CLASSIFYING STUDENTS' MISTAKES IN CALCULUS

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ABSTRACT

This article analyses some structural errors in calculus problems from first year mathematics undergraduates. They arise for reasons related to generalisation, intuition, inadequacy of concepts, instrumental understanding, problems of language and symbol manipulation. The lack of metacognitive control is also an important factor.

1. Introduction

There are many accounts of mathematical errors, which have an underlying logical explanation. In a pioneering study Brown and Burton (1978) catalogued many such errors in the domain of arithmetic. There have been similar studies since, for example Van Lehn (1980), and Maurer (1987). In the *Concepts in Secondary Mathematics and Science* project, reported in Hart (1981) misconceptions in other areas of school mathematics were investigated.

In many situations what appears to happen is that a procedure is learned instrumentally (Skemp 1976) in a way which does not reflect the underlying mathematical structure but which gives the correct answers in a particular set of examples. It is then extrapolated, but gives incorrect results, because of the structural mismatch, which the instrumental learning cannot adapt to. A common undergraduate example is

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}\ \text{(correct)} \quad \rightarrow \quad \frac{\partial y}{\partial x}=\frac{1}{\frac{\partial x}{\partial y}}\ \text{(incorrect).}$$

Maurer's (1987) article, which discusses mainly subtraction, considers generalisation, and makes the point that this seems to happen purely syntactically, ignoring semantic considerations. Norman & Pritchard (1994) relate errors to Krutetskii's (1980) ideas about generalisation.

In undergraduate mathematics Orton (1983) discussed errors in basic calculus. Following Donaldson (1963) he focused on Structural Errors (as distinct from mistakes in calculation (Executive Errors) which nevertheless sometimes have a structural explanation). Orton's work concentrated on basic concepts and calculations in one-variable calculus. He explored things like limits, the meaning of *dy* and *dx* and the differential quotient, rates of change and turning points. In dealing with integration he looked at the integral as the limit of a sum, and at area and volumes of revolution. The examples he reported use simple polynomials, concentrating on basic conceptions. In this paper we explore structural errors occurring in first year university calculus arising as a result of procedural extrapolation as described above. We have chosen examples which emphasise algorithmic procedures, and which are some way beyond the basic ideas which Orton discussed.

Cipra (1989) gives examples of student errors in his book on mistakes in calculus, and suggests methods of checking and monitoring. This relates to the ideas of Schoenfeld (1985) concerning metacognitive *control*. Cipra does not analyse individual student errors to categorise them structurally, as will be discussed in this article. He gives some hypothetical explanations, for example for Fractional Inversion (p.61).

All the examples below were encountered in students' written work, or during problem classes. In the former case the written work was followed by discussions, where students explained their (erroneous) procedures. In problem classes one was able to interrogate students' thinking as they worked on problems. In the examples we give a condensed version of the students' solutions, and a résumé of their explanations, using their own language, to clarify the observations.

It is important to realise that these are not isolated errors. All the examples here were encountered during a one semester first year university calculus course. They are a small but representative sample, not only of that course but of many of the structural errors one has observed teaching calculus over many years.

The examples below are split into three categories: procedural extrapolation, pseudolinearity and equation balancing. These are not designed to be a definitive taxonomy, but to indicate that one can observe common features among the student errors one encounters.

2. Procedural extrapolation

We give three examples involving differentiation, and then two on integration, where the second has several integrals giving rise to similar errors.

Example 1: Find the first five derivatives of $f(x) = \exp(x + x^2)$.

Solution: $f'(x) = \exp(x + x^2); f''(x) = \exp(x + x^2);\ldots$ and so on: they are all the same.

Explanation: *Well, the derivative of the exponential function is always the same.*

Comment: The student has used the fact that the derivative of the exponential function is the exponential function. This has however been used as if it were a universal procedure. One can observe this particular extrapolation in many similar contexts. The students appear to be operating on the (exponential) function as an object, having lost sight of its process or action attributes (Thompson 1994, pp. 26-7). However, as Thompson points out

> " it is easy to be fooled - to think that students are reasoning about functions as objects when it is actually the function's literal representation (i.e. marks on paper) that are the objects of their reasoning."

In fact one might also refer to *oral* representation since they say *the exponential function* in their explanations. The kind of error in this example is encountered both when the function is written as e^{x+x^2} and also as $\exp(x + x^2)$, emphasising that the students associate a name (the exponential function) with what they see on paper, and then operate with the name (verbal symbol) and the properties they associate with that. Evidence from students' written work in the context of this error suggests that they also operate internally in this way. Subsequent discussions confirm that their internal verbalisation of their procedures follows the same pattern as that which they offer when they work "out loud".

