# MATHEMATICS THAT CHANGES LIVES

**Austin MELTON & Beverly M. REED**
Kent State University, Ohio, USA
melton@mcs.kent.edu; reed@math.kent.edu

**A. Bathi KASTURIARACHI**
Kent State University -- Stark Campus, Ohio, USA
bathi@stark.kent.edu

## ABSTRACT

We are developing a general education mathematics course that will introduce students to mathematical reasoning and applications. The course will cover the history of, the motivation for, and an introduction to cryptography, fuzzy set theory, graph theory, and non-Euclidean geometry. Three weeks (nine discussion-lecture hours) will be devoted to each topic, and the remaining three weeks will be used for student group work and project presentations. Through discussion and exploration, students will experience some of the insights, frustrations, and excitement experienced during the development of new concepts.

The cryptography unit emphasizes the classic cryptosystems and focuses on the mathematics behind the *how* and *why* the schemes actually work. The fuzzy set theory unit will emphasize applications. Topics in graph theory will highlight applications in management science, including Euler and Hamiltonian circuits, the traveling salesman problem, minimum spanning trees, and ideas in scheduling and planning. The non-Euclidean geometry unit will teach the basics of spherical geometry and focus on developing an understanding of an axiomatic system.

Pre-service elementary and secondary school teachers will be encouraged to take this course in order to expand their understanding about the nature of mathematics. The hope is they will begin to experience mathematics as a process of finding patterns and making and verifying conjectures. We will encourage them to work on projects that can be useful in their own classrooms.

*Mathematics Subject Classification* (2000): 00A35, 97C90, 97D50

# 1. Introduction and Rationale

According to Schoenfeld (1992), mathematics instruction goals depend on beliefs about what mathematics is:

> At one end of the spectrum, mathematical knowledge is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and relationships among them; knowing mathematics is seen as having "mastered" these facts and procedures. At the other end of the spectrum, mathematics is conceptualized as the "science of patterns," an (almost) empirical discipline closely akin to the sciences in its emphasis on pattern-seeking as the basis of empirical evidence. (p. 335)

Our conception is the latter, and we hope to induce our students to engage in a "hands-on" version of *doing mathematics*. Classroom activities will focus on *sense-making* with the hope of developing a predilection to think mathematically, i.e., to explore patterns, formulate conjectures, and test these conjectures. We hope our students will begin to develop the following attitudes toward mathematics and mathematics teaching: (Lampert in Shoenfeld, 1992)

- Mathematics is more about ideas and mental processes than about facts
- Mathematics can best be understood by rediscovering its ideas
- The main goal of the study of mathematics is to develop reasoning skills that are necessary to solve problems
- The teacher needs to create and maintain an informal classroom atmosphere to insure students' freedom to ask questions and explore their ideas
- The teacher should encourage students to guess and conjecture and should allow them to reason things on their own rather than show them how to reach a solution or answer
- The teacher should appeal to students' intuition and experiences when presenting the material in order to make it meaningful
- Coming to grips with uncertainty is a major part of the learning process
- Students should be induced to reflect on their own thought processes.

We plan to assess the change in students' beliefs about mathematics as well as their level of enjoyment of mathematics by having them complete a questionnaire on the first day of class and at the end of the semester. (Please see Tables 3 and 4 following the references.) Additionally, we will assess the students' abilities to do mathematical modeling. In our pre-course evaluation, we will ask the students to choose a situation and model it mathematically. We expect that most students will claim they cannot model anything mathematically. In the post-course evaluation, we will ask them to choose one or more situations and to model them mathematically in as many ways as they can.

# 2. The Cryptography Module

A unit on cryptography is ideal for enticing students to explore patterns, formulate conjectures, and test these conjectures. Initially, we will give students enciphered messages to decode using frequency tables of letters of the alphabet and common digraphs. Working in small groups, they will experience uncertainty and formulate conjectures about possible decoding schemes. Testing their schemes will come naturally as they try to make sense of the message before them.

**Monoalphabetic Substitution Ciphers**

The first two weeks will emphasize monoalphabetic substitution ciphers. In this type of cipher, each character of a written message (hereon refereed to as the *plaintext* message) is matched with a unique alternate character to obtain an encryption (a *ciphertext* messsage).

