ANALYZING FUNCTIONS' BEHAVIOUR IN A COMPUTATIONAL ENVIRONMENT

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ABSTRACT

This article reports the research conducted with first year Calculus students. During the last five years, the authors have been investigating whether exploring functions in a computer environment would improve students' performance. This is a research report on the "Analysis of the Behavior of Functions," with the computer as the main instrument in this methodology. This tool must be used bearing in mind certain criteria. The teacher must control not only the content, but also the software used. In the first attempt, the activities were carried out in the computer laboratory right after the discussion of the subject in a theoretical class, and students insisted on presenting "exact" results. Not satisfied with this type of behavior, the researchers decided it would be important to change students' attitudes. How could this be encouraged? By using, for example, an instrument of analysis, such as the effects of the "didactic contract." Later, new activities were prepared in an attempt to promote the computer → theory → computer dynamic. This dynamic proved to be highly efficient in fostering behavior that was active, critical, investigative and more independent of the teacher.

Keywords: Behavior of function – Graph – Image – Didactic Contract – Computer environment

Introduction

Over the last six years, the authors have investigated whether the exploration of each of the topics of differential and integral calculus in a computer environment helps improve students' performance in this discipline. To this end, they prepared activities which were applied and, after revision, reworked. This is the report on the research conducted into "Analysis of the Behavior of Functions."

At first, the class plan was: theoretical considerations followed by computer-based activities. As this dynamic did not prove to be satisfactory in achieving the objectives proposed, we decided to change the focus. Our group believes that effective learning occurs when the situations proposed provide a reciprocal exchange, thus favoring the construction of knowledge. As such, the following teaching procedure was used: *computer* $\rightarrow$ *theory* $\rightarrow$ *computer*. This new approach was put in practice, breaking a didactic contract (Brousseau, 1986) in the process.

The main tool in this methodology is the computer. Especially in the study of the functions, it enables one to show that image plays a partial role in realization: the graphs allow one to see the function. If the resources of this tool are fully exploited, students may construe basic elements for forming concepts regarding differential and integral calculus. Nevertheless, the group believes that the computer cannot substitute theoretical classes, much less the teacher, but is an ally in making conjectures, testing hypotheses and validating student results.

Description of the Study

At the beginning of the course, the students study the behavior of affine, quadratic, cubic, exponential, logarithmic, and trigonometric functions. New tools are necessary to extend the study: limits and derivatives. In educational books, algebraic expressions are generally used to determine the items necessary for preparing graphs of functions.

According to the Theory of Conceptual Fields (Vergnaud, 1990), it is important to offer a variety of situations for students to identify the invariants of a particular concept. As such, it may be desirable for students to be able to recognize and identify the following elements in graphs: domain, image, symmetry, parity, critical points, maximum and minimum values, inflection points, tangent lines, asymptotes, behavior close to infinity and points where the function is not defined.

For these purposes, the group posed itself the following question:

"How can analysis of the behavior of functions be conducted in a computer environment?"

Two activities were initially prepared to be worked on in the computer laboratory, directly after a discussion of the subject in a theoretical class.

For the first activity, functions were chosen which allowed one to observe a variety of behaviors, the objective being to have the students recognize the critical points ($f'(x) = 0$ or $f'(x)$ does not exist) in the graph of the function, in addition to the sign of the derivative, in order to determine both ends of the function. The students were also asked to decide whether the graph had asymptotes. With the software used, students were able to draw the tangent to the curve at each point; analyzing the angle formed by the tangent line and the axis of the abscissae, and were able to decide on the sign of the angular coefficient and, consequently, the derivative at this point.

The first activity is presented below.

I. Graph Sketching

For each of the functions below.

(a) Plot the graph using a curve sketcher.

(b) Identify the points where $f'(x) = 0$ occurs.

(c) Looking at the graph, identify the x values for which $f'(x) > 0$ occurs.

(d) Looking at the graph, identify the x values for which $f'(x) < 0$ occurs.

(e) Calculate algebraically the points for which the derivative is zero.

(f) Compare the answers obtained in (b) with those obtained in (e): what happened?

(g) Looking at the graph, check whether the function has: local maximums, absolute maximum, local minimums, absolute minimum, inflections, asymptotes, points where the function is not derivable.

