CONNECTING UNDERGRADUATE NUMBER THEORY TO HIGH SCHOOL ALGEBRA:
A Study of a Course for Prospective Teachers

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ABSTRACT

Most universities in the US require prospective high school mathematics teachers to major in mathematics. In most cases, these students will encounter a course in abstract algebra and number theory, usually in the third year. Though the topics studied in these types of courses are closely related to those of high school mathematics, research on teacher education indicates that students generally do not see these connections and regard these courses as completely unrelated to the mathematics they will be teaching in the future. For example, the students in my study did not appear to view linear congruences as being analogous to equations. When solving a congruence such as $5x \equiv 3 \pmod 7$, they did not tend to think of "dividing" both sides of the congruence by 5, or of using a "guess and check" strategy. A method for solving linear Diophantine equations was viewed by the students almost exclusively as an algorithm to be memorized, and they generally did not recognize the connection between this method and the solving of equations in elementary algebra. This study has several implications for teacher education. In line with current recommendations for teacher preparation, I believe that we should make explicit for future teachers the connections between the abstract algebra and number theory that they study as undergraduates and the high school algebra that they will teach. Placing emphasis on the connections between the mathematics they are learning at the undergraduate level, the mathematics they already know, and the mathematics they will be teaching will emphasize the importance of understanding why algorithms and processes work. We expect them to emphasize this understanding with their own students; thus expecting it of them is important .

1 Introduction

The teaching of algebra is arguably the largest component of the job of a secondary school mathematics teacher. Most secondary schools in the US offer at least three levels of courses in algebra, and most universities in the US require that students have completed at least two years of algebra study. In addition, algebra is the foundation for much of the mathematics that secondary school students will study. According to the National Council of Teachers of Mathematics, algebra is an "essential component of contemporary mathematics and its applications in many fields" (NCTM, 2001).

Many researchers have emphasized that in addition to studying a good deal of mathematics at the undergraduate level, prospective teachers need to develop *knowledge of mathematics for teaching* – an understanding of the underlying processes and structure of concepts, the relationships between different areas of mathematics, and knowledge of students' ways of thinking and mathematical backgrounds (Fennema and Franke, 1992, Ma, 1999, MET, 2001). However, it has become clear in recent years that this knowledge of mathematics for teaching is not easily developed. For most prospective teachers there is what Cuoco calls a *vertical disconnect* between the undergraduate mathematics that they study and the mathematics that they will teach, and that "this is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra" (Cuoco, 2001). Undergraduates do not automatically recognize that the topics studied in abstract algebra provide explanations for why certain equations can be solved and others not, and provide rationale for many of the processes of high school algebra (Usiskin, 1988). *The Mathematical Education of Teachers* recommends that prospective teachers take courses in abstract algebra and number theory in order to examine the mathematical structures foundational to algebra and number systems, noting that these connections may need to be made in other courses (MET, 2001). If these connections are not made, then teachers must rely upon their own precollege algebra education, "an experience that is likely to have been focused on an algorithmic approach to mathematics and unlikely to have contributed to conceptual understanding" (p. 441, Ball and McDiarmid, 1990).

My dissertation study focused on students' understanding of congruence of integers developed during a unit on modular arithmetic in an introductory number theory course. The topics studied in this course were chosen by the instructor because they are closely related to those of high school mathematics. For example, the students were introduced to various methods for solving linear Diophantine equations, including the method of reduction of moduli. In order to understand how to use this procedure, the students were first introduced to solving linear congruences of the form $ax \equiv b \pmod{n}$. In general, the students did not appear to view congruences as being analogous to equations. When solving a congruence such as $5x \equiv 3 \pmod{7}$, they did not tend to think of "dividing" both sides of the congruence by 5, or of using a "guess and check" strategy. Reduction of moduli was viewed by the students almost exclusively as an algorithm to be memorized, and they generally did not recognize the connection between this method and the solving of equations in elementary algebra.