Several types of monoalphabetic substitution ciphers exist: *additive ciphers, multiplicative ciphers, affine ciphers,* and *keyword ciphers.* In an *additive cipher,* a plaintext character is replaced by another plaintext character whose position in the alphabet is a certain number of units away (mod 26) from the plaintext character. This number of units is called the *key*. The mathematics involved includes modular addition, additive inverses, and elementary statistical analysis. In a *multiplicative* cipher, rather than adding a number (the *key*) to the position of a plaintext character, one *multiplies* its position by a number mod 26. Students working on decoding such a cipher would be using modular multiplication, multiplicative inverses, prime numbers, the division algorithm, and relatively prime numbers. *Affine* ciphers combine both of the above ciphers by having two *keys*. One first applies an additive cipher with a key $r$ to obtain an intermediate cipher and then applies a multiplicative cipher with key $s$ to produce the cipher text. For *keyword* ciphers, one chooses a word and a letter as keys to create the monoalphabetic substitution cipher.

Once the groups successfully decode short encrypted messages with each of the above ciphers, they will come together as a whole class to discuss the problem solving processes they used to decipher each message as well as the mathematics behind their work. We will discuss the mathematics informally, hoping that the students find them natural to understand given the applications.

**Polyalphabetic Substitution Ciphers**

A polyalphabetic substitution cipher is a cipher in which the correspondence between the characters in the plaintext and those in the ciphertext is not one-to-one. Since these ciphers are more difficult to decipher, classroom activities will require more direction from the instructor. We will emphasize one example, the Vigenère Square. Students will initially *encode* a message before beginning work on a short *deciphering* assignment. Using the Friedman Test, which entails elementary probability, they will determine if a ciphertext has been encrypted using a monoalphabetic or polyalphabetic substitution cipher. If the ciphertext is polyalphabetic and a string of characters appears repeatedly in the message, the distances between occurrences *may be* a multiple of the length of the keyword. This observation is known as the Kasiki Test, and with it students will attempt to determine the length of a keyword.

**R.S.A. Algorithm**

A brief discussion of the RSA algorithm will conclude the unit, time permitting. The algorithm derives its name from its creators R. Rivest, A. Shamir, and L. Adelman and requires that each participant have two keys, a private key and a public key. The private key is a positive integer and the public key consists of *two* positive integers.

**Group Projects**

Student projects will entail researching the history of cryptography, encryption methods not discussed in class, or a more detailed exposition of the RSA method.

# 3. The Fuzzy Set Theory Module and Fuzzy Modeling

**Introducing Fuzzy Sets**

After reviewing the classical concepts of set and set operations, we will pose a question like: “Suppose you are working with the set of average daily temperatures for our area. What is the set of *warm* temperatures?'” After discussions about such a subset, we will, if necessary, ask the students if it would make sense for some temperatures to be *partially* in the subset of *WARM* temperatures. We will use the sets {0,1}, {0,0.5,1}, and [0,1] as ranges for our membership functions.

**Operations on Fuzzy Sets**

Our review of classical set operations will use a computer program and Venn diagrams. The program will work so that subsets of the universal set will be labeled, outlined, and shaded in a dark color, for example, in dark blue. The area within a Venn diagram but outside the sets in question will be in white. We will perform set operations, and the results will appear in dark blue with labels and with a border outlining the original sets used in the operation(s).

We will then consider fuzzy (sub)sets with the membership set {0,0.5,1}. Elements with a 0.5 membership value will have a medium shade of blue. We will ask the students to represent the union, intersection, and difference of sets using the program's color scheme. We hope it will not be difficult for the students to understand these operations in terms of the color scheme, and then we'll discuss the operations using standard mathematical notation. Next we will generalize our results to the membership set [0,1]. The program will represent membership values from 0 to 1 by using color shades from white to dark blue, respectively.

**Fuzzy Conditionals**

As many fuzzy control applications are based on fuzzy conditionals or fuzzy If-Then statements, we will introduce fuzzy conditionals where both the antecedent and the consequent involve fuzzy sets, i.e., fuzzy linguistic variables. Once the concept of a fuzzy set with a membership function is understood, understanding fuzzy conditionals is relatively straightforward.

