IMAGINATIVE DEPLOYMENT OF COMPUTER ALGEBRA IN THE UNDERGRADUATE MATHEMATICS CURRICULUM

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ABSTRACT

Much of the author's recent experience is attempting to teach Mathematics primarily to undergraduate students following degree programmes in Electronics or Audio Technology. Increasingly, it is found that although such students may be able to perform mechanistic steps such as obtaining a simple derivative, or evaluating a straightforward definite integral, they have little idea as to what these quantities mean. Very few (if any?) would know that these results are connected to a limiting process.

Unless the student's understanding of basic calculus is strengthened, they have little chance of subsequently dealing with the solution of differential equations or the construction of Fourier series. This paper shows how imaginative deployment of computer algebra (*DERIVE*) can substantially assist the understanding of calculus and its applications in the aforementioned areas. In particular, the paper will demonstrate the advantages of using computer algebra as an on-line teaching aid in the classroom compared with using traditional methods of teaching topics such as solving differential equations.

1. Introduction

Mathematics is increasingly perceived as being a difficult subject with the inevitable consequence that many students will try to avoid its study if at all possible. However, it is also well known that knowledge, understanding and competence in certain areas of Mathematics are required for the successful study of many undergraduate courses in Science and Engineering.

Instructors are frequently facing an audience of students, normally with weak mathematical backgrounds [1], who are obliged/forced to study more Mathematics to support their chosen degree programmes. This situation presents considerable challenges to instructors who have the difficult task of motivating reluctant students and of finding ways to facilitate understanding so that such students end up being reasonably competent in the areas taught.

The author believes that imaginative deployment of computer algebra in the undergraduate Mathematics curriculum can greatly assist the understanding of many concepts and applications encountered therein. Using the software package *DERIVE*, this is achieved by the use of built in commands, bespoke user defined commands and visual graphics. In the classroom/lecture theatre, the form of tuition is a combination of traditional methods – white board etc., and interactively generated computer algebra images provided via a notebook PC linked to a data projector.

In this paper, the author gives examples of how computer algebra can be imaginatively deployed to assist with the teaching and learning of differential and integral calculus, solving differential equations and construction of Fourier series. Bespoke user defined commands will be presented for the benefit of instructors. In practice, **the definitions of such commands are normally hidden from students** who simply need to know how to supply the values of the arguments contained in these commands for their own use during workshop sessions.

2. Differential Calculus

When introducing differential calculus, it is customary to begin with the simple function $y = u(x) = x^2$. We obtain a value for the gradient function (rate of change function, derivative etc.) at some fixed point e.g. $x = 3$, by drawing a series of chords with ends anchored at (3, 9) that are decreasing in length and then calculating their gradients. We conclude quite straightforwardly that the gradient function has the value 6 when $x = 3$.

In order to demonstrate this approach for a wide range of different functions, we can employ the User Defined Command (UDC) GRAD_FUNC_POINT(u, x, a) which simplifies to a vector containing two entries namely a and the value of the gradient function evaluated at $x = a$.

$$\text{grad\_func\_point}(u,\ x,\ a) := \left[a,\ \lim_{h \to 0} \frac{\left(\lim_{x \to a+h} u\right) - \lim_{x \to a} u}{h}\right]$$

This UDC was authored as:

$$\text{GRAD\_FUNC\_POINT}(u, x, a) := [a, \lim((\lim(u, x, a+h) - \lim(u, x, a))/h, h, 0)]$$

Examples of its use are $\text{grad\_func\_point}(x^2,\ x,\ 3) = [3,\ 6]$

(This was obtained by authoring the command followed by an "equals sign", then selecting simplify).

$$\text{grad\_func\_point}(\text{LN}(t),\ t,\ 4) = \left[4,\ \frac{1}{4}\right]$$

$$\text{grad\_func\_point}\left(\text{SIN}(x),\ x,\ \frac{\pi}{3}\right) = \left[\frac{\pi}{3},\ \frac{1}{2}\right]$$