1. $f(x) = x^4 - 2x^2$ 2. $f(x) = \dfrac{x^2 - x + 1}{x^2}$ 3. $f(x) = \dfrac{x^2}{x^2 - x - 2}$

4. $f(x) = xe^{-2x}$ 5. $f(x) = \sqrt[3]{x^2 - x^3}$ 6. $f(x) = e^{\frac{1}{x}}$ 7. $f(x) = e^{-x^2}$.

Most students answered the question satisfactorily, apparently just through observing the graph, as requested. Some carried out algebraic calculations, especially to determine the ends, using the sign of the derivative.

It was noted that the last item involved many concepts simultaneously, which is not advisable from a pedagogical perspective.

In the second activity, seven functions were chosen and for each one, questions were selected which were more appropriate to their graphs.

Although the objective was to interpret the graphs, many students presented algebraic calculations and about half of them calculated the limits for determining the asymptotes.

The second activity is presented below.

II. Interpretation of the graphs

1. (a) Use a *graph sketcher* to plot the graph for: $f(x) = \exp(1/x)$.

(b) Give the algebraic expression of each asymptote.

2. (a) Use a *graph sketcher* to plot the graph for: $f(x) = xe^{-2x}$.

(b) Give the coordinates of the maximum points and maximum values.

(c) Give the coordinates of the inflection points.

3.(a) Use a *graph sketcher* to plot the graph for: $f(x) = \sqrt[3]{x^2 - x^3}$.

(b) Give the coordinates (x_0, y_0) of the minimum point. What is the value of $f'(x_0)$?

(c) Write the equations of the tangent lines at the critical points.

4. (a) Use a *graph sketcher* to plot the graph for: $f(x) = x^4 - 2x^2$.

(b) Does the graph of f present any symmetry? If so, what type?

(c) Is the function even? Odd? Neither? Why?

(d) Give the angular coefficient of the tangent lines to the graph of f at the abscissa points x = -1, x = 0 e x = 1.

(e) Give the image of f.

5. (a) Use a *graph sketcher* to plot the graph for: $f(x) = \frac{x^2}{x^2 - x - 2}$.
(b) Give the domain of f.
(c) Give the algebraic expressions of the asymptotes.
(d) Give the coordinates of the maximum and minimum points.

6. (a) Use a *graph sketcher* to plot the graph for: $f(x) = e^{-x^2}$.
(b) At which points is the function negative?
(c) What is the behavior of f when close to $+\infty$ and to $-\infty$?
(d) Give the algebraic expressions of the asymptotes.
(e) Give the image of f.

7. (a) Use a *graph sketcher* to plot the graph for: $f(x) = (x^2 - x + 1) / x^2$.
(b) What is the behavior of f when close to $+\infty$, $-\infty$ and $x = 0$?
(c) Give the maximum value of function f.
(d) Give the minimum value of function f.

Upon analyzing the outcome of the activity, we realized that the choice of function in question 2 did not to make it easier to view the inflection points, making it difficult to answer. This may have led many students to study the first and the second derivatives.

In an overall analysis of the two student activities, one may observe the students' difficulty in realizing that the current "didactic contract" had been broken: in the interpretation of a graph, approximate answers are expected, however, there was an insistence on presenting exact results, resorting to calculations based on the algebraic expression of the function.

According to Brousseau, the "didactic contract" is a set of behaviors that each of the participants of a teaching/learning relationship expects from the other in terms of mathematical knowledge. One "clause" of the Contract, very engrained in students' minds, is that every mathematical problem has only one solution, known by the teacher beforehand and, to discover it, the student must find, in the details of the problem, the best means of attaining the solution.

It is importance to debunk this notion so the students may identify the invariants suggested by Vergnaud in is his Theory of the Conceptual Fields:

"How does one foster this change in perception?"

Firstly, by exploring graphs of functions before theoretical considerations; secondly, by making deep changes in the structure of the activities. With the first change, we encouraged the breaking of another "clause" of the didactic contract, that is, the students were only allowed to answer the questions after the theory had been explained by the teacher. The second change was a result of the first and was also due to the large number of items involved. It was decided that they should be explored separately, and in the following year, this resulted in the preparation of five new activities to be applied instead of the two previous ones.

These activities contain leading and open-ended questions in an attempt to provide an opportunity for students to work autonomously. In addition, at the end of each one, the theoretical result concerning the concepts studied is included.

