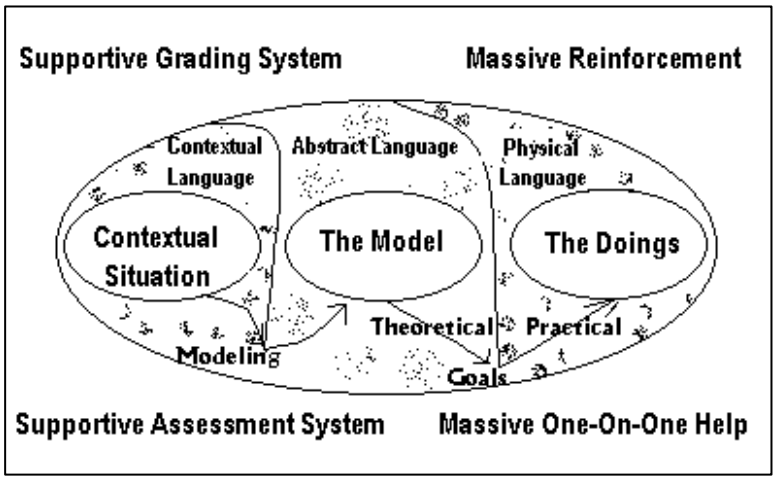
f Employing a holistic approach working up through the contextual to the Abstract and then 'back' to the "Abstract-Physical" plan and again to the contextual stratums.

The Learning Space

We perceive the core of the Learning Space in which the learning-teaching experience is taking place as consisting of three sub-spaces: The contextual, The Abstract and The Physical-Abstract sub-spaces. These three sub-spaces differ in the kinds of reasoning that are employed in each of them; in the language that is used in each of them and in the kinds of activities that are performed in each of them. The transition among

the three sub-spaces is done by goal-driven means of re-phrasing into the language of that used in the 'new' sub-space. This model also provides the frame for relating the contextual concepts (Altogether), the abstract-mathematical concepts (Addition), the Abstract-Set concepts (Union) and Physical-Abstract concepts (Join). The core of the Learning Space is surrounded by a comprehensive Support System, as it is shown in Figure 1:

Figure 1 The Learning Experience



The Support System

We used intense Verbal Reinforcements in order to boost the student's self-image and their confidence as well as to reduce their anxiety level. To illustrate, we frequently used sayings such as: '… the future of the mathematics knowledge is in your hands…' or '...If you were just an engineering student I wouldn't mind, but I expect more from you...', or '…you can't understand it NOW but you WILL in a short while…', or "…anything I can do YOU can do better…". A relaxed and an informal atmosphere encouraged students to discuss as openly, and as frankly as they could their mathematical ideas, their feelings and their attitudes towards the learning experience.

Furthermore, the students had as many one-on-one instructional sessions as they needed and many more by e-mail consultations. Though it is noteworthy that few of the students perceived the 'generous' office hours policy as a negative one, some expressed feelings that could be summed up as: 'we paid for learning in class, so YOU need to make sure that WE will not need office hours'...

In addition, we employed a supportive grading system in which the 'learning processes' (rather than the "results" or "products") were assessed and graded and in which most of the points were given on 'proven hard work'. To illustrate, the final exam was only 25% of the total grade, 10% was assigned to the weekly Home Work assignments (which were returned fully checked and with relevant comments), 12.5% was assigned to two papers, and approximately 50% was assigned to the quizzes and the exams that students could re-do.

The Contextual Sub-Space

By the contextual sub-space we refer to the sub-space of the mathematical story-problem. Thus the language here is contextual and is related to the 'story' (Apples and oranges in one instance or velocity and distance in another), while the activities here are 'real' (Picking or eating in the apples case or driving in the

other) and so is the reasoning (We have less apples since few were eaten or -3 can't be the velocity since the car is moving forward).

Though there is much to be said about the nature and the constructs of this sub-space (choosing the problem, sequencing issues etc.) we will limit our current presentation to the two other sub-spaces.

