OLD IDEAS: A NEW TOOL FOR TEACHING BASIC MATHEMATICS

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ABSTRACT

Incorporating the History of Mathematics in everyday teaching can be much more than just giving anecdotic events a place in our classroom expositions. In fact, there are ideas and methods used in the past which may be no longer of practical usefulness today, but may nevertheless help our students to grasp the meaning of a theory, an algorithm or a simple formula. Just as, for Elementary School children, manipulating pebbles is a useful way of visualizing some properties of the basic operations, like commutativity, for example, also working at the High School level with Euclide's " Geometric Algebra" or the egipcians' " False position rule" may be of great help for understanding relations and concepts more clearly. In fact, the possibility of realizing that concepts and ideas in different topics of Mathematics are connected is one of the most important benefits for the student exposed to the historic development of some mathematical ideas. On the other hand, the teaching of Mathematics as a field of knowledge which is ever changing, instead of as a rigid set of formulas and algorithms, is, besides a tribute to truth, a way of encouraging our students to interact with the ideas developed in the classroom, since they will necessarily be exposed to several different ways of solving problems, and therefore their creativity will be stimulated. After a 4- year experience in Mérida, Venezuela working with High School teachers who introduced their students to Algebra and Geometry using the approach mentioned above, there have been positive results, especially in their general attitude towards learning Science. Some examples of the use of certain elements of the History of Mathematics in the exposition of basic topics in Algebra and Geometry of the High School level will be shown.

1 Introduction

The teaching of Mathematics at the High School level in Venezuela is, generally speaking, a task which faces great difficulties due to many factors, the two most important of these being : 1)The curricular design, and 2) the academic background of the teachers.

Since the year 1.996, and after obtaining a Doctorate degree in Algebra, the author has participated, as a Professor of Mathematics at the Mathematics Department of the Facultad de Ciencias, Universidad de Los Andes, in a special service program of this University called " Proyecto Palestra". This program was created as a contribution to the improvement in the teaching of Science at the High School Level in our city, and, eventually, in our country. At the beginning, the work was concentrated in the curricular revision of the Mathematics courses taught in Venezuela at 7th, 8th and 9th grades. Also, for the academic year 1996-97, weekly visits to the classrooms were made, watching the Mathematics teachers working with their students in one of the most important High Schools of the city: Liceo Libertador. This institution has made an official agreement with Universidad de Los Andes to incorporate our suggestions on the teaching of Mathematics, including curricular modifications of the official curriculum, into some of their regular Mathematics courses.

In this work we will explain the main ideas that support those curricular modifications, among which the introduction of " Historic Mathematics" in the curriculum, which means more than just adorning the exposition in the classroom with a few anecdotes, is one of the most important aspects of the proposed curriculum.

Liceo Libertador has 4 sections of each course in 7th, 8th and 9th grade, and the Palestra proposal was applied for the first time with one section of 7th grade which began in September of the year 1.997. It was Section C. The students of this section were kept together in the three following years, and the proposal was applied at their Mathematics courses in 8th and 9th grade as well. There were, then, three other sections taking the regular Mathematics courses, and at the arrival of all 4 sections to 10th grade (in the venezuelan system this is called the 1st year of Sciences), observations were made to compare the performance of the Section C students with that of the other sections, not only in Mathematics courses, but in other Sciences, especially Physics.

It may be of interest to say that Liceo Libertador has 10 sections of the 1st and 2nd year of Sciences (10th and 11th grades). This means that our Section C students were also compared to students coming from High Schools other than Liceo Libertador. We should also add that the curricular changes proposed do not affect the total list of topics included in the official curriculum of 7th, 8th and 9th grades , in the sense that , by the end of 9th grade, the students of section C and all the other students of the same level have studied the same topics. The curricular changes in our proposal regard the order in which the topics are taught, the incorporation of some aspects of the History of Mathematics, the special emphasis on the connections between different topics, and the general orientation of the work in class. We will comment on these changes later.

The results that were observed in the academic year 2.000- 2.001 were very much encouraging. The students in Section C had in general a much more positive attitude towards Mathematics and Physics in relation to most students in other sections. The teachers of these subjects observed a clear eagerness of students in Section C to ask questions and participate with their opinions during their classes, in contrast with very passive students in most of the other sections. In other words, the level of motivation for the comprehension of Physics and Mathematics topics was clearly higher in the group of Section C.

As for the grades of the students, the information gathered is still not sufficient for a complete analysis, since only one group of students has been subject to the experimental curriculum, during three years of their studies .

2 Main aspects of the Palestra Experience

In this section, we will briefly comment on the basic curricular changes which were introduced in the teaching of the Mathematics courses of Section C, Liceo Libertador, from 7th through 9th grade.