**Fuzzy Modeling**

Our fuzzy modeling will be based on a fuzzy partitioning of the domain space, on defining fuzzy conditionals relative to the partition, on unioning the results of the conditionals, and on defuzzifying via a modified center-of-area method. The fuzzy partitioning will be explained and justified informally.

We will work though examples including a fuzzy model for designing a personal savings plan. The fuzzy conditionals will be defined with respect to age, number of dependents, and annual income, and the consequents will be defined in terms of a percent of income to be saved and/or invested. Using this fuzzy model, we will be able, for example, to suggest what percent of John's income should be invested if he is 38 years old, has 4 dependents, and earns $42,000 per year.

# 4. The Graph Theory Module

Topics on graph theory are chosen with a commitment to helping students acquire knowledge about the basics of management science. It is our view that such knowledge can be built with virtually no previous mathematical training and with relatively little pain. Granted that a working knowledge of basic statistics would be quite useful in dealing with complex problems, our choice of topics will include very little statistics, if any. Management science is a many-faceted subject. Its aim is to provide analysis, advice, and support to decision-makers. While our introduction to this vast subject will not be in depth, we expect that our students will find it easier to understand lots of commonsense examples, and be able to appreciate the beautiful mathematics that describe the solutions. The three main themes of the graph theory module are Euler Circuits, Hamiltonian Circuits, and Planning and Scheduling.

**Euler Circuits**

The first week of this module will delve into Euler circuits motivated by practical problems of street networking. Equipped with only the basic definition of an Euler circuit, namely circuits that cover each edge only once, the students will be encouraged to try out different solutions to given networking problems. The problems will be carefully chosen to incorporate a variety of possible cases. Our emphasis will be on discovery and innovative methods of solution. Collaboration will be allowed and encouraged through small group discussions. Based on their solutions, they will be asked to make conjectures about the existence of an Euler circuit for a given network. Through such a discovery-based approach, the groups (at least some) would come up with a statement similar to Euler's Theorem. After refining the students' conjectures, we will formally present Euler's Theorem, which describes a necessary and sufficient condition for the existence of an Euler circuit in a given graph. A simple elementary proof will be given. The section on Euler circuits will conclude with a few exercises on eulerizing graphs (by reusing edges). It is at this stage that the students would realize that some solutions are better than others, and this realization would lead into a discussion about "optimal solutions."

**Hamiltonian Circuits**

In Hamiltonian circuits, one starts at a given vertex and visits each vertex exactly once and returns to the starting vertex. Both Euler and Hamiltonian circuits are similar in that they both prohibit the reuse of some entity of the graph (edges in the case of Euler and vertices in Hamiltonian). Several beginning activities under this topic would come in the form of in-class worksheets. The reasons are two fold. Firstly, we want *discovery* to be at the helm of learning and cooperative activities can help accomplish this goal. Secondly, we want the prospective teachers among the students to gain some experience in designing in-class worksheets that are created as part of an outcome-based lesson plan.

The topics covered in the beginning of this section will include construction of non-Hamiltonian circuits, weighted and minimum cost Hamiltonian circuits, and the fundamental counting principle. The latter half of the section will include topics ranging from the traveling salesman's problem (TSP) and nearest neighbor algorithm to NP complete problems. We will use Kruskal's algorithm and critical path analysis to launch our discussions into ideas in Scheduling and Planning. Throughout this entire section we will make available to the students computer programs written on graphing calculators. With the aid of these programs the students will be able to check solutions, experiment, and confirm conjectures - practices that often form the backbones of mathematical research.

**Planning and Scheduling**

The final section of the graph theory module will focus on applications. We will guide students through simple optimization problems arising from applications in planning and scheduling. The emphasis will be on solving simple problems with a deep understanding of methods and principles involved. We will keenly observe the students throughout this stage to glean information about their individual skill levels. Our observations will then be incorporated into the designing of the final group projects for the course. It is our hope that each project from this module will have some aspect that would appeal to each student. The problems solved in class as well as the project problems will be similar to the ones found in Brucker (1999), COMAP (2000), Dolan and Aldus (1993), Heizer and Render (1999), and Roberts (1978). Samples of such topics include scheduling exams, planning meeting times, allocation of hospital resources, and efficient banking practices.