Example 2: Find the Maclaurin expansion of $f(x) = \ln(1 + 2x)$.

Solution and explanation:

You need to work out the derivatives and then put x = 0. The first one is $f'(x) = \dfrac{1}{1+2x}$.

This is a fraction so you have to use the quotient rule. First you square the denominator. On top you have two terms. The first is the denominator times the derivative of the numerator, and there are no x terms so the derivative of that is 1. The second term is minus the numerator times the derivative of the denominator, which is 2. So you get

$$f''(x) = \frac{(1+2x)-2}{(1+2x)^2} = \frac{2x-1}{(1+2x)^2}.$$

You do the next one the same way, by the quotient rule

$$f'''(x) = \frac{(1+2x)^2 \cdot 2 - (2x-1).2(1+2x)\cdot 2}{(1+2x)^4}.$$

This is getting too complicated. It must be easier but I can't find a mistake.

Comment: This example was observed in a tutorial class with the student being asked to think out loud. The student had seen so many applications of the rule that he could not imagine the possibility that the denominator term should be multiplied by zero in applying the quotient rule. He was easily able to follow the alternative solution, writing the first derivative in the form $f'(x) = (1+2x)^{-1}$ and continuing by using the chain rule successively, but he could still not find his mistake. When it was pointed out that the derivative of his original numerator is zero and not 1 he was genuinely surprised, responding

So the quotient rule doesn't always have two terms on top?

He took a lot of convincing that he did have two terms, but one of them was zero.

But nought is the same as nothing. So there is really only <u>one</u> term is there?

Here the form of the result of the procedure as well as its description is being extrapolated.

Example 3: Find the first and second partial derivatives of $f(x, y) = \exp(x^2 y^2)$.

Solution: $\dfrac{\partial f}{\partial x} = 2xy^2 \exp(x^2 y^2)$; $\dfrac{\partial^2 f}{\partial x^2} = \left(2xy^2\right)^2 \exp(x^2 y^2)$. [The other second order partial derivatives were subject to the same error.]

Explanation: *When you use the function of a function rule the derivative of the exponential function is the same again. Then you differentiate what is in the brackets and so you multiply by* $2xy^2$. *You do the same again to work out the second partial derivative, so you multiply by another* $2xy^2$.

Comment: The first partial derivative has been calculated correctly. The problem seems to be that this procedure has been formulated in the instrumental form

multiply by $2xy^2$.

The first step does not involve the product rule and so the student performs the following steps by extrapolating the procedure used at step 1, namely

multiply by $2xy^2$, *and the exponential function is unchanged.*

Example 4: Evaluate the indefinite integral $\int x \cos x dx$.

Solution A: $\int x \cos x dx = \frac{1}{2} x^2 (-\sin x)$.

Explanation: *Integration by parts is the reverse of the product rule for differentiation. In the product rule you differentiate both functions, so for integration by parts you must integrate both functions.*

Comment: What is interesting is that in this example many of the students making the error were able to apply the product rule for differentiation correctly during the course of discussion. What appears to be extrapolated here is not so much the procedure but an informal verbal description of the procedure. The linguistic register (Pimm, 1987, Chapter 4) has been shifted from that of mathematical English to everyday English, where the fuzziness of ordinary discourse is a factor.

Other students arrived at this kind of error by asserting that

the integral of a product is the product of the integrals.

Solution B: $\int x \cos x dx = \frac{1}{2} x^2 \cos x + x(-\sin x)$.

Explanation: This student also said that

integration by parts is the reverse of the product rule for differentiation.

She continued

For differentiation $(fg)' = gf' + fg'$, *so for integration*

$\int fg = g \times \int f + f \times \int g$.

Comment: As in example 2, preservation of form appears to be a factor here.

Example 5: Evaluate the following indefinite integrals:

$$\int \frac{1}{1+3x}dx; \int \frac{1}{1+x^2}dx; \int \cos(x^3)dx; \int (t^2+1)^{\frac{1}{2}}dt.$$

Solution:

$$\int \frac{1}{1+3x}dx = \frac{\ln(1+3x)}{3}; \int \frac{1}{1+x^2}dx = \frac{\ln(1+x^2)}{2x};$$

$$\int \cos(x^3)dx = \frac{\sin(x^3)}{3x^2}; \int (t^2+1)^{\frac{1}{2}}dt = \frac{(t^2+1)^{\frac{3}{2}}}{\frac{3}{2}\cdot 2t}.$$

Explanation: *If you differentiate ln(1+3x) you get the function we are supposed to be integrating, except for an extra 3, so we have to divide by the 3 to get the answer. The others are similar. In the second one "1 over" gives you a log again. This time if you differentiate ln(1+*x^2*) you get an extra 2x, so you have to divide by it like we did in the first one. In the next one the integral of cos is sin, and this time if you differentiate you get an extra 3*x^2*, which you divide by. In the last one, if you integrate* x^n *you get* x^{n+1} *over n+1. But it isn't x, it's* t^2*+1, so again if you differentiate you get an extra 2t, so this gets divided as well as the* $\frac{3}{2}$.