1) The order in which the topics are taught:

The official curriculum of 7th, 8th and 9th grades courses of Mathematics in Venezuela has several mistakes regarding the order of the topics, especially in Algebra. For example, students in 8th grade are introduced to polynomials, in some cases in several variables, before they have been exposed to quadratic functions or quadratic equations, because these topics belong to the 9th grade curriculum.

In the Palestra proposal, the order of these and other topics was rearranged with the purpose of adjusting to the following premises:

a) The topics should be arranged in such a way that the difficulties due to complexity or degree of abstraction are in a non decreasing order. As obvious as this may seem, the example mentioned above shows that it is not always taken into account.

b) The teaching of Mathematics should emphasize the relations of the discipline with other disciplines and the historical relations between the ideas that originated different topics within Mathematics . The connection between two subsequent topics in the curriculum should be highlighted. The official curriculum does not contribute to this practice. Rather it is a good example of what has been called an " atomic" curricular design: isolated topics to be taught without an explicit connection between them. The student enjoys realizing that there are " hidden" relationships between concepts seemingly belonging to completely different topics of Mathematics.

The introduction of some important events of the History of Mathematics has proved to be very valuable for this purpose, as we will see in Section 3.

c) Mathematics should be taught as we would teach a foreign language: the meaning of each symbol, of each expression, is to become clear, either before or while engaging in learning the basic rules of grammar. Mathematics are taught at the High School level in Venezuela, in most cases, as if the symbols had no meaning, and only the rules of grammar are to be memorized for the next evaluation, and forgotten soon after it. Again, some historic ideas in Mathematics help the students grasp the meaning of symbols and algorithms.

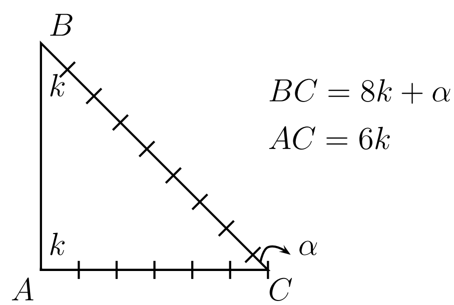
d) The presentation of the cultural context in which some ideas were developed contributes to give the student a different and more realistic perspective over what Mathematics is: a living, ever changing body of knowledge, instead of a rigid set of formulas, algorithms and rules.

3 Some examples of the introduction of Historic Mathematics in the curriculum

I. Crisis in pythagorean Mathematics as the first irrational numbers were considered.

The pythagorean conception of the Universe as being ruled by the natural number, including ratios, collapsed as the isoceles right triangle proved to have a hypotenuse which could not be expressed as $\frac{a}{b}$ with a, b natural numbers. Discussing this episode with 8th grade students who have just looked at right triangles with natural numbers as lengths of the sides, and in that context learned the formula associated with Pythagoras, gives a good opportunity to regard mathematical ideas as connected historically to other cultural aspects of the time: philosophical beliefs in this case created resistance to the evidence of a new kind of number. And this number arose in the simple example of the length of the side of a triangle. After being exposed to this episode, the students engage in working with square roots and irrational numbers in general, having arrived at the topic through geometry .

On the other hand, watching closely the geometrical construction which showed, empirically, that there was no way of expressing the hypotenuse BC of the triangle ABC as a ratio of two natural numbers, gives the student a chance to grasp further the concept of ratio and also to have a geometric intuition of what irrationality means: if we choose a unit of measure such that the sides adjacent to the right angle are exact multiples of this unit, then the hypotenuse is not an exact multiple of that same unit, no matter how small the unit is chosen:



II. Euclid's "Geometric Algebra".

The historical interaction between algebra and geometry has one of its most beautiful exhibitions in Euclid's methods for studying geometric solutions to linear and quadratic equations .

Students, in general, find these methods amusing and the austerity of Algebra is somewhat lightened when Geometry comes into play.