# 5. The Non-Euclidean Geometry Module

The purpose of this unit is to expand students' thinking and teach the basics of spherical geometry. They will review the concepts and definitions of plane geometry by comparing them to their corresponding ideas in spherical geometry. We assume that students will have had a high school course in geometry.All work will be done with manipulatives, presumably a class set of Lénárt spheres which have smooth write-on surfaces and tools for drawing and measurement.

Initial activities would concern basic geometric concepts: straight lines and distances, equators and pole points, angles, and parallel and perpendicular lines. Activities will be those suggested in Lénárt (1996).

**Straight Lines on a Sphere**

After finding the shortest distance between two points on a plane, students will sketch two points on a sphere and stretch a piece of string to find the shortest path between them. Using a spherical ruler, they will continue drawing the line, thus sketching a great circle. Class discussion would guide students to recognize great circles on the earth, with a possible connection to ancient astronomy. Students will be asked to compare characteristics of lines on the plane with those on the sphere.

**Distance on the Sphere**

Students will sketch pairs of points on a sphere and use a spherical ruler to measure the length of each arc. They will compare distance on plane with distance on sphere, particularly noting the units of measure. They then will find length of entire great circle and perhaps find a place on the globe that is $90^{o}$ from Kent, $180^{o}$ from Kent, etc.

**Parallel lines**

Through a guided activity, students will review lines in the plane. For example, given a line *l*, they would be asked to draw another straight line that has no point in common, exactly one point in common, exactly two points in common, more than two points in common with *l*. Then they will try to do the same on sphere. Which constructions are possible? A discussion of parallel lines will ensue, with the instructor giving some history about Euclid's fifth postulate and its role in the development of non-Euclidean geometries. Ultimately, we will learn that the first four postulates of Euclidean geometry also hold true on the sphere.

**Triangles on the Sphere**

- Students will investigate how many different triangles they can create by connecting three non-collinear points on a sphere and compare their results to those obtained on the plane.
- Students will investigate the sum of angles of triangles first on the plane and then on the sphere. They will try to construct a triangle with more than one right angle, then one with three right angles. They will then investigate the sum of the measure of the angles of quadrilateral.
- Students will investigate whether two triangles must be similar if their corresponding angles are congruent.

The culminating activity concerning triangles on the sphere will be finding the *qibla* (the angle one must turn from a given location in order to face Mecca) using spherical trigonometry (the law of sines and rule of four quantities).

**Hyperbolic Geometry**

A brief mention of hyperbolic geometry will conclude the unit. Time permitting, we would define line and distance using the Lobachevskian model. Students would be asked to compare and contrast properties in the three geometries. They will be encouraged to pursue an independent study project on the history of non-Euclidean geometry, its applications, or perhaps an elementary inquiry into hyperbolic geometry.

# 6. Assessment

The introduction of innovative pedagogy often prompts reevaluation of traditional classroom assessment practices. As described earlier in the Introduction and Rationale Section, our course is designed to promote mathematical reasoning and expand understanding about the nature of mathematics. Our philosophy is to minimize the anxiety that students typically associate with mathematics. We will continuously monitor students' learning, constantly provide important feedback about their progress, and encourage them through the power of systematic inquiry. With these standards in mind, we will evaluate our students in the following manner:

- During the first twelve weeks of class, i.e., during the presentations-discussions of the four main topics, assessment will be based on in-class worksheets and homework assignments. A student who completes this portion of the course successfully will receive a "C" grade.
- The last three weeks will be devoted to group projects based on material selected from each module. A student, upon earning the "C" grade, may successfully complete a single project to move to the next grade level "B."
- In order to earn an "A" for the course, a student must complete a second project (chosen from a different module) successfully.

# 7. Changes

Due to scheduling constraints this course has not yet been taught; it is planned that it will be taught during fall 2002. Thus, we cannot report on how students have changed as a result of taking this course. We can, however, do, at least, two things. We can report on opinions and beliefs of students like those who will take the course, and we can elaborate on the types of changes which we hope to see in our students.

To better know and understand the opinions and beliefs of the students who will take this course, we have given the "Beliefs about Mathematics" (given at the end of this article) questionnaire to 16 students like those who will take this course. The responses are summarized below.

