DESIGN OF THE SYSTEM OF GENETIC TEACHING OF ALGEBRA AT UNIVERSITIES

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ABSTRACT

For teaching on the basis of the genetic approach,, one should accomplish the analysis consisting of two stages: 1) genetic elaborating of a subject matter and 2) analysis of arrangement of a material and possibilities of using various ways of representation and effect on students. The genetic elaborating of a subject matter, in turn, consists of the analysis of the subject from four points of view: a) historical; b) logical; c) psychological; d) socio-cultural. In designing of the system of genetic teaching very important is to develop problem situations on the basis of historical and epistemological analysis of a theme.

Keywords: mathematics education, teacher education.

For teaching on the basis of the genetic approach, we offer to construct didactical system of study of a mathematical discipline (a part of a mathematical course, important concept or system of concepts) consisting of two parts 1) preliminary analysis of the arrangement of the contents, of didactical means and 2) concrete design of the process of teaching.

The preliminary analysis consists of two stages: 1) genetic elaboration of the subject matter and 2) analysis of the arrangement of a material and possibilities of using various ways of representation and effect on students. The genetic elaboration of a subject matter consists of the analysis of the subject from four points of view:

historical;

logical;

psychological;

socio-cultural.

The historical analysis frequently encounters with large complexities because of insufficient knowledge of the history of the origin and development of many branches of modern mathematics included in university curricula, inaccessibility of the literature on the given subjects. Therefore, it is necessary to conduct research of the history both of appropriate areas of modern mathematics, of their inclusion in the university curricula, to study educational literature, text- and problem-books, the history of the teaching of modern mathematics. As more or less accessible sources for the teachers and students the monographs and other scientific works – books and articles, books on the history of mathematics and mathematics education, manuals and encyclopaedias can serve. Very important is also to study original works of great mathematicians, classical textbooks, popular scientific literature, journal and magazine articles. The purpose of the historical analysis is to reveal paths of the origination of scientific knowledge underlying the educational material; to find out, what problems have generated need for this knowledge, what were the real obstacles in the process of the construction of this knowledge.

In designing the system of genetic teaching very important is to develop problem situations on the basis of historical and epistemological analysis of a theme.

The major aspect of rational (in the sense of Toulmin, 1972) organisation of an educational material consists in organising a material so that to reveal the necessity of the construction and of development of concepts and ideas. It is necessary to arrange problem situations or tasks, for which the important concepts or ideas, which should be studied, would serve as the best solutions. It is necessary to analyse those problems of knowledge, for which the considered concepts and ideas serve as the necessary solutions. For this purpose, both historical analysis and epistemological considerations, and special search for appropriate problem situations and tasks can help.

In our view, for the logical organisation of a system of concepts and propositions of a theme, of the teaching unit of a mathematical discipline, one should carefully analyse the deductive structure of such system, required, for example, for the construction of a concept or for the statement of a proposition. We will name the results of such analysis a logical genealogy of a concept or a proposition. In the university mathematics, especially in higher algebra, such genealogies may be rather complicated.

Clearly, such complicated structure of concepts and statements, needed for understanding the theorems of large difficulty, requires well-designed activities for successful learning.

Therefore, very important is also the psychological aspect of the genetic approach to the teaching of mathematical disciplines.

The psychological analysis includes determination of the experience and the level of thinking abilities of the students (whether they can learn concepts, ideas and constructions of the appropriate abstraction level?), possible difficulties caused by the beliefs of the students on mathematical activities (for example, the students can bear from school views on mathematics as mere calculations aimed at the search of (usually unique) correct answers with the help of ready instructions etc.). The psychological analysis has also the purpose to plan a structure of the activities of the students on mastering concepts, ideas, algorithms, to plan their actions and operations, and also to find out necessary transformations of objects of study.

When studying university algebra courses, the students usually are encountered with sequentially growing steps of abstraction - with a «ladder of abstractions».

À. À. Stolyar (1986, p. 58-60) has revealed 5 levels of thinking in the field of algebra and has noted, that “the traditional school teaching of algebra does not rise above the third level, and in the logical ordering of properties of operations even this level is not reached completely”. The following is the description of the third, fourth and fifth levels according to À. À. Stolyar (ibid., p. 59):

“On the 3-d level the passage from concrete numbers expressed in digits, to abstract symbolic expressions designating concrete numbers only in determined interpretations of the symbols is carried out. At this level the logical ordering of properties is carried out “locally”.