The objective of the first is to relate the sign of the first derivative to the increase of the function. Two functions were used, a polynomial and a rational function. For the graph of the first, the students were asked to: choose three points to draw a tangent line - one of them with a positive angular coefficient, another with a negative angular coefficient and a third one with an angular coefficient value of zero -, identify and algebraically calculate the points at which f '(x) =

0, identify the intervals where f ’(x) > 0 (f ’(x) < 0), and relate them to the increase (decrease) of the function.

Activities 1

I.

1. Use the graph sketcher to obtain the graph of the function $f(x) = x^4 - 2x^2$.
2. Draw a tangent line to the graph of f with a positive angular coefficient. Give the coordinates of the point of tangency. What is the angular coefficient of this line?
3. Draw a tangent line to the graph of f with a negative angular coefficient. Give the coordinates of the point of tangency. What is the angular coefficient of this line?
4. Draw a tangent line to the graph of f with an angular coefficient value of zero. Give the coordinates of the point of tangency. What is the value of the derivative of function f at the abscissa of this point?
5. At abscissa point x = 1/2, is the angular coefficient value of the tangent line positive, negative or zero?
6. Looking at the graph, identify all of the x values for which f ´(x) = 0.
7. Calculate these values algebraically.
8. Looking at the graph, identify the intervals for which f ´(x) > 0.
9. In these intervals, is function f increasing or decreasing?
10. Looking at the graph, identify the intervals for which f ´(x) < 0.
11. In these intervals, is function f increasing or decreasing?

For the graph in the second activity, students were asked to establish this relationship, though without drawing the tangent lines.

II.

1. Use the graph sketcher to obtain the graph of the function $f(x) = (2x^2 - x + 1) / x^2$.
2. Looking at the graph, identify the intervals in which the derivative has a positive sign.
3. In these intervals, is function f increasing or decreasing?
4. Looking at the graph, identify the intervals in which the derivative has a negative sign.
5. In these intervals, is function f increasing or decreasing?
6. Identify the abscissa points x where f ´(x) = 0 occurs.
7. Calculate algebraically the x values in which the derivative is zero.
8. In these two examples, what relationship do you see between the increase of a function and the sign of its derivative?

The objective of the second activity is to identify the maximum points through the increasing/decreasing of the function. Three functions were chosen: a rational function with R domain and an absolute maximum, a polynomial function with a relative maximum and a modular function with a relative maximum at points in which the derivative does not exist. Students were initially asked to identify the highest value of the first function and then study its increase/decrease close to this point. For the second function, students had to invert the process, i.e., choose an interval in which the function showed an increase followed by a decrease, then find its relative maximum point. The procedure for the last function was identical to that of the second, except that there was no derivative at the maximum point. For the three functions, students were asked to find the value of the derivative at the maximum point. After studying these functions, students had to describe and test an algebraic method to determine the relative maximum points of a function (if there were any). At the end of the activity, four statements were presented to the students, who had to decide whether they were true or false. In the final discussion, the content of the activity (minimum point) was also institutionalized.

The activity is set out below.

I. 1. Sketch the graph of the function $f(x) = x/(1+x^2)$. D = Im =
2. What is the greatest value of f ?
3. For which value of x does it occur?
4. What is the value of f′ for this x ?
5. Study the increase/decrease of f close to this x value.
6. What is the sign of f′ close to this point?

II. 1. Sketch the graph of the function $f(x) = x^3 - 3x^2 + 1$. D = Im =
2. Choose an interval in which the function displays an increase followed by a decrease.
3. What is the greatest value of f in this interval?
4. For which value of x does it occur?
5. What is the value of f′ in this x?
6. What is the sign of f′ in the interval chosen?
7. Is the number found in question 3 the highest value of the function?

III. 1. Sketch the graph of the function $f(x) = \left||x - 2| - 3\right|$. D = Im =
2. Choose an interval in which the function shows an increase followed by a decrease.
3. What is the greatest value of f in this interval?
4. For which value of x does this occur?
5. What is the value of f ′ in this x?
6. What is the sign of f ′ in the chosen interval?
7. Is the number found in question 3 the highest value of the function?

III. In each of the examples studied, you identified a f(x) number which was the highest value of f close to x.

This f(x) is called the *local (or relative) maximum of f*; and the corresponding x *local (or relative) maximum point of f.*

Algebraically, how would you determine the local maximum points of a function f (if there are any)?