The transition from the contextual sub-space to the Abstract sub-space is motivated by the "problem" which is first stated in the "contextual language" (How many fruits do I have altogether? Or How far did he drive? Etc.) and then modeled by the Abstract Model and Re-phrased into Abstract Sets-language. This is also the point in which we relate the Formal mathematical concepts to the contextual concepts.

The Abstract Sub-Space and the Abstract Model

By modeling the contextual situation we move to work in the Abstract sub-space and to use Abstract Sets-Language. Complex situations involve staging (breaking down) procedures and using the model iteratively, but the Basic Model refers only to the basic binary mathematical operations ($+, -, \times, \div$) and it consists of three components, and two types of goals.

The Model Components are:

a. First Component - The Number of Disjoint Sets that are involved in the situation.
b. Second Component - The Number of Elements in each of the disjoint sets.
c. Third Component - The Total Number of Elements in the situation.

The Model Goals are:

a. The Theoretical Goal is: Either to 'expose' the 'omitted' value of one of the model's components (i.e. # of sets or # of elements) or to 'describe' the relations between two of the model's components.
b. The Practical Goal is: To 'physically' do something in order to achieve the theoretical goal (i.e. to JOIN sets, to TAKE-AWAY elements, to PUT-EQUALLY into a few empty sets, to compare/correspond sets, to measure one set using the other, etc)

Hence we define two basic types of the model: The *Additive Model* and the *Multiplicative Model*. While a **CONSTANT NUMBER** (2) of disjoint sets that are involved in a situation characterizes the *Additive Model*, it is the **EQUAL SIZE** of the disjoint sets that characterizes the *Multiplicative Model*. In both models the third component is the number of elements of the **Union Set** of all the disjoint sets.

The Contextual Space determines the Theoretical Goal for the Abstract Space, which is therefore expressed in Sets-Language. If the theoretical Goal is to 'expose' the third component, then the model represents an addition (*Additive Model*) or multiplication (*Multiplicative Model*) situation, whereas if the theoretical Goal is to 'expose' the second or first component, then the model represents a subtraction (*Additive Model*) or a division (*Multiplicative Model*) situation.

Furthermore, in the case of finding the First Component in a multiplicative situation, the model describes what is usually referred to as The Measurement Division approach, while if it is to find the second component (# of elements in each of the disjoint sets) then the model describes what is usually referred to as The Sharing or Partitative approach.

Relational Theoretical Goals in the *Additive Model* could describe either additive relations (bigger-smaller) that are basically subtraction situations or Multiplicative relations, which are Ratio situations.

The reasoning in the Abstract Space is based on the relations between the Sets and it is done in order to support the doings in the Physical-Abstract sub-space. For example a contextual situation of 2/3 x 15 will consist of 2/3 of a set that contains 15 elements, hence since the Union set is comprised of less than one such

set it must be that its number of elements is less than the number of elements in one set, which is 15. Similarly in the case of 15 x 2/3; each of the 15 sets "contribute" to the Union Set less than 1 element, so it must be that the number of elements in the Union-Set is less than the number of sets. Hence, if in the Physical-Abstract sub-space we get a solution of more than 15 elements, we'll know that we are mistaken.

The Physical Abstract Sub-Space or the Doings

The theoretical Goal determines the practical Goal in the transition to the Physical-Abstract sub-space, where the 'Doings' take place. The most significant features of this space are the 'physical' doings and the language that are employed on the abstract objects (Sets and Elements), such as "Put (elements) Equally", "Put Proportionally", "Take Away (elements)", "Make sets of size x", "Break Down (sets)", "Stage (the doing)" etc.

These physical doings also serve to reason "practically" about the situation. For example: '$3 \div \frac{1}{2} = 6$ since I've MADE 6 sets, each having ½ elements until I've exhausted my resource set of 3 elements'. Alternatively; 'I've PUT-EQUALLY all my 3 elements into all of my ½ empty sets. So now each of the (whole) sets in the situation has 6 elements, since each of its 2 halves has 3 elements'.