In the case of $\ln(t)$, GRAD_FUNC_POINT can be used for several suitable values of t and, invariably, students are able to conclude that if $t = a$ where $a > 0$, then the gradient function will have value $\frac{1}{a}$. However, the aim is to be able to obtain the gradient function for an arbitrary given function at an arbitrary point. The UDC GRAD_FUNC_POINTS(u, x, b, e, s) simplifies to a matrix of coordinates corresponding to discrete points of the gradient function for $u(x)$, beginning with $x = b$ and ending with $x = e$ in steps of s.

```
grad_func_points(u, x, b, e, s) := VECTOR(grad_func_point(u, x, a), a, b, e, s)
```

We demonstrate the use of this command on $y = u(x) = \sin x$ by authoring:

```
grad_func_points(SIN(x), x, 0, 2·π, π/6)
```

The matrix of co-ordinates (not shown here but obtained via the ≈ button) can now be plotted to see:

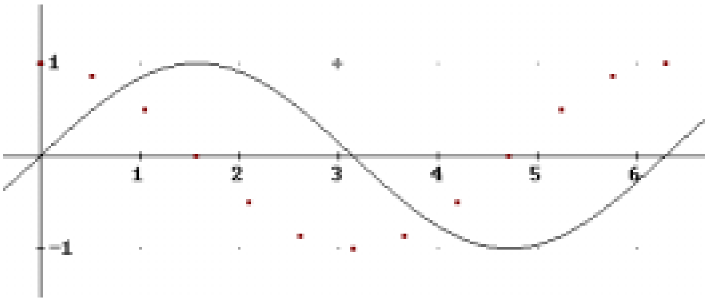


Figure 1 - $\sin x$ *plotted along with discrete points of its gradient function.*

From the plot, it should be apparent that the derivative of $y = u(x) = \sin x$ is $\cos x$. This can now be reinforced by returning to the first UDC and not specifying a numerical value for a.

```
grad_func_point(SIN(x), x, a) = [a, COS(a)]
```

or even

```
grad_func_point(SIN(x), x, x) = [x, COS(x)]
```

!

Hence, CAS has been used to generate the derivative of $\sin x$ using a graphical/visual approach as opposed to solely using an abstract/rigorous approach that students often struggle with. (The reader will recall that students will not be exposed to the definition of the command GRAD_FUNC_POINT).

3. Integral Calculus

It would be unwise for an instructor to launch into definite integration for non specialist Mathematics students (or others?) by starting with the definition:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}[f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n],$$

where P is a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$

yielding the n sub-intervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ with lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$,

x_i^* is a point taken from $[x_{i-1}, x_i]$ for $i = 1,\ldots,n$, and $\| P \| = \max \Delta x_i, i = 1,\ldots,n$.

A much gentler approach, which will make the above more palatable if the instructor later chooses to expose this to their students, is to associate definite integration with the area under a curve by means of a simple (i.e. equal length subintervals with $x_i^* = x_{i-1}$ or x_i) Riemann sum.

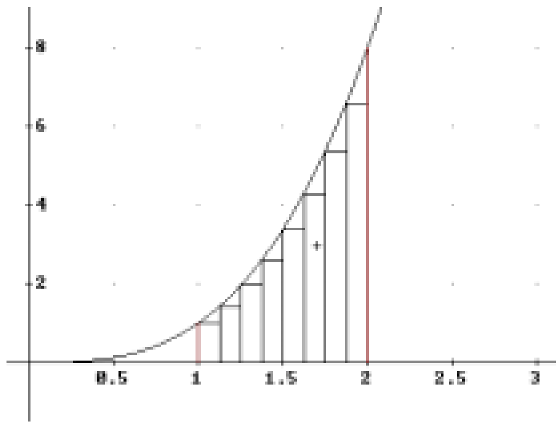
This can be achieved by employing the UDC MAKE_RECTS(u,x,a,b,n) which simplifies to a vector containing (4×2) matrices whose elements are the coordinates of the four corners of the n rectangles, with width $(b-a)/n$, under the curve $u(x)$ between $x=a$ and $x=b$, arranged in such a way as to provide a lower bound for the exact area under the curve for a monotonically increasing function.