2 Review of Relevant Research

There is a small body of research on the learning of abstract algebra, most of which focuses on elementary group theory. Dubinsky (1994) writes that "constructing an understanding of even the very beginning of abstract algebra is a major event in the cognitive development of a mathematics student" (p. 295). Dubinsky also argues that since a significant proportion of mathematics majors will become high school teachers, this course plays a critical role in developing teachers' knowledge of and attitudes toward mathematical abstraction. Clark *et al* (1997) write that "many who are to be ambassadors and salespersons for mathematics at the secondary level develop a negative attitude towards mathematics in general and a fear of abstraction" (p. 182). There seems to be general agreement that this type of course is a turning point in the mathematical careers of many students, and that serious investigation into the teaching and learning of abstract algebra is of critical importance. Recently, research on the development of concepts in elementary number theory has begun to appear, though this has for the most part focused on concepts related to divisibility and proof. To my knowledge, no research has focused on the topic of congruence of integers.

Research on children's interpretations of algebraic equations and the process of solving these equations reveals that there are many conceptual difficulties. Booth (1988) says that "in algebra, the focus is on the derivation of procedures and relationships and the expression of these in generalized, simplified form" (p. 21). Students have difficulty accepting algebraic expressions as "answers," preferring to pick values for the variables in order to give a numerical answer. Kieran (1981) and Wagner (1977) showed that secondary school students typically regard the equals sign operationally – as "a unidirectional symbol preceding a numerical answer" (p. 24, Booth, 1988), instead of relationally – indicating that two quantities are the same. Kieran (1988) reported that when solving equations, beginning algebra students tended to rely on a memorized procedure that appeared to disregard the role of the equals sign in the equation. Wagner and Parker (1999) describe the difficulty that students with an operational view of equality often face when solving equations in algebra, noting that most solution methods assume a relational view of the equals sign, so that students must work with the entire relation as they transform it into equivalent relations. They state that "few students fully appreciate the fact that solving an equation is finding the value(s) of the variable for which the left- and right-hand sides are equal" (p. 333).

Bernard and Cohen (1988) write that understanding how to solve equations by the equivalent-equations procedure is a conceptually sophisticated task that requires a good deal of cognitive preparation. They claim that methods typically used in pre-algebra such as guess-and-check, the "cover-up" method (viewing equations as arithmetic identities with one value covered up), and the "undoing" method (viewing equations as a sequence of reversible steps that have been applied to a number), though important activities, are not adequate preparation for learning to solve equations using equivalent-equations. Note that a student with an operational view of equality can be successful learning to solve simple equations by such methods, since the relational aspect of equality is not necessary to guess the value of the missing number, and then perform arithmetic operations to check if the result is correct. Herscovics and Kieran (1999) also report that students have a great deal of difficulty solving equations by the equivalent-equations procedure. Kieran (1999) states that though research shows that

many students become quite adept at solving equations in an automatic, procedural fashion, "these studies demonstrate that the same students are generally not aware of the structure underlying the manipulations they perform" (p. 351).

3 Study Background and Methods

Since the topic of congruence is virtually unstudied, I decided to use an exploratory case study design in my dissertation study. In the spring of 2001, I was a teaching assistant in a third-year introductory number theory course at a large state university in the southwestern US. The course was taught by Dr. Thomas, the professor who had originally designed the course. The students enrolled in the course were primarily prospective secondary mathematics teachers.

Modular arithmetic was introduced in the course as a tool for solving linear Diophantine equations, and students were first taught to solve them graphically, by guessing, and by using the Euclidean algorithm. Congruence was defined in two ways: a is congruent to b modulo n if 1) a and b have the same remainder upon division by n, and 2) n divides $a - b$. Reduction of moduli was then introduced as a means to find all solutions to linear Diophantine equations.

Dr. Thomas and I chose six students that we viewed as above-average based on exam scores and our perceptions of their attitudes towards the course. I interviewed the students three times over the course of the semester about their conceptions of statements of congruence. The first interview took place approximately three weeks into the modular arithmetic unit, the second took place after the exam, approximately three weeks later, and the third at the end of the semester. The interviews were transcribed, and then these data were triangulated with written questionnaires and exams, and field notes from observations in class. Analysis of the data was primarily done via open and axial coding, followed by a modified discourse analysis.