Comment: The students appear to have formulated the instrumental procedure "divide by the derivative of what is in the brackets". This works in the case when that derivative is a constant, where the "inner function " is *linear*. It is being extrapolated to situations where the inner function is non-linear. This is a very commonly encountered error. Many students display it, and their explanations usually follow similar lines to the one reported here. One can speculate as to how this extrapolation might be a consequence of instruction as follows. Students are urged always to check their integration by differentiating the result. When they first encounter differentiation of composite functions they are given examples like the first one, where the inner function is linear, in order to keep the calculations straightforward initially. For example the textbook Adams (1995) presents the Chain Rule in §2.5 (p.121), but prior to that in §2.3 there is a separate explanation "A Special Case of the Chain Rule" for derivatives of functions of the form $g(x) = f(ax+b)$. In the context of integrals like the first one in this example what they observe is that when they differentiate the inner function they always obtain what is in the denominator. The resulting cancellation gives the correct result. We have encountered students who have consciously extrapolated this aspect of the procedure when checking other results. For example in the second integral, when checking by differentiating $\frac{\ln(1+x^2)}{2x}$ they do not use the quotient rule. Instead they follow a sequence of steps which imitates what happens with $\frac{\ln(1+3x)}{3}$, so that the procedure

> *differentiate ln(1+3x) and you get 1 over (1+3x) times 3, which cancels with the 3 in the denominator*

is transformed to

> *differentiate ln(1+*x^2*) and you get 1 over (1+*x^2*) times 2x, which cancels with the 2x in the denominator*.

When the error is pointed out, it is not uncommon for students to query the first (correct) integration. Having been shown that a procedure they have used gives an incorrect result in one case, they feel that it must be wrong in all cases (a further extrapolation), as their own comments indicate.

As well as using written work, where students are asked for explanation some time later, situations like this have been observed directly in a tutorial setting. Students give verbal explanations along the lines reported here, and when asked “is that how you thought about it” respond affirmatively. In many cases the students provide observable evidence that their subvocal (self-talk) explanations (Pimm, 1987, p. 24) are very close to what they say out loud.

3. Pseudolinearity

A well known class of extrapolations is the erroneous use of linearity in such examples as

$\ln(x+y) = \ln x + \ln y$; $\mathrm{e}^{x+y} = \mathrm{e}^{x} + \mathrm{e}^{y}$; $\tan(x+y) = \tan x + \tan y$;

$(a+b)^2 = a^2 + b^2$; $(\sin x + 1)^2 = \sin^2 x + 1$;

$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$; $\sqrt{4t^2+4} = 2t + 2$; $\sqrt{1+\frac{1}{x}} = 1 + \frac{1}{\sqrt{x}}$.

It is clear from discussions with students that they do not consciously think of the various functions involved (square root, tangent, etc.) as linear. Errors such as these occur before they encounter linearity in an overt, systematic manner as linear operators in differentiation and integration, linear transformations in linear algebra etc. One of the underlying possibilities is extrapolation of the distributive rule, and Norman & Pritchard (1994) label such examples unequivocally as generalised distributivity, as does Maurer (1987). One does in fact come across students who say

log <u>times</u> x plus y is log <u>times</u> x plus log <u>times</u> y.

One also encounters, in connection with the square root error for example, explanations like

well, when you do something to a + b you get the same as doing it to a and doing it to b. It’s the same as with a times b.

Norman & Pritchard formulate this as $F(a*b) = F(a)*F(b)$, where $*$ is some binary operation (extrapolated from situations such as $\sqrt{a\times b} = \sqrt{a}\times\sqrt{b}$, where it *is* true). In practice the situation is more complicated than just the single category of *distributivity* would imply. We find the use of the generalised rule $f(a*b) = f(a)\circ f(b)$, with different binary operations on each side, for example $\ln(a+b) = \ln(a)\times\ln(b)$, which is an erroneous extrapolation from the two (correct) relationships $\ln(a\times b) = \ln a + \ln b$ and $\exp(a+b) = \exp(a)\times\exp(b)$. The focus of attention seems to be the *a* and the *b*. These seem to be the primary objects, with the binary operations not being regarded as objects to the same extent. What is clear is that the binary operation plays a lesser role than the algebraic variable. With many of these errors students will spontaneously correct them when challenged. They often put it down to memory,

Oh! I never remember whether it’s plus or times,

rather than the structural considerations above. This is not surprising, because they do not have the language to describe these things in structural terms.