The command is authored as :

MAKE_RECTS(u,x,a,b,n) := VECTOR(LIM([[$x+r(b-a)/n$,0],[$x+r(b-a)/n$, LIM($u,x,a+r(b-a)/n$)],[$x+(r+1)(b-a)/n$, LIM($u,x,a+r(b-a)/n$)], [$x+(r+1)$(b-a)/n,0]],x,a),r,0,n-1)

Plotting this vector of matrices, i.e. the rectangles, gives a visual display which is easy to understand. We demonstrate this by approximating to the area under the curve $u(x) = x^3$, bounded by $x=1, x=2$ and the x-axis using 8 rectangles.

MAKE_RECTS(x^3, x, 1, 2, 8) (The matrix of coordinates is not displayed here).



The figure was produced by simplifying the previous command and then plotting the resulting matrix of coordinates.

Display options need to be set to suppress colour changes and to join the vertices of the rectangles in order to construct the rectangles shown.

Figure 2 – *A lower bound approximation to the area under the curve $u(x) = x^3$, bounded by $x = 1, x = 2$ and the x-axis using 8 rectangles.*

The UDC SUM_RECT_AREAS(u,x,a,b,n) simplifies to a left Riemann sum of the areas of the rectangles produced by MAKE_RECTS.

$$\text{SUM\_RECT\_AREAS}(u, x, a, b, n) := \frac{b-a}{n} \cdot \sum_{r=0}^{n-1} \lim_{x \to a + (b-a) \cdot r/n} u$$

The command is authored as:

SUM_RECT_AREAS(u,x,a,b,n) := $(b-a)/n$ * SUM(LIM($u,x,a+(b-a)r/n$),r,0,$n-1$) .

Applying this command to the above example gives:

SUM_RECT_AREAS(x^3, x, 1, 2, 8) = 3.32

By increasing the number of rectangles to say, 100, we obtain the following:

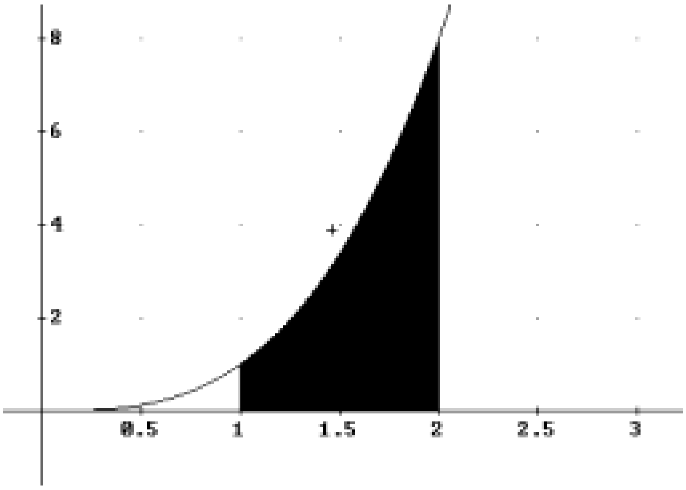


Figure 3 – *A lower bound approximation to the area under the curve* $u(x) = x^3$*, bounded by* $x = 1, x = 2$ *and the x-axis using 200 rectangles.*

This treatment should clearly demonstrate the limiting process inherent in the definition of a definite integral since, visually, we can see that an infinite number of rectangles must correspond to the exact area when summed. The area shown in figure 3 is readily calculated, yielding

$$\text{SUM\_RECT\_AREAS}(x^3, x, 1, 2, 200) = 3.73$$

Leaving the number of rectangles, n, arbitrary yields the closed form sum:

$$\text{SUM\_RECT\_AREAS}(x^3, x, 1, 2, n) = \frac{15 \cdot n^2 - 14 \cdot n + 3}{4 \cdot n^2}$$

It is clear that the right hand side can be expanded as $\frac{15}{4} - \frac{7}{2n} + \frac{3}{4n^2}$, with limiting value $\frac{15}{4}$ as $n \to \infty$.

At this stage, students could be shown the relationship $\int_1^2 x^3 dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$.