For example, the following method for finding a geometric solution to the linear equation

$$2x + \frac{x}{3} + \frac{x}{2} = 8$$

shows the typical creativity of Euclid's reasoning:

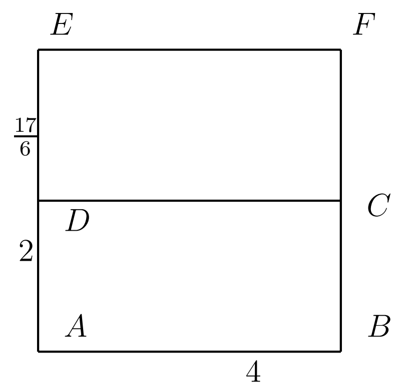
If we start by writing the equation in this way:

$$x\left(2 + \frac{1}{3} + \frac{1}{2}\right) = (2)(4)$$

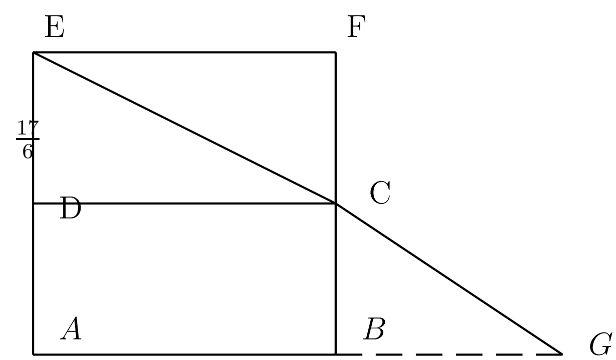
then we may interpret it as stating the equality of the areas of two rectangles: one which has sides of lenghts x, $2+\frac{1}{3}+\frac{1}{2}$ and the other having sides of lengths 2, 4.

We begin by constructing a rectangle with sides of lengths 2,4:

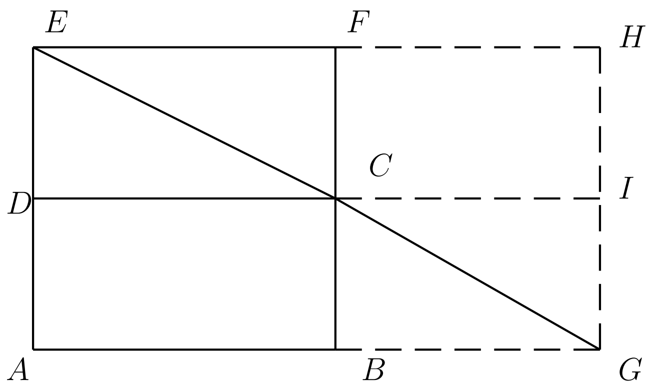
We will now find a rectangle which has the same area as the preceding one, and with one of its sides of length equal to $2+\frac{1}{3}+\frac{1}{2}=\frac{17}{6}$. The conclusion will be that the length of the other side of the new rectangle is x. In order to do this, we will add to the original rectangle, a new one of sides 4 and $\frac{17}{6}$:



We now draw the straight line that contains the diagonal EC of the upper rectangle and call G the point where it meets the extension of the base AB of the lower rectangle:



We draw over this figure the rectangle $AGHE$:



By construction, EC is the diagonal of $EDCF$ and CG the diagonal of $CBGI$. Therefore, the triangles EDC and CFE are congruent, and also the triangles CBG and GIC.

On the other hand, since EG is the diagonal of $EAGH$, the triangles EAG and GHE are congruent. Therefore, the rectangle $CIHF$ has the same area as the rectangle $ABCD$, and the side CI represents the geometric solution to the original equation, because the length of FC equals $\frac{17}{6}$.

In general, students enjoy trying out the same construction, but beginning with a different factorization of 8, and then checking out that the solution segment has the same length as in the first construction.

Usually, we use algebra as a tool for solving geometric problems, but examples such as this one, of the use of geometry for solving algebraic equations are rarely shown in the classroom, and we have found a positive reaction in the students exposed to them, such as an awakened curiosity and a desire to interact with the teacher or the other students in the classroom.

The following construction appears also in Euclid's "Elements" and was learned on the IX century A.D. by al-Khowarizmi, the great arabic mathematician who wrote an important treatise on Algebra, and used it to check out his own algebraic solutions for quadratic equations.

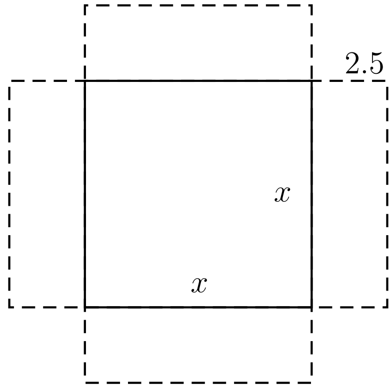
For the equation

$$x^2 + 10x = 39$$

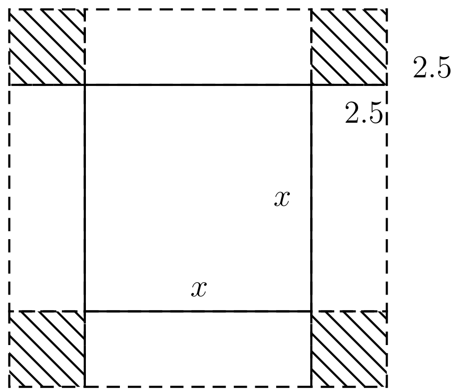
we have the following geometric interpretation:

If x^2 represents the area of a square of side x, and $10x$ the area of a rectangle of sides 10 and x, then the equation states that the sum of the two areas is equal to 39.