4 Results and Discussion

In general, the students demonstrated an operational view of congruence. They tended to view a congruence statement as a transformation from $\mathbb{Z}$ to what they called the "mod n world." For example, Chris interpreted the statement $5x \equiv 1 \pmod{11}$ as:

> "I think of this [left] side of the congruence as being any possible number and this [right side] is the class of number it is, it's a 1 mod 11. That [right] side of the congruence thing means something specific to me and that [left] side of the congruence thing means something that's in that same class. But it's not as specific."

In fact, there was a shift *towards* this operational view as the semester progressed. At the time of the first interview, Chris and Barbara had demonstrated a relational view of congruence, considering congruences as statements that showed when two integers could be considered "the same." However, by the second interview, both seemed to be viewing congruences operationally. The other students had held operational views at the time of the first interview, and this did not change. This finding is interesting in light of children's tendencies to view the equals sign operationally.

Before reduction of moduli was introduced, many of the students had been struggling to understand how to operate in the "mod n world." When they realized that one could rewrite a congruence of the form $a \equiv b \pmod{n}$ as $a = b + nk$, the students seemed to grasp onto this interpretation as an alternative to the earlier definitions they had been given. Fran said, "I think that now I have a better understanding of how to put it into an equation which makes a lot more sense to me than being in mod world." Dan suggested that most of the students in the course felt this way. "People are uncomfortable with congruence arithmetic and they look at this and they say, I don't really understand the rules of congruence arithmetic. But if you put it straight out in normal equation setting, it's not a problem."

The students began to display a tendency to automatically rewrite congruences as equations, and then work with these equations as much as possible. Once this practice emerged, the students appeared to have stopped trying to understand what was going on in "mod world" and to deal primarily with equations in $\mathbb{Z}$. In some cases, the students appeared to view the "mod n" term as merely different notation for "$+nk$". When solving $75x + 27y = 12$, Fran said, "I guess that would be just 12 minus $27y$. So $75x$ is equal to 12 mod 27. Can I do that with a negative?" At this point, Fran was not sure if she could rewrite $12 - 27y$ as 12 mod 27. When Barbara was asked if she viewed $5x \equiv 1 \pmod{11}$ as similar to an equation, she responded, "I actually look at this in terms of an equation. Like when I look at that, Im thinking to myself, five x equals one plus 11 y."

Overall, there were many parallels between the students' views of the reduction of moduli procedure and children's difficulties solving equations in algebra. The fact that the procedure of reduction of moduli is analogous to the equivalent-equations method of solving algebraic equations was not seen by the students, and they had little understanding of how this process worked or why it produced a set of solutions. Barbara's comment was typical:

> "I guess I understand why we're reducing it down, but when we start introducing other variables and you know, keep trying to reduce it, reduce it, and then probably where I get lost is when we go back to unraveling it. I'm trying to figure out like why that's important to solve it. You know to me, it seems like once we get down here, that would seem like a solution but it's not because you have to go back and do that so, that's what's the mystery to me."

In general, the students did not understand that this procedure was a process of repeatedly transforming the original equation several times by viewing it modulo one of the coefficients n (and thus mapping the equation to $\mathbb{Z}/n\mathbb{Z}$), and then deriving a related equation (mapping the equation back to $\mathbb{Z}$), the solutions of which were related to the solutions of the original. Instead, they viewed reduction of moduli as a complex procedure to be memorized and applied with great care, since mistakes were easy to make. They frequently expressed frustration with this procedure, not understanding why they were getting incorrect answers or even when they were making a mistake. Dan said, "I just don't understand ... there's always a small answer. I mean half the stupid homework problems we did there was a smaller answer than the way if you did it with the reduction. So like I don't ... am I doing it wrong?"

In class, Dr. Thomas had attempted to guide the students towards viewing congruences as analogous to equations in the sense that one could operate on both sides of a congruence, but his attempts were generally met with silence and confusion. The students instead chose to rewrite congruences as more familiar equations of two variables in $\mathbb{Z}$.

