

**INTEGRATING SYMMETRIES OF POLYHEDRA IN THE CONTEXT OF REAL SPACE
EXPERIENCE: STUDENTS' WORK ON A GEOMETRICAL THEME**

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Keywords: Thematic Approach, Project Work, Symmetry, Polyhedra, Mathematics in Context, Art and Mathematics.

ABSTRACT

Polyhedra and their Symmetries have appeared since Antiquity in diverse studies of philosophy, art, science and mathematics (especially geometry), but a lack of discussion of this context in some depth is apparent in Mathematics Education. This paper discusses related project work on the theme «Polyhedra and Symmetry» in the University classroom. Students' work is analyzed according to the didactical intentions, the project issues proposed and students' own choices and thinking processes. A case study is also included, by taking into account the particular students' «background» and «foreground», which helps in interpreting the students' choices.

1. Background

1.1 Geometrical “Order” in Art and Nature; Symmetry of polyhedra.

There is a vast literature on mathematical “order” in Art and Nature, from Antiquity and Renaissance to our epoch. The themes are rich and diverse, but most of them reveal a mystic ideology and a Platonic or Pythagorean conception of the world. Even in the *Bauhaus*, the famous modern art school in the Democratic Republic of Weimar (1919-1932), the mystic and idealistic tendencies were renewed by some of the “Masters of Form” who taught there, as for example by the Swiss Johannes Itten, an Expressionist painter and unconventional art teacher with mystic beliefs, who was an opponent of every materialist interpretation of the world¹. However, the “Masters of Form” (sculptors and painters) were not the only teachers at the *Bauhaus*. Walter Gropius, the famous architect who founded the School, had conceived a radically new kind of art education, in which the fine arts and the crafts would not be considered as two fundamentally different activities but as two varieties of the same thing. Therefore Gropius also appointed the “Workshop Masters” in order to equip students with manual skills and technical knowledge, while the “Masters of Form” were to stimulate the students’ minds and encourage creativity².

If Geometry is considered as a theoretical ingredient of some creative activities in Art, Science and Technology, then a lack of a similar vision to that of the *Bauhaus* is apparent in Mathematics Education. Teachers and textbook’s writers sometimes generally refer to the virtues of geometrical thought as introducing “order” in natural contexts of experience, but such general verbal descriptions usually fail to encourage any creativity in the students. Instead of discussing concrete and non-trivial examples, which could attract the students’ interest and illustrate the power and applicability of geometrical methods in particular contexts, university textbooks usually contain an excessive amount of formal definitions and technical proofs. “Order” is then restricted to Pure Mathematics as an abstract and separate subject closed in itself.

On the other hand few «popular» books, in our opinion, manage to penetrate both (modern) Geometry and Art (or Nature) in such a way, that their deep mutual relationships are made apparent to the reader. One of these books –and by no means an “easy” one – is H. Weyl’s *Symmetry*³. In this book Hermann Weyl discusses symmetry as an idea, which in his words «was essential through the ages in human efforts to understand beauty and order». After a first chapter devoted to bilateral symmetry, the book introduces the concept of a group of transformations as a key mathematical idea suitable for the study of the general notion of symmetry. More specifically, given a figure F in space (for example: a regular polygon or polyhedron) the one-to-one transformations of the space onto itself which leave F invariant form a group depending on F , $\tilde{A}=\tilde{A}(F)$ and this group describes exactly the “type of symmetry” possessed by F . Groups of rotations and translations are discussed as the most important examples, which are then applied to polygonal or other shapes of 2-dimensional ornaments and to polyhedra and natural crystals in 3 dimensions. Thus, to search for all *symmetries* of a given polyhedron P means to try to determine the group $\tilde{A}(P)$.

¹ Whitfort, 1991, p. 52

² *ibid*, pp, 47-48

³ Weyl, 1964, p. 52

1.2 “Themes” and Project Work in the University Classroom.

A “*thematic approach*” to Mathematics Education is described by Skovsmose (1994, pp.59-90). This approach has been adopted since 1994 in Aalborg University, Denmark, where the curriculum is organized in several “themes” (normally covering a semester) in such way that increased knowledge and cognition can be obtained progressively during the educational process. A corresponding innovation in teaching practice is *project work*, which must provide students with special professional skills (F. Kjersdam et al, 1994).

Our approach is a little different. Although we adopt the idea of “theme” as central, we conceive it as an autonomous subject of scientific and didactical discourse rather than a curriculum unit (that is why we shall not deal with the curriculum in general in this paper).