V. Apply this procedure to the function $f(x) = 4x^3 + 15x^2 + 12x + 5$.

VI. Decide whether the following statements are true or false and justify your answers.
1. If c is a local maximum point, then f ′(c) = 0.
2. If f ′(c) = 0, then c is a local maximum point.
3. If f is increasing to the left of c and decreasing to the right side of c, then f(c) is a local maximum of f.
4. If f ′(c) does not exist, then c may be a point of local maximum of f.

The third activity presents two polynomial functions. The students were asked to relate the concavity of the graph of the function to the sign of its second derivative. To do so, they had to apply the results obtained in the previous activities regarding the study of a function to the derivative function.

For institutionalization purposes, at the end of the activity, the following theoretical result was presented:

If f is the derivable function up to the second order in]a, b[; then:

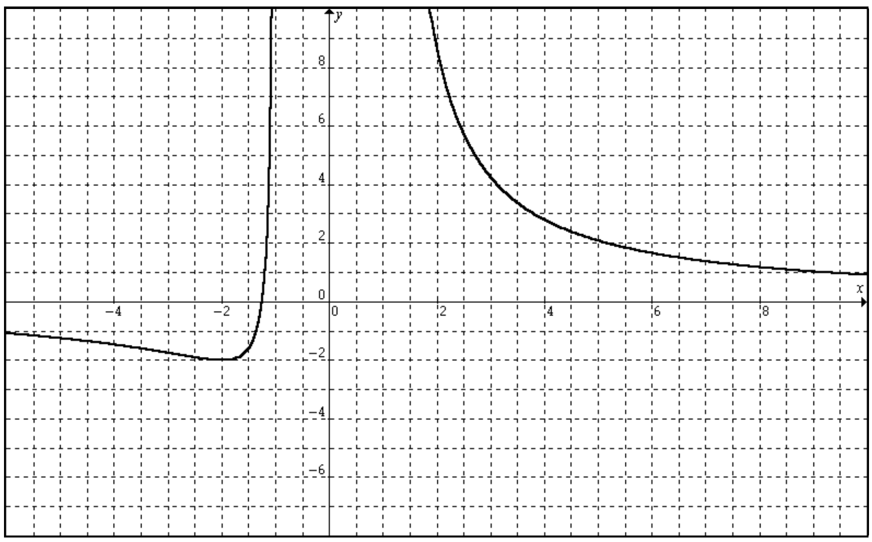
a) if $f''(x) > 0$ for every x in]a, b[, then the graph of f is concave upwards in]a, b[;
b) if $f''(x) < 0$ for every x in]a, b[, then the graph of f is concave downwards in]a, b[.

Observation: the point at which the graph of a function changes its concavity is called the *inflection point*.

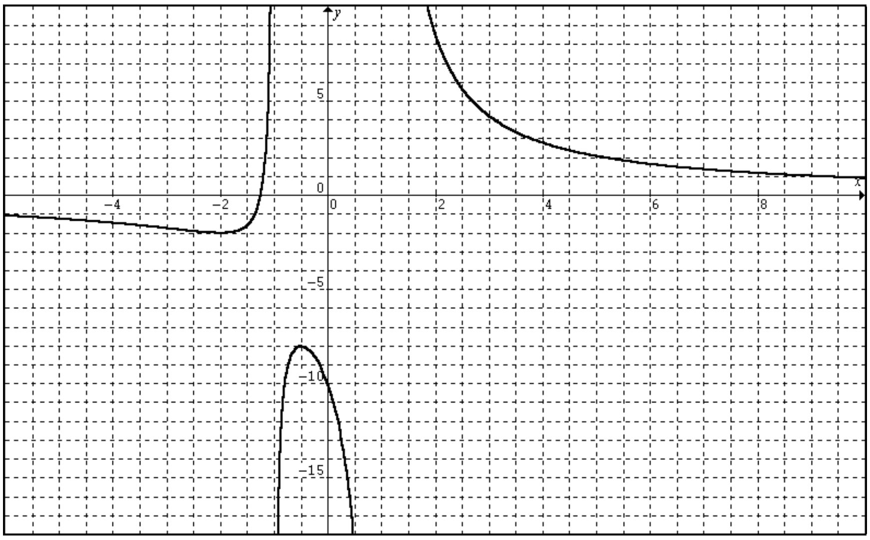
The fourth activity was a reformulation of the second activity from the previous year. The last function, whose graph did not meet the requirements of the study, was eliminated and some items of the other functions were removed and instructions rewritten.