Closed form sums are nice to see. Of particular interest is to calculate the left Riemann sum for the area bounded by $\sin x$, $x = 0, x = \pi/2$ and the x-axis using an arbitrary number of rectangles.

$$\text{SUM\_RECT\_AREAS}\left(\text{SIN}(x), x, 0, \frac{\pi}{2}, n\right)$$

Expanding the above command gives: $\dfrac{\pi \cdot \text{COT}\left(\dfrac{\pi}{4 \cdot n}\right)}{4 \cdot n} - \dfrac{\pi}{4 \cdot n}$

We can use *DERIVE*'s limit command (from within the calculus menu) to obtain the exact value

for the area i.e. $\lim_{n \to \infty} \left[\dfrac{\pi \cdot \text{COT}\left(\dfrac{\pi}{4 \cdot n}\right)}{4 \cdot n} - \dfrac{\pi}{4 \cdot n}\right] = 1$

This result also demonstrates that $\lim_{\alpha \to 0}(\alpha \cot \alpha) = 1$!

We note further that

```
lim   SUM_RECT_AREAS(SIN(x), x, a, b, n) = COS(a) - COS(b)
n→∞
```

This result can be used to introduce the concept of the anti-derivative. Koepf and Ben-Israel [2] pursue this approach showing that an indefinite integral can be regarded as a definite integral over a variable interval.

It is the author's experience that even students who have encountered integral calculus prior to embarking on their undergraduate course have rarely appreciated that definite integration is connected to a limiting process. CAS enables this important concept to be presented both visually and algebraically by generating, where possible, closed form sums.

4. Differential Equations

Students can often be intimidated by the term "differential equation" and expect these to be difficult at the outset simply because of the presence of one or more derivatives in an equation. It is useful to begin with a very simple example such as $\frac{dy}{dt} = 2t$. Most students will be able to say that "the" solution is $y = t^2$ and the instructor then normally has to interject to coax out the infinite number of solutions given by $y = t^2 + c$, where c is an arbitrary constant. *DERIVE* can be used here to demonstrate diagrammatically that, in the absence of any boundary conditions, a differential equation will have an infinite number of solutions that can cover the whole real plane.

This is readily accomplished by authoring, simplifying and then plotting the command

```
        2
VECTOR(x  + c, c, -4, 4, 0.5)
```

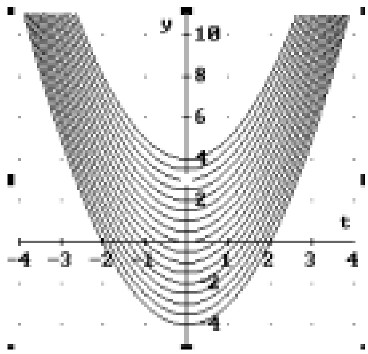


Figure 4 - *Solution curves for* $\frac{dy}{dt} = 2t$

If we only consider tangent line segments drawn at regular points on these solution curves, then the resulting diagram should give a very good indication as to what the actual solution curves look like.

The tangent field can be obtained via the BIC

$\text{DIRECTION\_FIELD}(f(x,y), x, x_0, x_m, m, y, y_0, y_n, n)$

where $\frac{dy}{dx} = f(x,y)$, x varies from x_0 to x_m in m steps and y varies from y_0 to y_n in n steps.

We now author, approximate, then plot the command:

```
DIRECTION_FIELD(2·x, x, -2, 2, 9, y, -4, 4, 15)
```

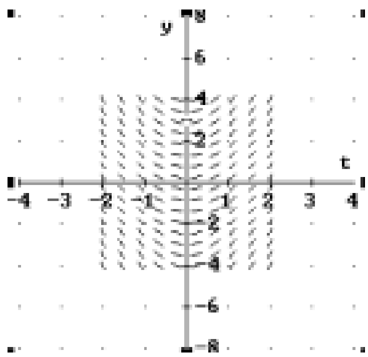


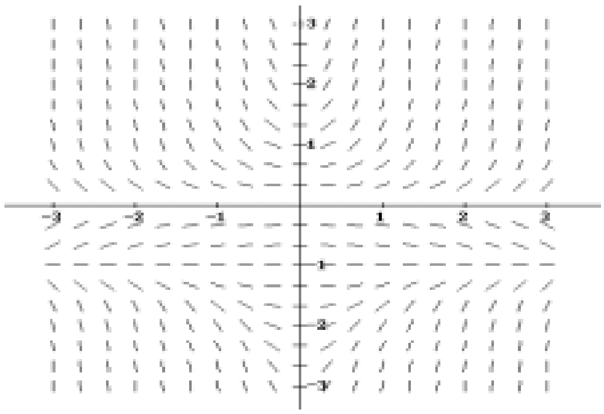
Figure 5 – *Tangent field for* $\frac{dy}{dt} = 2t$

Thus, via this very simple example, students can appreciate that much information about the general solution of a differential equation can be obtained from the initial differential equation without the need to solve it. It would be very difficult to convey these ideas to students without the use of a software package.