As well as the rule $F(a*b) = F(a)*F(b)$ being applied when *F* is a real function, it is also applied when *F* is an operator such as differentiation. A typical example is the assertion that the derivative of a product is the product of derivatives.

Example 6

$$\frac{d}{dx}\ln\left(\frac{\sin x}{x}\right) = \frac{x}{\sin x}\left(x^{-2}\sin x\right)\left(x^{-1}\cos x\right)$$

Explanation:

Well because it's ln you have one over what's in the brackets. Then you have to differentiate the bracket, so it's easier to write it as $x^{-2}\sin x$. *So you have to differentiate the first and the second and multiply by the one over bit.*

Comment:

This student performed well on basic differentiation exercises using the product rule, and did not use pseudolinearity. However in previous exercises on the chain rule the "inner function" did not involve products or quotients, but simple polynomials or trigonometric or exponential functions. The expectation was therefore established that the answer would always be in the form of a product of expressions. In extrapolating this expectation we see that pseudolinearity comes to the surface again. The earlier exercises on differentiating products (and quotients) had not eradicated this deep-seated structural misconception.

It is sometimes unclear whether examples in this category are structural (e.g. application of linearity), or arbitrary (randomly mis-placing of the binary operations), in the sense of Orton and Donaldson. This would benefit from further research.

4. Equation balancing

How often do we emphasise in elementary algebra the principle "you do the same thing to both sides of an equation and they are still equal"? (Pimm, 1987, p.20)

Well here are some situations where the students' explanations involve this principle. What may be significant is that on many occasions in their comments the students replace the phrase "*to* both sides" with "*on* both sides".

Example 7:

$$\int \frac{1}{(1+u)^3}\,du = \ln|1+u|^3; \quad \int \frac{1}{x^2+4x+7}\,dx = \ln\left|x^2+4x+7\right|$$

Explanations:

$\int \frac{1}{(1+u)}\,du = \ln|1+u|$, *and you cube on both sides.*

$\int \frac{1}{x}\,dx = \ln|x|$, *only it's* x^2+4x+7 *on both sides instead.*

Comment: The students are not applying a general rule of the form $\int \frac{1}{f} = \ln|f|$, for if they are given an example where f is a trigonometric function they do not respond in this way. One does not find errors like $\int \cos\left(\frac{1}{x}\right)dx = \cos(\ln x)$, or $\int \frac{1}{\cos x}dx = \ln(\cos x)$ with anything like the frequency with which this type of error appears when simple polynomials are involved as in these two examples. So there are some limits to the extent to which procedural extrapolation occurs. (It is tempting to talk about a "Zone of Proximal *Extrapolation*", *à la* Vygotsky.)

Finally we have an example from the examination paper on the course from which all the errors in this article come.

Example 8: Find the Maclaurin expansion of $f(x) = \frac{1}{(1+2x)^{\frac{1}{4}}}$, by any method.

Solution:

$$\frac{1}{1+2x} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots,\ \text{and so}$$
$$\frac{1}{(1+2x)^{\frac{1}{4}}} = 1 + 2x^{\frac{1}{4}} + 4x^{\frac{2}{4}} + 8x^{\frac{3}{4}} + \ldots$$

Comment: In this case there wasn't a verbal explanation because it was an examination question, but the results of the application of the principles of equation balancing and pseudo-linearity can be clearly seen.

5. Conclusions

Many studies concerning students' mistakes analyse elementary mathematics (Brown and Burton (1978), Van Lehn (1980), Hart (1981), Maurer (1987)) or basic concepts of more advanced topics - Orton (1983).

In this study we have discussed mistakes relating to algorithmic processes in one variable calculus, lying beyond the basic principles. The study demonstrates that mistakes occurring here reflect structural errors, which Donaldson (1963) found in elementary mathematics. These involve confusion between object, action and process (Thompson (1994)), mis-application of language (Pimm (1987)), generalisation (Krutetskii (1980), Maurer (1987)), confusion between syntax and semantics (Norman & Pritchard (1994)), and inadequate metacognitive control procedures (Schoenfeld (1985)). This provides evidence that the types of error present in elementary mathematics continue into more advanced mathematics. This confirms the suggestions of Maurer (1987) and Norman & Pritchard (1994) that such structural errors cannot be avoided. In teaching mathematics we emphasise qualities such as flexibility, reversibility, generalisation and intuition, and so paradoxically it seemes that these very qualities can give rise to structural errors. From a constructivist viewpoint they will happen in the course of learners constructing their own meanings.

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