At this point, I will present a few (partial) examples that best illustrate our method and than we will describe a typical class discussion to provide the context in which we use this approach:

1. Abstract Space: 145+324= , Sets Language - 2 sets of 145 (Set A) and 324 (Set B) elements - An Additive Model. The Theoretical Goal is to reveal the number of elements in the Union Set. Hence in the Physical-Abstract Space the Practical Goal (expressed in "physical" Language on Abstract objects) is to JOIN both sets. The Doing of the JOIN will be STAGED:
   a. Reasoning: since I need to JOIN all sets of the situation I can do it in any way that I wish to as long as all the elements of all the sets will 'get' into the Union-Set eventually, so:
   b. First Stage – BREAKDOWN Set B into 3 Sub Sets B_1 of 300 (Ones) elements, B_2 of 20 (Ones not Tens) elements and B_3 of 4 (Ones) elements.
   c. Reasoning - I am comfortable with these "nice-round" numbers which I can manipulate mentally, so:
   d. Second Stage - JOIN A and B_1 (445 elements) and then JOIN this with B_2 (465 elements) and finally JOIN this with B_3 to have the Union-Set with 469 elements.

2. 364-79= Abstract Space - 2 sets of 79 (Set A) and of ? (Set B) elements and the Union-Set have 364 elements - An Additive Model. The Theoretical Goal is to reveal the number of elements in Set B. Hence in the Physical-Abstract Sub-Space the Practical Goal is to "Take Away" the 79 elements of Set A from the Union-Set so only the elements of set B will be left there. Figure 2 illustrates partial doing in this situation. It is noteworthy that the reasoning that leads the DOINGS is the wish to work with small numbers and so the doings involve Tens and Hundreds and not only Ones as in example # 1:

3. $3\frac{1}{4} \div 2\frac{1}{5} =$: First Component- # of Sets-?- Second Component- # of elements in each set - $2\frac{1}{5}$, Third Component - Total Number of elements $3\frac{1}{4}$ Hence the theoretical Goal is to reveal The value of the first Component. So the practical Goal is to MAKE SETS of $2\frac{1}{5}$ elements each TO EXHAUST our RESOURCE SET of $3\frac{1}{4}$ elements, as is shown in Figure.3.

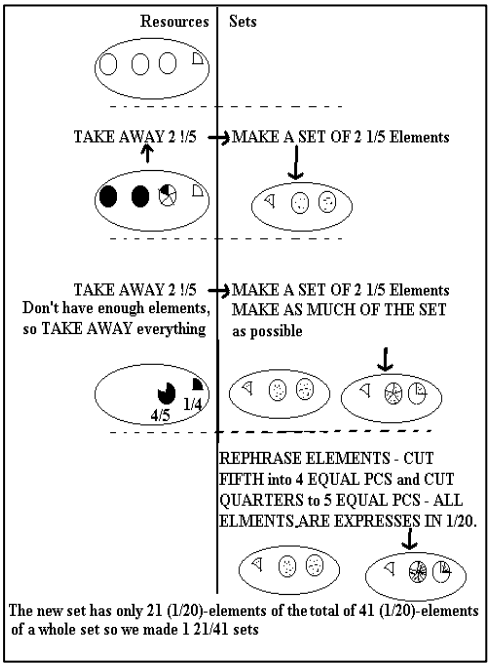


Figure 3 Example No.3

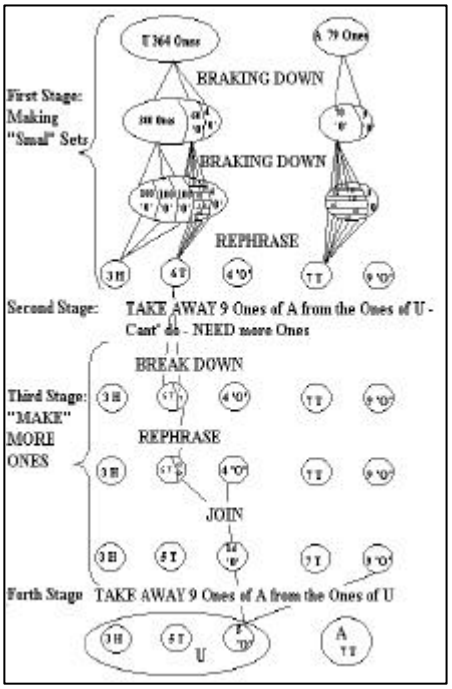


Figure 2 Example No.2

4. $12\frac{1}{6} \div 1\frac{3}{4} =$: First Component- # of Sets -$1\frac{3}{4}$ - Second Component - # of elements in each set -?, Third Component - Total Number of element $12\frac{1}{6}$. Hence the theoretical Goal is to reveal The value of the Second Component. So the practical Goal is to PUT EQUALLY all the $12\frac{1}{6}$ elements, to EXHAUST our RESOURCE SET, into all the ('empty') sets ($1\frac{3}{4}$). Partial doings are described in Figure 4.