A “theme”, in our approach, is an open-ended problem situation in a real context or field of experience⁴, or a generic example⁵, which becomes the subject of a scientific discussion in the university classroom, starting from historical roots and leading up to the present situation of research. A “theme” as above must be general and rich enough to generate several questions (or aspects), sub-problems and issues (or topics), which may be interesting to students. These questions may become the subject of separate projects, which can be progressively integrated into the central “theme”.

2. Analysis of the educational process

We now proceed to analyze project work as an actual educational process in the university classroom. Our *theme* will constantly be «Polyhedra and Symmetry» with two main aspects to be covered, namely (i) *Symmetry in Art and Nature from Antiquity to Modern Times* and (ii) *Symmetries of Polygons and Polyhedra*. Our analysis is based on empirical data (participant observation, interviews and examples selected from the students’ work). The whole process of students’ work consists of the following general phases:

- Starting from students’ own experience
- Stating a problem as an open task
- Working (usually in groups)
- Presenting the results
- Discussion and evaluation

We are going to analyze the project work specifically analysis, according to the aspects proposed, the intentions of the supervisor and the students’ own choices and thinking processes. The projects actually carried out in the classroom were titled as follows (we shall refer to them as «Project I», «Project II» and «Project III» respectively):

I. Symmetry in Art and Nature from Antiquity to modern times -with emphasis on regular polyhedra.

II. Symmetry of flat figures and polyhedra.

III. The symmetry group of the cube.

⁴In the sense of Boero, 1989. See also Patronis, 1997, for a revision of the notion of context in Mathematics Education

⁵As e.g. it is the case with Bernoulli Trials for Probability Theory, or polyhedra for Topological and Combinatorial Topology.

These titles, which more or less cover the two aspects mentioned above, were proposed to students within an optional course on «Transformational Geometry and its Teaching» which is offered in the mathematics department of Patras University. Several students were involved in these projects, usually in groups of two (or more).

The intentions of the supervisor of these projects were different from each case to another and accordingly different were also his suggestions to students. Project I was to cover a vast bibliography area, which had a risk of “being lost” within the subjects and the variety of the approach. However, the supervisor proposed this project to students with an actual interest in history of art and its interrelations with the developments of mathematics-especially geometry. He intended to make these students *to search for the relevant literature* and think, moreover, in their own terms⁶. On the other hand, the supervisor mentioned no bibliographic references for Projects II and III and his general suggestion to students was to try to “experience” the symmetry transformations of figures; thus *to search the subject directly by themselves* instead of searching the bibliography. Moreover there is an evident difference between Project II and Project III, namely that the scope of the former is wider and less special than that of the latter.

As a result, the supervisor’s didactical choices had a remarkable effect on most students’ work and especially on their thinking processes. Students participating in Projects II and III, who used no bibliography and followed the supervisor’s suggestions, proceeded inductively in their own investigation, by numbering of cases of figures or of symmetries of one and the same figure. Students carrying out Project II generally remained into a “static” and particular conception of symmetry. They did not discover rotations and they treated (axial or central) symmetry as an internal displacement of points within the figures. On the other hand, a group of four students who together carried out Project III developed a more “dynamic” conception of symmetry in space. This group, as well as another student carrying out Project I, has visualized rotation of polyhedra around axes as a movement in (real 3-dimensional) space. In the case of Project I, however, many books and Internet sites have been used in addition to students’ own investigation, but this additional information was not always relevant (see the case study in Section 3 below). For an analytic description of students’ thinking aids and processes, the producing of drawings, the style of presentation and the character of knowledge finally obtained see Table 1.

3. Students’ background and foreground: a case study

Knowledge of students’ «background» and «foreground» is indispensable for a better interpretation of students’ choices as related to students’ own overall views and plans in their life.

We borrow from Skovsmose (1994) the concepts of (a person’s) «background» and «foreground» as a framework of analysis and interpretation of students’ intentions and choices. Choosing a project and choosing a particular style of study and exposition as well depends on a set of dispositions of a person. This set can be divided into the person’s «background» and «foreground»:

⁶ H. Weyl’s *Symmetry* (Weyl, 1964) was the only book that was named as “important” by the supervisor. Two or three other titles were mentioned as free to the choice of students (among them Ghyka, 1971 and some sources for Platonic Solids).

«A *background* can be interpreted as that socially constructed network of relationships and meanings, which belong to the history of the person (...). But the background is not the only source of intentions. Equally important is the *foreground* of the person. By this expression I refer to the possibilities, which the social situation makes available for the individual to perceive as his or her possibilities. It is not open to me to have the (realistic) intention of being the next president of Mexico. It is not part of my *foreground* and only if I were a madman would I produce intentions of this kind»⁷.