Of course, another obvious advantage of this approach is to emphasise that seeing the tangent field determined by a differential equation is possibly the best we can see with regard to the complete solution curves if analytical techniques cannot be employed to solve the equation. Indeed, we may only be able to generate points for particular solutions using numerical techniques.

Several of the aforementioned concepts can be encapsulated by the following example. We shall consider the solutions of the differential equation $\frac{dy}{dt} = y(1+y)t$, and begin by obtaining a plot of its tangent field.

DIRECTION_FIELD(y·(1 + y)·t, t, -3, 3, 18, y, -3, 3, 18)

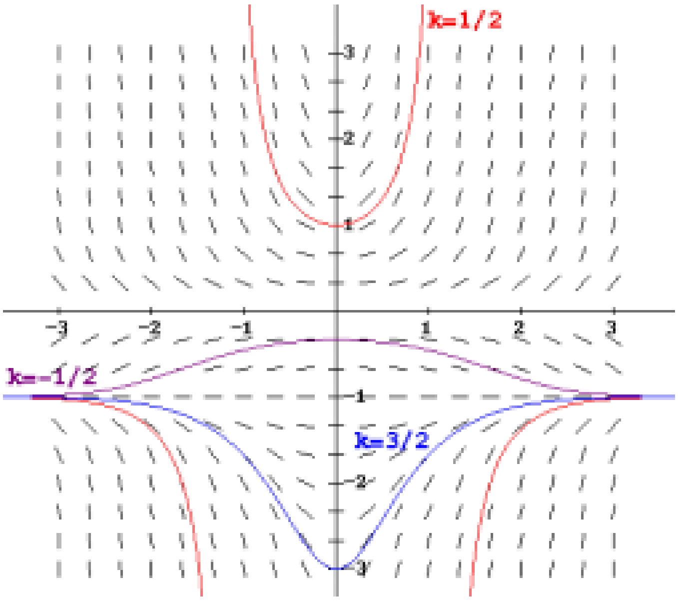


Using the approximate command, we obtain a large matrix of coordinates (not shown here) which can now be plotted.

This rather interesting diagram shows the flow of the solution curves and also indicates asymptotic behaviour.

Figure 6 – *Tangent field for* $\frac{dy}{dt} = y(1+y)t$

It is a straightforward matter to analytically obtain the general solution $y = \frac{ke^{\frac{t^2}{2}}}{1-ke^{\frac{t^2}{2}}}$. This now presents the instructor and students with a rich mathematical investigation. We may pose the question "for which values of k do we obtain solutions in that part of the plane where $y < -1$, where $y \in [0,-1)$ and $y > 0$?" Using *DERIVE*'s SUB command, by experimentation, we can discover that if $k > 1$, we obtain solution curves in the region where $y < -1$. If we choose $k < 0$, we obtain solution curves in the region $y \in [0,-1)$. Both these ranges for k show solution curves that are asymptotic to the line $y = -1$. For $k \in (0,1)$, we obtain solution curves each consisting of three pieces with two vertical asymptotes and the horizontal asymptote $y = -1$. The case $k = 1$ yields a solution curve with different characteristics to the previous cases. A selection of these solution curves is shown below.



For completeness, at this stage students can be informed that sometimes only numerical techniques are available to obtain a numerical solution of a differential equation. *DERIVE* supports a variety of numerical techniques the simplest of which is $\text{EULER\_ODE}(f(x,y),x,y,x_0,y_0,h,n)$ and approximates to a vector of $n+1$ solution points of the equation $\frac{dy}{dx}=f(x,y)$ with $y=y_0$ and $x=x_0$ using a step size of h.