On the 4-th level the possibility of a deductive construction of the entire algebra in the given concrete interpretation is become clear. Here the letters designating mathematical objects are used as variable names for numbers from some given set (natural, integer, rational or real numbers), and the operations have a usual sense.

At last, on the 5-th level distraction from the concrete nature of mathematical objects, from the concrete meaning of operations takes place. Algebra is being built as an abstract deductive system independent of any interpretations. At this level, the passage from known concrete models to the abstract theory and further to other models is carried out, the possibility of existence of various algebras derived formally by properties of operations is accomplished”.

Thus, to the 5-th level the deductive study of groups, rings, linearly ordered sets etc. corresponds. The highest degree of abstraction here is the study of general algebraic systems with various many-placed operations.

To the 4-th level corresponds, for example, a systematic and deductive study of the sets of natural numbers or integers. Therefore, taking into account, that in school teaching even the 3-rd level is not completely reached, it would be certainly a big mistake to omit in pedagogical institutes the 4-th level (systematic study of an elementary number theory) and immediately pass to the deductive study of groups, rings and even of general universal algebras (as is done in a text-book by L. Ya. Kulikov, 1979). Therefore, systematic study of the elementary number theory can serve as a good sample of the construction of a deductive theory for preparation for the further construction of the axiomatic theories.

À. À. Stolyar built his classification of levels from the point of view of teaching school algebra. In our view, development of algebra as a science in the last decades (after the World War II, under the influence of works of S. Eilenberg and S. MacLane, 1945, and A. I. Maltsev, 1973) allows to distinguish one more higher, the 6-th level of algebraic thinking - we will name it the *level of algebraic categories*. At this level the entire classes of algebraic systems together with homomorphisms of these systems - varieties of universal algebras, categories of algebraic and other structures (for example, topological spaces, sets and other objects) are considered. Thus, the abstraction from concrete operations in these structures and from the nature of homomorphisms and generally of maps takes place; morphisms between objects of categories are considered simply as arrows subject to axioms of categories – for example, the associativitiy law for the composition. Moreover, the functors between categories – certain maps compatible with the laws of the composition of morphisms, and natural transformations of functors are considered.

Note that J. Piaget in the last years of his life was interested in the theory of categories as the highest level of abstraction in the development of algebra (Piaget and Garcia, 1989).

The teaching of algebra at this level (theory of categories and varieties of universal algebras) is not included into the obligatory curricula even of leading universities and happens only on special courses. But, nevertheless, the presence of this level demands that the students should master algebraic concepts in obligatory courses in a sufficient degree for understanding the algebraic ideas on the highest level of abstraction.

Essential in teaching algebra and number theory in pedagogical institutes are the 4-th and 5-th levels in the classification of À. À. Stolyar. First of all, the 4-th level (which is already beyond the school curricula) should be reached. Therefore, during the first introduction of the definition of group in the beginning of the algebra course, one should not immediately begin the full deductive treatment of the axiomatic theory of groups. Only after the experience of the study at the 4-th level of thinking in the field of algebra, namely of the study of the elements of number theory, it is possible to consider a deductive system of the most simple constructions and statements of the group theory, and the systematic account of complicated sections of the theory should be postponed to a later time, after studying at the 4-th level of such themes as complex numbers and arithmetical vector spaces.

J. Piaget who developed the classification of levels for thinking in the fields of geometry and algebra ("intra", "inter" and "trans"), noted that it is possible to distinguish sublevels inside each level (Piaget and Garcia, 1989).

According to the theory of A. N. Leontyev (1981), actions on learning concepts, as well as any actions, consist of operations, which are almost unconscious or completely unconscious. These operations are essentially «contracted» actions with the concepts of the previous level of abstraction. As M. A. Kholodnaya (1997) noted, «a contraction is immediate reorganisation of the complete set of all available … Knowledge about the given concept and transformation of that set into a generalised cognitive structure».

The theories of E. Dubinsky (1991) and A. Sfard (1991) are close to the Soviet conceptions of actions and operations as contracted actions in mathematics teaching.

In our view, for reaching a contraction of an action with algebraic objects into (automatic) intellectual operation it is necessary, after sufficient training with of this action, to include it in another action, connected with the construction of objects of the next step of abstraction.

One more purpose of the psychological analysis of the subject matter is finding out the ways of the development of motivation of learning.

The socio-cultural analysis has a purpose to establish connections of the subject with natural sciences, engineering and economical problems, with elements of culture, history, public life, to reveal, whenever possible, non-mathematical roots of mathematical knowledge and paths of its application outside of mathematics.