In the fifth activity, the function $f(x) = (8x + 10)/(x^2 - 1)$ was chosen so that one branch of its graph would not appear on the screen (unless certain changes were made to the axes scale, which was not requested) in order to observe whether the students were able to critically analyze the answers obtained. The first intention was for them to use this screen to identify the characteristics of the function and its graph. They were then asked to recognize the same characteristics, however, using an algebraic expression. Finally, they were asked to compare the results obtained in both screens: graph and algebraic. These formulations perplexed the students who observed a clear contradiction between the computer results and the theoretical results obtained via the algebraic calculations.

Below is the graph presented on the computer without any alteration of scale.



After some alterations of scale, we obtained the following graph.



Some students exchanged the "dubious" results for the "correct" results. The dubious results were considered less legitimate. When it is the teacher's word against the computer, the teacher is correct (the effect of the didactic contract); but when it is the student against the computer....

The second function of this activity was chosen so that the students, using the algebraic expression (with the computer turned off), could answer questions regarding its behavior. The students were only allowed to turn on the computer after this study, to see the answers.

The student protocols provided a wealth of information regarding both aspects of the didactic contract and the limitations of the software used.

As regards the didactic contract, the following categories were identified:

- The students who did the algebraic calculations from the outset, showing that they had not noticed the breaking of the contract.
- The students who initially worked with the graphs, as requested, but who, after the algebraic calculations returned to the previously given answers and "corrected" them. This attitude shows that the clause of the didactic contract, according to which the answers obtained through algebra are the true ones, is the strongest.
- The students who worked with the graphs and algebra, who noticed that the answers were conflicting but were not surprised. In this case, they felt satisfied for having fulfilled their part of the contract by answering the questions proposed by the teacher, regardless of the mathematical knowledge involved (for example the fact that the function had two different domains).
- The students who worked with the graphs and algebra, who noticed that the answers were conflicting and tried to discover the reason for this conflict, seeking mechanisms which would allow them to view all of the "branches" of the graph. This last category includes the students who noticed the teacher's breaking of the contract.

Many students found ways of overcoming the limitations of the software used either by changing the scale or the "zoom" command. The opportunity was taken to reinforce the discussion with the students about the advantages, limitations and “dangers” of using a computer tools in the teaching-learning process.

Conclusion

When introducing the subject by exposing students to theory, the teacher may intend to “transmit knowledge.” Our group believes that knowledge cannot be transmitted, rather, it is construed by the student. Teachers contribute to this process when they create learning environments which allow students to become more active, critical, independent of teachers, and more inquisitive.

The computer laboratory proved to be an excellent environment for the development of these characteristics in students and the activities prepared for analyzing the behavior of functions proposed a breaking of the conventions of the didactic contract (open-ended questions, conjectures, ...), which, in conjunction with constant renegotiation, prompted a change in attitude. For example, some students used the graph of the derivative function to obtain information about the increases, decreases and end points of the primitive function, despite the fact that the focus suggested for this was the tangent line, showing an independent attitude.

The application of these activities within the proposed dynamic allowed the concepts necessary for analyzing the behavior of functions to be construed. Consequently, producing a sketch of the graph of a function was no longer a “magical” feat performed by the teacher.

BIBLIOGRAPHICAL REFERENCES

BROUSSEAU, G., 1988, "Le contrat didatique: le milieu", R*echerches en Didactique des Mathématiques,* vol. 9, n.3, p.309-336, Grenoble.

MANRIQUE, Ana ... et al, 1998, "Teaching function in a computational environment", *Conference of the International Group for the Psychology of Mathematics*, Proceedings of the 22nd PME, vol.4, p. 273-, Stellenbosch, South Africa.

PONTE, J.P., MATOS, J.M., ABRANTES, P., 1998, "Investigação em educação matemática: implicações curriculares", *Ciências da Educação,* vol. 22, Lisboa: Instituto de Inovação Educacional.

VERGNAUD, G., 1990, "La Théorie des champs conceptuals", *Recherches en Didactique des Mathématiques*, vol. 10, n.2.3, p. 133-169, Grenoble.