5. Contextual Situation of Share $27 according to 3:2 Ratio- Model- 2 sets (additive situation) - # of elements in both sets are different and unknown, Total # of elements in the situation 27. Theoretical Goal to reveal the # of elements in each set – The Practical Goal – To PUT PROPORTIONALLY (to EXHAUST our RESOURCE SET of) all the elements in the 2 empty sets. Figure 5 Describe one way of 'Doing" it:

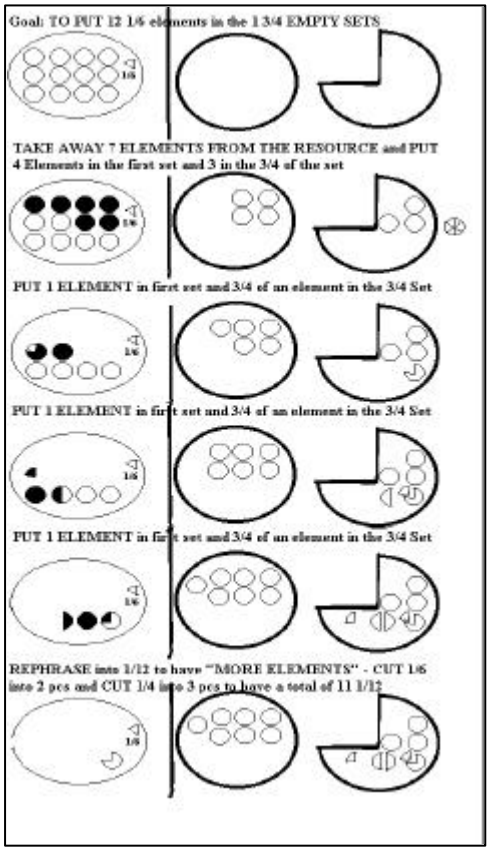


Figure 4 Example No.4

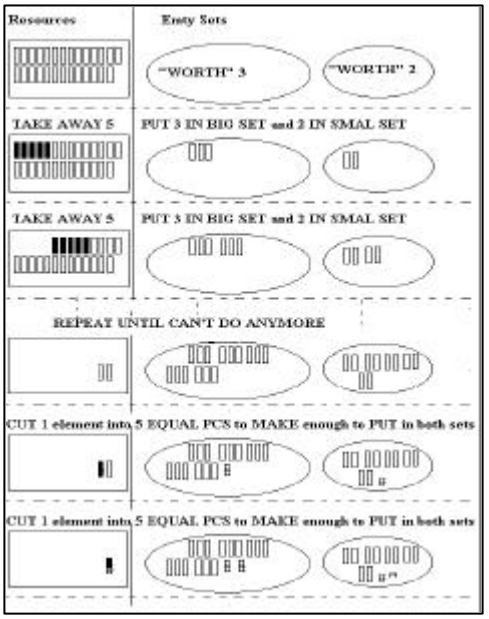


Figure 5 Example No.5

A typical Class Experience

The class discussion usually begins by presenting one or more contextual situations that lead to a specific mathematical operation, or to a few operations depending on whether my objective is to investigate an operation or if it is to compare a few operations. These situations are brought up by me or by the students as a response to my challenge.

We discuss the use of the contextual language and how it affects the way in which one perceive the mathematical situation. To Illustrate: 'I had 15 candies and I ate 7 of them. How many more…' or 'I ate 7 candies and I'm allowed to eat 15. How many more…' or 'I have 7 candies to give to my 15 guests. How many more candies…'. The language will lead us to different mathematical representations (in the Abstract-mathematical sub-space) that are all 'summed' in the 'mathematical sentence': 7+8=15.