Hermes and Orpheus

We shall focus on *Hermes* and *Orpheus* (pseudonyms of the students who together have carried out Project I) as a case study. Hermes and Orpheus were together, at that time, in the last year of mathematics in Patras University. Orpheus was mostly interested in the art aspect of the project, since in parallel to mathematics he studies music. Hermes' background was different: being an expert in the computer and passionate user of the Internet, had never thought of art in relation to mathematics before this project. By «searching» in the web and the department's library as well, Hermes and Orpheus had gathered the following list of books (written at the beginning of Hermes' diary):

- (1) «Is God a Geometer?», by M. Golubitsky and Ian Steward.
- (2) «Symmetry», by H. Weyl.
- (3) Plato's «Ὅβιᾶείδ».
- (4) Encyclopedia Britannica, 1999.
- (5) Encyclopedia Encarta, 1999.

Only book (2) of this list had been suggested by the supervisor of the project. Not all of the books were relevant and both students (especially Hermes) had a difficulty in grasping what was essential to the theme of the project. The students' mathematical background -as Hermes himself admitted when interviewed- was rather poor for a good mathematical understanding. It seems that "popular" literature as in the above list cannot help in this direction, with the exception of Weyl's book, which probably the students did not read very carefully.

However, some mathematical definitions and classifications (as e.g. that of symmetry groups) are found in Hermes' diary, together with aesthetic questions such as the possibility of determining "beauty" by means of (mathematical) symmetry. Does a beautiful thing need to be symmetrical? Conversely, is any symmetrical thing necessarily beautiful? Hermes was a beginner in both Transformational Geometry and the History of Art, therefore his approach of these matters was rather absolute and naïve, but he finally tended to answer both above questions in the negative.

In any case, philosophical questions concerning Platonism and the "way of existence" of mathematical structures in Art and Nature have not been discussed in the project, and this is largely due to a lack of historical and philosophical background in both students. Although their final essay has superficially an "historical" structure, Hermes' and Orpheus' views of their theme have not yet been critically developed. Hermes only seems to have gained some knowledge of regular polyhedra and their symmetries, but this knowledge is mainly integrated into a *visual-empirical* context and not into a *critical-historical* or a *theoretical-mathematical* one.

Will Hermes and Orpheus "meet" this theme in their life again? Such a possibility seems now distant to both of them, which explains why there was no continuation in their enquiry. Today, a year after doing this project, Hermes is not so enthusiastic as at the beginning of the project work. Hermes

⁷ Skovsmose, 1994, p. 179. Our emphasis.

is a graduate student in the department but without a specific direction (hesitating between Computer Science and Mathematics Education), while Orpheus has postponed his graduate studies until he finishes his service in the Army. At least Orpheus has (until now) kept his enthusiasm for the art -and-mathematics issues of the theme. He now studies the same theme in relation to modern physics and cosmology, which (in his words) «introduces Pythagorean harmony to the universe once again».

4. Concluding Remarks

The character (and status) of knowledge finally obtained by the students seems to depend on many more factors than simply the intentions and suggestions of the supervisor. Project work clearly differs from usual teaching-learning processes. It is not easily adopted and understood by students, who need a continuous care and encouragement. Especially within such a traditional institution as usually a Greek mathematics department looks like, the status of project-generated knowledge seems to be low compared to technical mathematical knowledge obtained by usual rote learning. The knowledge obtained in the process of project work generally seems “unfinished”, incomplete and insecure to students’ own eyes. A girl described her experience with Project II as «exploring something unclear, uncertain, perhaps varying, but anyway unknown». Yet, in the words of the same student, «it is the object itself that, without doubt, challenges and fascinates». It is through this meaningful experience that the students gradually gain real self-confidence in their learning of (modern) mathematics.

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TABLE 1

	“MATHEMATICAL” ASPECT		“HISTORICAL” ASPECT
	“Dynamic” conception Projects III+I	“Static” conception Project II	Project I
Use of Bibliography	No use of bibliography	No use of bibliography	Many book titles and Internet sites
Thinking Aids and Processes	Using Visualization of “movement” of figures in space; empirical verifications and (some) calculations.	Making “direct” conjectures from the (static) drawings with empirical verifications and explanations.	Using visualization; also metaphors and comparisons of various examples from art and life.
Style of Presentation	“Inductive” (numbering of cases) and semi-formal	“Inductive” (numbering of cases) - not formalized at all	Narrative
Geometrical Drawings	By hand or computer graphics	By hand or photocopy	Use of computer graphics
Character of Knowledge obtained	Empirical towards an abstraction	Empirical but remaining particular; trying “direct” generalizations	Contextual