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Mean** | **2.75** | **1.56** | **2.44** | **2.38** | **2.56** | **2.25** | **2.81** | **3.56** | **4.38** | **2.19** |
| **SD** | **1.18** | **0.73** | **1.41** | **1.15** | **0.96** | **1.06** | **1.33** | **1.03** | **0.72** | **1.42** |

**Table 1.** Means and Standard Deviations of Responses to "Beliefs About Mathematics" Questionnaire

The relatively low mean and standard deviation for responses to question #2 (which concerns the uniqueness of a solution to a problem) pleasantly surprised us, though we are concerned about the responses to question #3. Eight students (50%) responded to this question with an answer of 3, 4, or 5. At least two of these are pre-service elementary school teachers. Responses to question #4 confirm what students indicated to us in the qualitative portion of the survey. When asked whether or not they saw advantages of working on mathematics in groups, 13 of 18 (72%) students responded positively. The comments in the table on the following page are typical.

| **Question: Do you see any advantages to working in groups? If so, what are they? If not, why not?** |
|---|
| • I think group work is great. Sometimes it is just one little concept that was missed that makes a particular problem confusing. Group work helps to clear the confusion. |
| • Yes, working in a group in math can help in many ways. Not only are there other ideas, but solutions can be explained through many different viewpoints. Different ways of knowing/solving exercises in math often help those that may be having difficulty. |
| • Yes…Working in groups helps students who are high achievers understand the mathematics by explaining it to others. Sometimes people go through the motions without truly understanding the reasoning behind mathematics. For example, smart students might memorize. |

**Table 2.** Sample Student Responses about Group Work in Mathematics

The types of changes in which we are especially interested are beliefs about the nature of mathematics, confidence in understanding public press articles involving mathematics, and (for those who aspire to be teachers of mathematics) a willingness to freely and openly think about appropriately difficult mathematical concepts.

For the most part, our target audience believes that mathematics is the solving of equations often involving one or two variables, that mathematics deals with well known and clearly understood (though not by most students) concepts, and that mathematics in the form of equations supports science and technology. Student responses to the question "In your opinion, what is mathematics?" are listed in Table 5 at the conclusion of this paper. Fourteen of the 18 respondents referred solely to numbers, numerical expressions, or numerical calculations. Interestingly, 7 students think of mathematics only as calculating or manipulating numbers, variables, or formulas. These comments support our contention that student thinking needs to be developed and changed. Some bright spots did emerge in the data, however. Three students mentioned the study of relationships and six at least alluded to modeling. One student, albeit a masters level economics student, indicated that "Mathematics is a way of studying how things behave."

By the end of the course, we want the students to understand that mathematics embodies a rich and ever growing field of knowledge and concepts that in many cases *mold* the sciences and technology and that in most cases the basic nature and purpose of these concepts can be understood by educated individuals.

Further, we want to create a classroom environment that encourages questions and inquiry so that those who plan to teach mathematics, either as mathematics teachers or K-6 teachers who will teach mathematics as one of several subjects, will feel comfortable allowing their students to freely think and openly inquire about mathematical ideas and concepts even if it means that the teachers themselves will not be able to answer or solve all the questions.

## 8. Summary

In this course we want to expose our students to mathematics that has changed people's lives, and we want to present this mathematics in a way that will change the way our students think and, thus, will also change their lives.


### REFERENCES

-Aiken, L., 1974, "Two scales of attitude toward mathematics," *Journal for Research in Mathematics Education*, **5** (2), 67-71.

-Brucker, P., 1995, *Scheduling Algorithms*, Heidelberg, Germany: Springer-Verlag.

-COMAP, 2000, *For All Practical Purposes: Mathematical Literacy in Today's World,* New York: W. H. Freeman Company.

-Dolan, Alan, and Aldus, Joan, 1993, *Networks and Algorithms: An Introductory Approach,* Chichester: Wiley.

-Heizer, Jay, and Render, Barry, 1999, *Operations Management: Fifth Edition*. Prentice Hall.

-Lénárt, I., 1996, Non-*Euclidean Adventures on the Lénárt Sphere,* Berkeley, CA: Key Curriculum Press.

-Lewand, R., 2000, Cryptological *Mathematics*, Washington, DC: The Mathematical Association of America.

-Malkevitch, J., and Meyer, W., 1974, *Graphs, Models, and Finite Mathematics*, Englewood Cliffs, N.J.: Prentice Hall.