During the second part of analysis, considering the succession of study, it is necessary, in accordance with the principle of concentrism (Safuanov, 1999), to find out, on the one hand, which earlier studied concepts and ideas should be repeated, deepened and included in new connections during the given stage, and, on the other hand, which elements studied at the given stage, anticipate important concepts and ideas that will be studied more completely, become clearer later, to check, whether there are possibilities of such repetitions and anticipations.

The principle of multiple effect requires also the finding out possibilities of multiple representation of concepts studied, of use of active, iconic and verbal-symbolical modes of transmission of information, of other means of effect on students (the style of the discourse, emotional issues, elements of unexpectedness and humour).

After two stages of analysis, it is necessary to implement the project of the process of study of an educational material. We divide the process of study into four stages. The first two stages (construction of a problem situation and statement of new naturally arising questions) constitute the process of the rational organisation of the educational material confronted to the 3-d stage of the logical organisation of the educational material.

1) Construction of a problem situation.

In the genetic teaching, we search for the most natural paths of the genesis of processes of thinking and cognition.

According to the activity approach to the process of teaching, usually "the initial moment of the mental process is the problem situation … This problem situation involves the person in the thinking process; the thinking process is always directed to the solution of a problem" (Rubinshtein, 1989, p. 369). Therefore, the main purpose of the teacher is to construct a problem situation. The necessity of the construction of a problem situation was underlined by many prominent educators – by constructivists (creation of "disequilibrium") and representatives of the "French didactique" ("didactic engineering", directed on the creation of the didactical situations, on determination of the "epistemological obstacles") as well.

2) Statement of new naturally arising questions.

According to the theory of the activity approach to teaching, "the arising of a questions is the first sign of the beginning work of the thinking and the first step to understanding … Every solved problem generates a lot of new problems; the more a man knows, the better he realises what more he should know" (Rubinshtein, 1989, p. 374-375). Therefore, it is important, after the solution of the initial problem situation, to constantly consider new, naturally arising questions. It was well understood by N. A. Izvolsky (1924) in his version of the genetic approach. Thus, in the design of the process of study of a subject the statement of new, naturally arising, questions is necessary.

Actually, both stages – construction of a problem situation and the statement of new, naturally arising questions – are aimed at the same purpose - to help students in the independent mastering of a concept. Therefore it is necessary to organise a construction of problem situations and also statement of new, naturally arising questions in such way that in a certain moment of time (we will name such moment "the hour of truth") the students could, independently or with the minimal help of the teacher, discover the new concept for themselves. It is similar to the moment of the selection in a subject of "the initial universal relation", leading to the theoretical generalisation in the theory of learning activity of V. V. Davydov (1986, p. 148), and also to the act of reflective abstraction (as the of interior co-ordination of operations of the subject in a scheme) in the theory of J. Piaget (Dubinsky, 1991), and also to a moment of a *reification* (Sfard, 1991). Such organisation of teaching frequently may be quite difficult and not always completely possible. For this reason we admit appropriate help from by the teacher.

3) Logical organisation of an educational material. Here, after the problem situation has been dealt with, the paths of its solution, various aspects and natural arisen questions have been discussed, the appropriate motivation has been reached, the construction of the elements of the theory - precise definitions, statements (axioms and theorems), conclusions takes place. At this stage deductive reasoning plays the great role.

4) Development of applications and algorithms. After the logical organisation of mathematical objects of a studied theory, it is possible to consider various interesting and useful applications of the theory in practice and in mathematics itself. According to the principle of multiple effect (Safuanov, 1999), it is necessary to solve the sufficient number of exercises on the variations of signs of concepts, on the inclusion of concepts in new connections and contexts, on various transformations of mathematical objects under study.

During all stages of study of the teaching unit or theme it is important to help the students to develop their own language for expression of their reasoning and ideas. For this purpose each proposition (definition or statement) should be stated (at lectures and in textbooks), whenever possible, in various languages: logical-symbolical and verbal (this suggestion complies also with the principle of multiple effect).

It is necessary also to give the students the exercises on development of mental operations (analysis, synthesis, generalisation, comparison, analogy, abstraction and concretisation). For example, the exercises on extraction of conclusions from theoretical positions will be useful. Such exercises promote development of abilities of the synthetic reasoning.

Finally, it is very important to encourage reflection in minds of students, i. e. the ability to realise the foundations of their own activities, reasoning and conclusions, to be aware of the structure of their thinking process.

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