Figure 7 – *Particular solutions of* $\frac{dy}{dt}=y(1+y)t$

We apply the EULER_ODE command to generate solution points on the particular solution passing through the point $(0,-3)$ and contrast these solution points with the exact solution given by $y=\frac{3e^{\frac{t^2}{2}}}{2-3e^{\frac{t^2}{2}}}$.

```
EULER_ODE(y·(1 + y)·t, t, y, 0, -3, 0.25, 12)
```

Simplifying EULER_ODE via the approximate command, yields:

0	-3
0.25	-3
0.5	-2.625
0.75	-2.091796875
1	-1.663581132
1.25	-1.387600869
1.5	-1.219527337
1.75	-1.119132491
2	-1.060802785
2.25	-1.028552903
2.5	-1.012033306
2.75	-1.004421989
3	-1.001368428

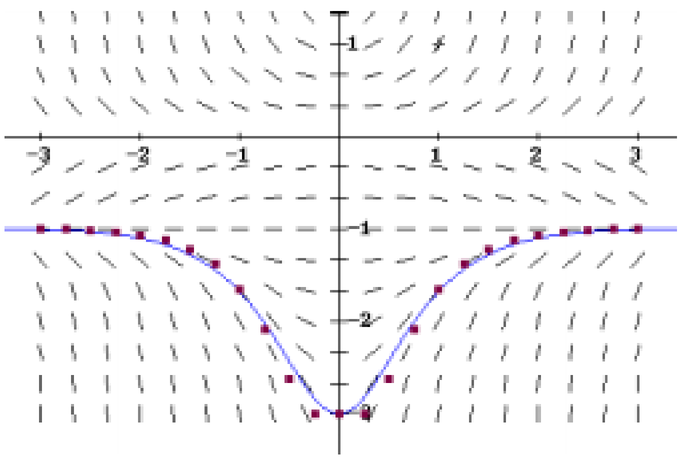


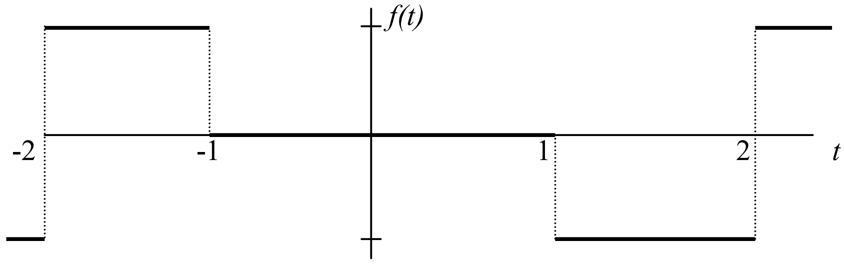
Figure 8 – *Numerical solution of* $\frac{dy}{dt}=y(1+y)t$ *passing through* $(0,-3)$.

Solution points for $t<0$ are obtained by replacing 0.25 with -0.25 in the EULER_ODE command.

Very few (if any) students have seen tangent fields associated with the solutions of differential equations even though they may already be familiar with solving simple differential equations. It is a revelation for them to see tangent field diagrams "on-line" by a CAS in the classroom and this stimulates them to engage with the topic with greater confidence and understanding.

5. Fourier Series

DERIVE is an indispensable tool for dealing with piecewise defined periodic functions and their associated Fourier Series representations. As an example, consider the function with graph:



Defined as:

$$f(t)=\begin{cases} 1 & -2<t\le -1 \\ 0 & -1<t\le 1 \\ -1 & 1<t\le 2 \end{cases}, \text{ where } f(t+4)=f(t)$$

It is useful to be able to plot the graph of this periodic function using *DERIVE*, so that, later, we can superimpose the graph of its Fourier Series and contrast the two.

Plotting the graphs of piecewise defined periodic functions is achieved by defining the function over the interval (0, *T*), where *T* is the period using *DERIVE*'s built-in function $\mathrm{CHI}(a,x,b)$, where $\mathrm{CHI}(a,x,b)=\begin{cases} 1, & a<x<b \\ 0, & x<a \\ 0, & x>b \end{cases}$, and then using the built-in MOD function to take care of the periodicity.

```
f(t) := CHI(1, t, 2)·(-1) + CHI(2, t, 3)·1
f(MOD(t, 4))
```

Plotting the latter expression produces the graph of the piecewise defined periodic function.