-Roberts, Fred, 1978, *Graph Theory and Its Applications to Problems of Society*, Philadelphia: Society for Industrial and Applied Mathematics,

-Schoenfeld, A., 1992, "Learning to think mathematically: problem solving, metacognition, and sense-making in mathematics", in D. Grouws (ed.), *Handbook for Research on Mathematics Teaching and Learning,* New York: MacMillan, pp. 334-370.

-Schoenfeld, A.,1994, "What do we know about mathematics curricula?" *Journal of Mathematical Behavior,* **13,** (1), 55-80.

-Yen, J. & Langari, R, 1999, *Fuzzy Logic: Intelligence, Control, and Information*, Prentice Hall.

***STUDENT QUESTIONNAIRE[1]***
***BELIEFS ABOUT MATHEMATICS***

***Students will be asked to describe their reaction to each of the following statements by using the following scale.***

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| ***strongly disagree*** | | | | ***strongly agree*** |

***1. Mathematics problems have one and only one right answer.***
***2. There is only one correct way to solve any mathematics problem – usually using the rule the teacher demonstrated in class.***
***3. When learning mathematics, I really don't expect to understand it; I prefer to memorize it.***
***4. Mathematics is best done by oneself.***
***5. Students who have understood the mathematics they have studied will be able to solve any mathematical problem in five minutes or less.***
***6. The mathematics I have learned in school has little or nothing to do with the real world.***
***7. Mathematics is less important to people than art or literature.***
***8. An understanding of mathematics is needed by artists and writers as well as scientists***
***9. Mathematics is needed in designing practically everything.***
***10. There is nothing creative about mathematics; its' just memorizing formulas and things.***

**Table 3. Student Questionnaire about Beliefs in Mathematics**

[1] Adapted from Lampert, in Schoenfeld, 1992.

***STUDENT QUESTIONNAIRE[2]***
***ENJOYMENT OF MATHEMATICS***

***Students will be asked to describe their reaction to each of the following statements by using the following scale.***

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| ***strongly disagree*** | | | | ***strongly agree*** |

***1. I enjoy going beyond the assigned work and trying to solve new problems in mathematics.***
***2. Mathematics is enjoyable and stimulating to me.***
***3. Mathematics makes me feel uneasy and confused.***
***4. I am interested and willing to use mathematics outside school.***
***5. I have never liked mathematics and it is my most dreaded subject.***
***6. I have always enjoyed studying mathematics in school.***
***7. I would like to develop my mathematical skills and study the subject more.***
***8. Mathematics makes me feel uncomfortable and nervous.***
***9. I am interested and willing to acquire further knowledge of mathematics.***
***10. Mathematics is dull and boring because it leaves no room for personal opinion.***
***11. Mathematics is very interesting and I have usually enjoyed courses in the subject.***

**Table 4. Student Questionnaire about their Enjoyment of Mathematics**

[2] Adapted from Aiken, 1974.

- study of numbers and objects that deal with numbers
- the study of laws concerning the physical world and the ways to understand it using relatively visible methods
- study of numerical expressions
- hard and complex problems that make a person think logically
- applying numbers and formulas to answer questions and solve problems
- using numbers to get a solution to a problem
- the study of numbers, their form, arrangements, and associated relationships, using defined literal, numerical, and operational symbols *(sounds like a dictionary definition)*
- manipulating numbers to understand everyday life and make it easier
- language of using numbers and formulas to describe why things work as they do.
- It is using numerical calculations to solve problems. It is very useful to a certain extent in everyday life, but is even more important in instances where exact values are needed.
- study of calculations, numbers, volumes, dimensions, and all kinds of other everyday measurements (*pre service elementary teacher)*
- numbers (*pre service elementary teacher)*
- numbers, signs, shapes, lots of stuff I don't understand (*pre service elementary teacher)*
- expressed numerical relationship between all things
- study of numbers and variables and how they can relate to each other through various manipulations (*pre service secondary teacher)*
- study of numbers, and how they apply to the world in which we live. ( *pre service secondary teacher)*
- the study of numbers
- Mathematics is a way of studying how things behave. This is a very broad answer but I don't feel that you can give a precise definition without leaving things out. I feel that mathematics by itself may not be able to accomplish very much but when used with other academic disciplines I feel it is essential

**Table 5.** Student Responses to "In Your Opinion, What Is Mathematics?"