We than move to the physical-abstract sub-space by introducing "our model" for the situation (2 sets, with 7 and ? elements each, 15 elements in total), the theoretical goal (to reveal the second component, number of elements in each of the sets), and the practical goal. The practical goal is motivated by the contextual space. Either it is to "take Away" (what I ate…), or it is to "fill in" (what I'll eat), or it is to "Pair" (candy to a guest).

We emphasize the connection between the contextual situations to the practical goals in the context of how a teacher can initiate a specific "physical-doing" by his students.

Also, when it is relevant, we discuss different ways of "Physical-Doing" to achieve the same Practical Goal such as the one in example No.2 for 364-79; in this case we compared 8 different algorithms of students, not all of which were mathematically correct. In each case we discuss All significant different options for a specific Doings, emphasizing the strengths and weakness of each one of them.

We also try to understand the connections between the motivations that lead an individual to his/her doings. For example the traditional addition/subtraction by columns could be understood as motivated by a desire to work with small numbers, not exceeding 10. Breaking down "ugly" numbers such as 364 to 300, 60 and 4 could indicate that this individual has no problem conceptualizing or manipulating 'big' numbers as long as they will be 'nice' and round. Also, using a 'counting on' (missing addend) algorithm for solving a subtraction problem (in our physical-abstract language it is referred to as 'fill in') could be understood by a strong inclination to addition algorithms and avoiding subtraction algorithms, which could be a sign of some weakness in this area. This kind of insight is something that a future teacher should be aware of while 'just' a mathematician could be satisfied with the fact that the 'mission had been completed'.

Other kinds of class discussion are constructed around a given solution to a specific mathematical sentence, which is provided by me or by the students themselves as a response to my challenge ($3 \div \frac{1}{2}$). Here the solution is purely mathematical, and we are trying to "reconstruct" the meaning, or the motivation to this solution by "justifying" each of the steps by means of our "physical language and our model". We ask: did this student think about many sets, each of them with exactly ½ elements (not half an element), and when joined together make a set with 3 elements. Perhaps he thought about a situation with as many as ½ sets (~2 sets), each of them having exactly the same number of elements (which we can't see at the moment), and 'all the ½ sets' are joined together to make a set of 3 elements. The first option would lead to a practical goal of "making" sets of ½ elements and we will look for 'evidence' of that (something like ½ + ½ + …) or may be we will look for evidence that he is "putting equally" all his 3 elements (resource) into all of his ½ sets, and than he looks at One set to determine how many elements are in each of his sets. The students seem to enjoy this kind of discussion and they usually are very active in these discussions.

Many times the discussions are based on group activities in which groups of students try to make sense of a given solution (or to 'physically do' in order to solve a problem). Sometimes each group works on a different solution and the discussion consists of presenting the different findings and trying to gain a comprehensive picture. In other instances the different groups will work on different problems (63+45, 63-45, 63x45, 63÷45) and the discussion consists of comparing the different doings in the different situations and how our model explain could these differences and similarities.

In addition to discussions of the more practical kind ("doing" to solve, analyzing and comparing different "doings") we also have theoretical discussions. In some of them we discuss the theoretical-mathematical rules (associative, commutative and distributive) and how we can "prove" them by our "practical-doing" methods.

While in other theoretical discussions we compare the mathematical concepts of the different basic binary operations of Arithmetic by means of 'our' model and 'language', we also consider the different approaches to teaching Arithmetic that exists in the literature and we 'connect' them to our models. For example, the Measurement approach to division is tied to our Multiplicative model where the theoretical goal is to reveal the first component - the number of sets in the situation. Furthermore, these approaches contribute a significant insight to our approach. For example, the "making of sets" could be understood as "using a measuring set/cup". These discussions offer the students opportunities for consolidating their knowledge in

which they can make sense of the many "different mathematics details" that they have collected through years of studying mathematics and to construct for themselves the 'big picture'.