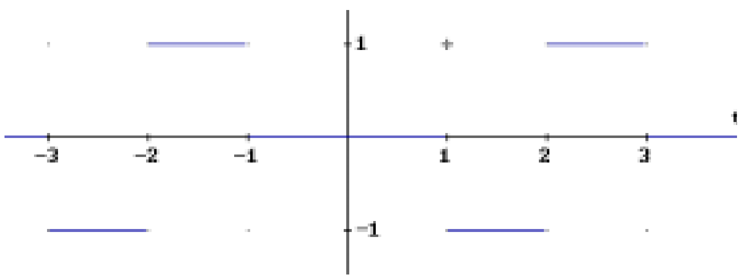


Figure 9 – *Plotting piecewise defined periodic functions using DERIVE's CHI and MOD functions*.

The standard Fourier Series representation for a function with period *T* is given by:

$$f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{2k\pi t}{T}+b_k\sin\frac{2k\pi t}{T}\right),$$

and the Fourier coefficients are given by:

$$a_0 = \frac{2}{T}\int_{t_1}^{t_2} f(t)\,dt$$

$$a_k = \frac{2}{T}\int_{t_1}^{t_2} f(t)\cos\frac{2k\pi t}{T}\,dt \quad \text{for } k \in \boldsymbol{N}^+$$

$$b_k = \frac{2}{T}\int_{t_1}^{t_2} f(t)\sin\frac{2k\pi t}{T}\,dt \quad \text{for } k \in \boldsymbol{N}^+$$

where $T = t_2 - t_1$.

Since the given example is an odd function, $a_0 = 0$, and $a_k = 0$ for $k \in \boldsymbol{N}^+$. In addition

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2k\pi t}{T}\,dt \text{ for } k \in \boldsymbol{N}^+$$

It is a straightforward matter to show that $b_k = \frac{2}{k\pi}\left((-1)^k - \cos\frac{k\pi}{2}\right)$

The required Fourier Series is therefore:

$$f(t) = -\frac{2}{\pi}\left[\sin\frac{\pi}{2}t - \sin\pi t + \frac{1}{3}\sin\frac{3\pi}{2}t + \frac{1}{5}\sin\frac{5\pi}{2}t - \frac{1}{3}\sin 3\pi t + \frac{1}{7}\sin\frac{7\pi}{2}t + \cdots\right]$$

As a check, or otherwise, we can use *DERIVE*'s BIC FOURIER$(f(t), t, t_1, t_2, n)$ to generate the first n harmonics of the Fourier Series for $f(t)$ defined over the periodic interval t_1 to t_2.

FOURIER(f(t), t, 0, 4, 7) which simplifies to the expression below:

$$-\frac{2\cdot\text{SIN}\left(\frac{7\cdot\pi\cdot t}{2}\right)}{7\cdot\pi} - \frac{2\cdot\text{SIN}\left(\frac{5\cdot\pi\cdot t}{2}\right)}{5\cdot\pi} - \frac{2\cdot\text{SIN}\left(\frac{3\cdot\pi\cdot t}{2}\right)}{3\cdot\pi} - \frac{2\cdot\text{SIN}\left(\frac{\pi\cdot t}{2}\right)}{\pi} + \frac{2\cdot\text{SIN}(3\cdot\pi\cdot t)}{3\cdot\pi} + \frac{2\cdot\text{SIN}(\pi\cdot t)}{\pi}$$

Superimposing this truncated Fourier Series onto the original piecewise defined periodic function, we obtain:

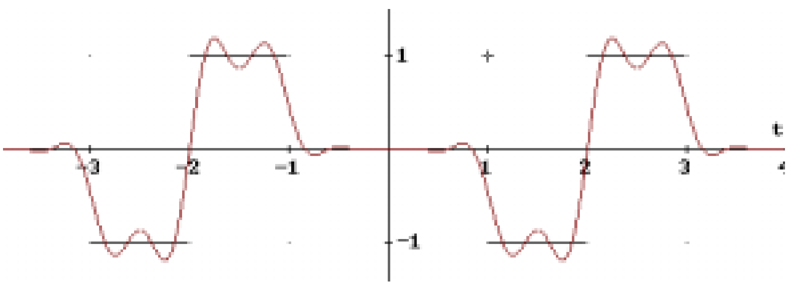
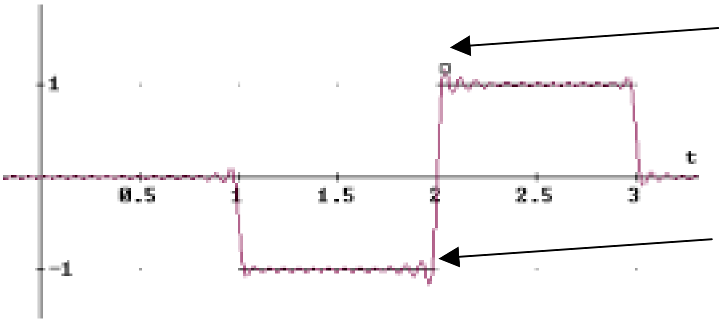


Figure 10 – *Plotting the truncated Fourier Series representation along with the original piecewise defined periodic function.*

At this stage, a discussion can take place over the behaviour of the synthesised function around the points of discontinuity. Classical theory states that in general, the magnitude of the combined

undershoot and overshoot together at a point of discontinuity amount to about 18% of the magnitude of the discontinuity. This is the so called onset of Gibbs' phenomenon.
We can "test" this theory using *DERIVE*'s **trace** facility to measure the lengths of the under and overshoots on a plot of the truncated Fourier Series containing 50 harmonics.



The pronounced "peak" has coordinates (2.04, 1.178) provided by *DERIVE*.

The pronounced "trough" has coordinates $(1.96, -1.178)$

Figure 11 – *Using DERIVE to explore Gibbs' phenomenon*

In the above example, at $t = 2$, the magnitude of the discontinuity is 2. We can from the coordinates obtained using *DERIVE*'s trace facility, that the distance from the trough to the peak is 2.356. Hence the magnitude of the combined under and overshoot is equal to $2.356 - 2 = 0.356$, and $\dfrac{0.356}{2} \times 100 = 17.8\%$

The main use of *DERIVE* here is to show, visually, how a Fourier series can generate a given periodic signal function even when it is piecewise defined. Moreover, the ability to measure the onset of Gibbs' phenomenon in such a straightforward manner is particularly appealing.

6. Conclusion

There is no doubt that the ability to perform tedious or repetitive symbolic manipulation using computer algebra focuses the student's mind on the concepts that are very often obscured by the time consuming process of carrying out the manipulation by hand. Furthermore, computer generated plots provide a powerful means of visualising concepts and applications.

Much of the treatment demonstrated in this paper would simply not be viable using traditional teaching methods. Certainly, the interactive use of computer algebra in the classroom both helps to "bring alive" the Mathematics being presented and stimulates interest. The very fact that a computer image is being projected catches the attention of the audience. This type of delivery, coupled with the enthusiasm and pedagogical skills of the instructor can result in a positive, productive and enjoyable experience for the students.

Whenever asked, students invariably welcome the deployment of computer algebra within the curriculum to assist their teaching and learning. This is further demonstrated by the many occasions where this style of exposition has provoked questions from the audience and has inspired dialogue between students and instructor. Common remarks have included statements such as "I never really understood calculus before" and "it is helpful to *see* what solutions to expect before actually finding them" etc.

The author's experience of this type of delivery has been to non-specialist Mathematics undergraduates where the emphasis has been on a less rigorous exposition of the Mathematics needed. However, the software can be used to address important and more rigorous aspects of calculus such as differentiability and continuity where the limiting processes need to be more controlled involving, for example, left and right limits.

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[1] The Engineering Council, London, 2000, *Measuring the Mathematics Problem*.

[2] Ben-Israel, A.., Koepf, W., 1994, "The definite nature of indefinite integrals", *The International DERIVE Journal*, vol.1, no.1, 115- 131.