We "add up" these two areas geometrically in the following way. We divide the rectangle in 4 rectangles of sides x and 2.5 each, and then place each of these rectangles on the sides of the square of side x, as shown in the figure:



So, the area of this figure equals $x^2 + 4(2,5)x = x^2 + 10x$, which is the same as 39 . Now we complete the square, placing the corners missing, which are, of course, squares of side 2.5 each:



Since the area of this new square is equal to $39 + 4[(2.5)^2] = 64$, then its side equals 8, but this side is equal also to $x + 2(2.5)$, so we get $x = 3$ as a positive solution to the equation.

This construction has proved to be very useful as a tool for introducing quadratic equations and their general solution.

III. The Egyptians"False position Rule" for solving linear equations.

The False Position Rule, used by egyptians circa 1,600 B.C. to solve certain linear equations is a good example of archaic methods that offer the students a chance to find relations between different topics learned in their basic Mathematics courses. In this case, as we explain to the students why the False Position Rule works, it is possible to show connections between linear equations , the idea of proportion, linear functions and similarity of triangles.

Let us use the mentioned egyptian method for finding the solution of the following equation:

$$x + \frac{x}{4} + 3x = 8$$

First of all, we choose an arbitrary value for the x and introduce it to evaluate the expression at the left side of the equation. For example, for $x = 4$, we get

$$4+\frac{1}{4}(4)+3(4)=17$$

Then we state the equality of the ratios:

$$\frac{x}{8}=\frac{4}{17}$$

and obtain

$$x=\frac{32}{17}$$

which is the solution to the equation.

We could explain to our students why this method works as follows:

Let $f(x)=\frac{17}{4}x$. We know that the inverse image of 8 by f is the number that satisfies the equation

$$\frac{17}{4}x=8$$

Since f is a linear function, the expression

$$y=\frac{17}{4}x$$

represents a relation of proportion between the variables x and y. In other words, the line $y=\frac{17}{4}x$ is the set of all points (x,y) in the plane such that

$$\frac{y}{x}=\frac{17}{4}$$

This is why the egyptian method works. As you choose a " false value" , in this case, 4, for the unknown, we discover what is the ratio between all pairs (x,y) that belong to the line associated to the equation

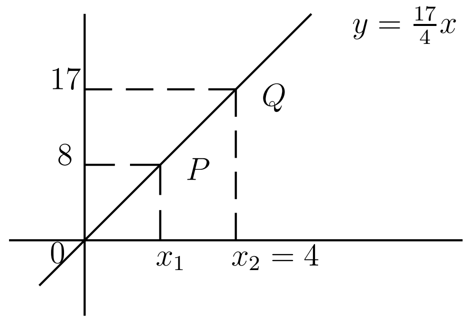
$$x+\left(\frac{1}{4}\right)x+3x=y$$

Once we have determined this ratio, we have

$$\frac{x}{8}=\frac{4}{17}$$

since the pair $(x,8)$ must also belong to the line.

We now use a representation of the situation in the coordinate plane for the purpose of emphasizing its geometrical meaning:



To solve the equation $\frac{17}{4}x = 8$ is to find the x- coordinate of the point P where the line $y = 8$ meets the line $y = \frac{17}{4}x$. On the other hand, we have two triangles to consider in the figure:

$$\triangle OX_1P \quad \text{and} \quad \triangle\, OX_2Q.$$

These triangles are similar, because their inner angles are congruent , so the ratios between the corresponding sides are equal:

$$\frac{X_2Q}{X_1P} = \frac{OX_2}{OX_1} = \frac{OQ}{OP}$$

So, we get

$$OX_1 = \frac{(OX_2)(X_1P)}{(X_2Q)} = \frac{(4)(8)}{17} = \frac{32}{17}$$

Showing the students various perspectives for considering a mathematical problem is always a good way of stimulating their own creativity.

IV. The use of the concept of similarity of triangles and its consequences in the calculation of unreachable distances.

The legend of Thales giving an exact measure of the height of one of Egypt's great pyramids using his knowledge of the properties of similar triangles illustrates the power of Thales' theorem when used as a tool for practical calculations.

Also, we can show the students, for the same purpose, the consequences that this idea had in the development of the Alexandrian Greeks astronomy, in particular,in the calculation of the distance from the Earth to the Moon done by Hipparchus, in the II century B.C.

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