Though some of the students had complained that this approach 'makes things harder than they really are', I believe that these types of comments reflect a misunderstanding of the main goal of a basic college-level Mathematics course for future elementary school teachers. I believe that the goal of such a course is not to 'teach' addition/subtraction/multiplication/division[1] but rather it is to offer a substantive basis for understanding of the knowledge or algorithms that the students already have (and which therefore they consider to be 'easy').

This approach offers informal "proofs" or justifications for the knowledge that the students already possess, but are unable to explain or to justify. The model, the "Physical-Doings" and the "Physical-Language" serve us instead of the formal theorems and logic which are used by mathematician to prove/understand their mathematical knowledge. By "physically" tracing each step of the 'statement' (solution algorithm, commutative rule, etc.) we prove it is "True" or "False". Moreover the 'physical-doing' serves as what mathematicians referred to as insightful proof, a proof that offers a 'deep' understanding of the situation on hand.

Also one of the student's tasks as future math-teachers will be to identify difficulties of their future student's doings of mathematics and to help their students to resolve these difficulties. Our approach provides them with a tool that makes tracking down and pinpointing these difficulties easy as well as offers them ideas to help their future students by means of "physical-doings".

Discussion

It is rather difficult to put into two-dimensional paper the full picture of a teaching-learning philosophy, which entails many dimensions simultaneously (mathematical, physical-doings, cognitive-reasoning, cognitive-procedural, affective, class interactions, individual aspects etc..).

The students were constantly engaged in verbally explaining each step of their contextual, abstract and physical doings and their motivation for doing it at different levels and in the different "languages". They were constantly required to relate various representations (I.e. contextual, formal-mathematical, the abstract - physical) and doings across situations and across concepts. Students were encouraged to construct their own individual 'doings' and they were challenged to try also 'awkward' procedures and not only the most 'efficient' one. For example: '…try to put ½ elements in each set first, even if it will make the 'leftovers' in the resource set an 'ugly' number…'(We were frequently using 'ugly' numbers).

Also, they had to 'finish' their colleagues' doings, or to come up with reasoning for their colleague's doings. Alongside, we had to work on the emotional dimension as well, since confusion and frustration were frequently threatening to interfere with the learning.

It seems that many of the students tend to appreciate the concise view of the situation that the model grants them, as well as the rigid frame it provides to lead the doings and the reasoning in a new situation. Also they seem to enjoy the flexibility that using the non-formal mathematical language permits. Nonetheless, they appear to not be very enthusiastic about the less structured and what they have conceptualized as less 'directed' teaching or "teaching-less" teaching.

[1] Though sadly enough too often we find ourselves in a position that we are obligated to verify that our students do know these basic mathematics concepts and algorithms.

By the end of the semester more than 70% of the students in both classes (50 students) were able to present good reasoning (>60%) while about 40% of the students presented very good to excellent reasoning capabilities (> 75%). Also, by the end of the semester it was evident that the quality of the discussions in the class was changed to the better, considerably. By then, students could discuss the whole mathematical situation from the constructing of a "good" problem-story up to the "doing" to solve it and they could reason about all its different aspects.

However the picture is not all so bright, and this approach calls for persistence in implementation in order to be effective, as 'It's too hard' for the students, and for the most part they prefer 'just tell me what to do and I'll do it' as can be seen in a few of the students' comments in the evaluation: '… She is very frustrating although she makes you think… I feel she makes things more difficult than they are, ..this class is too challenging for the type of class, …but be aware the course is a lot of hard work...'.

As mentioned before, we found this approach to be very efficient, particularly in promoting students' understanding of fraction's situations. Students that first resisted my 'extremist' initiative eventually 'discovered' that 'It's so simple, I can't believe I did not understand it before'. So the following collage of students' comments might offer an optimistic closure for this paper: '… Teaching was excellent and she puts everything into context and helped me understand why …have to admit that in the past I have been afraid of math - you have taught me that it can actually be "fun." …was my favorite class this semester, which really surprised me because I didn't think I liked math….at the beginning of the semester I gave you a below average grade of your teaching … at the end, not only me, but a lot of my other classmates… saw this pattern… After using it (the model) continuously, …it makes problems a lot easier to solve and easier to explain..'.