Dr. T: [writes $8x \equiv 4 \pmod{12}$ on board] "What do we do here?"

Student 1: "If you divide everything by 4, you get $2x$ congruent to 1 mod 3."

Student 2: "Just add 12 to four so that you have it congruent to 16. Then you can see you would have x is 2."

Dr. T: [following the second suggestion] "Well, if we did this we'd get that our answers are of the form $2 + 12k$. But we're missing ... 5. And if we do it the first way, we get five as an answer. [pointing to the congruence divided through by 4] Why does this work? How could we prove it?"

Student: "You could rewrite it as an equation, and then everything is divisible by 4."

Dr. T: [writes $8x = 4+12k$ and divides through by 4 to get $2x = 1+3k$, then rewrites as $2x \equiv 1 \pmod{12}$.

Students: [nod in agreement]

Barbara: "Can you divide the 8 and the 16, but not the 12?"

Dr. T: "Let's try that." [writes $8x \equiv 16 \pmod{12}$, and divides both sides by 8 to get $x \equiv 2 \pmod{12}$] "So can we do this?"

Chris: "There's something in the book that says the gcd of two of the numbers has to divide ... so the gcd of 8 and 12 must divide 16. If it doesn't, you can't."

Dr. T: [writes another example on the board: $8x \equiv 6 \pmod{7}$] "So here we can do what Barbara is suggesting ... the gcd of 8 and 7 is 1, and that divides 6, so we can divide by 2 on both sides. Barbara, did that answer your question?"

Barbara: "I think so ..."

The above transcript demonstrates that at this point in the course, the students generally dealt with linear congruences by rewriting them as equations in two variables. When asked how to prove that one can divide through a congruence by a common factor, a suggestion is immediately made to rewrite the congruence as an equation, and the students readily accepted this interpretation with no need for further justification. However, the students were reluctant to treat congruences as analogous to equations. When Barbara asked if one could divide both sides of a congruence by 8, the students did not know how to respond. Chris recalled an unrelated theorem about dividing through congruences by a common factor, but when Dr. Thomas redirected this comment towards an example in which one *could* divide both sides of the congruence (but not the modulus), the result was confusion.

When solving $5x \equiv 1 \pmod{11}$ in an interview, Chris said, "Then I just subtracted the 1 from both sides and I had to think about that one. I always, I still have to do that. I think about that. The subtracting, dividing or multiplying whether or not I can or can't." When solving the same congruence, Barbara was asked if one could approach solving it as one would solve an equation in algebra. She responded, "I think if I tried to solve for x, the thing that scares me is that I would get fractions and so since these are deal thingies or equations or whatever they're called, we want integers." Similarly, when asked if she viewed working with congruences as similar to working with algebraic equations, Fran replied, "To me it's not an equation like that, but I know I can convert it into an equation. But I don't look at that as an equation."

5 Conclusions and Implications for Teaching

It is striking that there were many similarities between the students' lack of understanding of the reduction of moduli procedure and children's difficulties solving equations in algebra. This may indicate that there are common underlying reasons for these difficulties. In addition, it is clear that these students did not make connections between the mathematics they were studying and the mathematics they will teach, as suggested by the research on abstract algebra. At the very least, addressing these difficulties with undergraduates in such a course may provide an opportunity to make connections with secondary mathematics.

Zazkis (1999) advocates having pre-service teachers re-examine familiar mathematical processes and objects in unconventional number systems as a means to get students to "reconsider their basic mathematical assumptions and analyze their automated responses. [These types of activities] constitute an essential tool for the development of critical thinking in mathematics teacher education" (p. 650). She uses a language analogy, saying that studying another language helps one to better understand the structure of one's own language. "Working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated."

I strongly agree with this perspective and suggest that the study of congruence provides an ideal opportunity to examine teachers' fundamental understandings of algebra. For example, studying the properties of functions and equations in the rings $\mathbb{Z}/n\mathbb{Z}$ could enable students to explicitly make connections with and deepen their understanding of the ways in which algebraic structures underlie the processes of secondary school algebra, such as modeling situations with functions and equations, finding roots of polynomials, and using various procedures for solving equations.

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