THE DIFFICULTIES AND REASONING OF UNDERGRADUATE MATHEMATICS STUDENTS IN THE IDENTIFICATION OF FUNCTIONS

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ABSTRACT

In this paper we investigated the difficulty levels of the identification of functions in different representations of mathematical relations. The relative difficulties associated with functions and developmental levels were examined through a written test administered to 38 first year undergraduate students. The results appear to support the assumption that there is a developmental pattern in students' thinking in identifying functions from their symbolic and graphical forms.

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1. Introduction

Representational systems are the keys for conceptual learning and determine, to a significant extent, what is learnt. The ability to identify and represent the same concept in different representations allows students to see rich relationships, and develop deeper understanding (Even, 1998). The difficulty of representing different topics in mathematics has been studied extensively. Some researchers interpret students' errors as either a product of a deficient handling of representations or a lack of coordination between representations (Greeno & Hall, 1997). A common conclusion in most of these studies is that students have deficient understandings in relation to the models and language needed to represent or illustrate and manipulate mathematical concepts (Tall, 1991).

Several researchers in the last two decades address the importance of representations in understanding mathematical concepts (Aspinwall, Shaw & Presmeg, 1997)). However, not enough attention has been given to the reasoning and difficulties of students in representing mathematical concepts at the university level. The primary goal of the present study is to explore students' understanding and reasoning of the concept of function through its multiple representations.

2. Theoretical Background and Literature Review

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community over the past decade (Dubinski & Harel, 1992). The understanding of functions does not appear to be easy, given the diversity of representations associated with this concept (Hitt, 1998). Aspinwall, Shaw and Presmeg (1997) asserted that in many cases the graphical (visual) representations can cause cognitive difficulties, because the perceptual analysis and synthesis of mathematical information presented implicitly in a diagram often makes greater demands on a student that any other aspect of a problem.

The standard representational forms of some mathematical concepts, such as the concept of function, are not enough for students to construct the whole meaning and grasp the whole range of relevant applications. Mathematics instructors, at the secondary level, have traditionally focused their instruction on the use of algebraic representations of functions. Most instructional practices limit the representations of functions to the translation of the algebraic form of a function to its graphic form. Vinner (1992) stated that a function, as taught at schools, is often identified with just one of its representations, either the symbolic or the graphical - the former can result in interpreting function as "formula". Sfard (1992), on the other hand, found that students are unable to bridge the algebraic and graphical representations of functions. Similarly, Norman (1992) found that even secondary school teachers pursuing their masters' degrees in mathematics tended to call up one particular representation of a function, often a graph. In general, they did not take into account verbal and intuitive representations. Furthermore, most teaching approaches do not take into consideration the movement from one type of representation to another, which is a complex process and relates to the generalization of the concept at hand (Yerushalmy, 1997).

Although there are a lot of studies dealing with students' conceptions of functions and their difficulties in coming up with the function concept (Tall, 1991), there remain issues to be examined in relation to the representations of functions and the connections between these representations (algebraic, graphical, verbal, tabular, etc.). This study purports to contribute to the ongoing research on representations in functions by identifying the levels of difficulty of fundamental modes of function representations. The literature does not provide the kind of coherent picture of undergraduate students' representational thinking in mathematical functions that is desirable for the improvement of current approaches to instruction. In this paper, we seek to define the difficulty level and the developmental trend of translations in the representations of a mathematical relationship. To this end, we used the SOLO taxonomy (Biggs & Collis, 1991). The SOLO taxonomy provides a systematic way of describing a hierarchy of complexity, which learners exhibit in the mastery of academic work.

SOLO describes five levels of sophistication, which can be found in learners' responses to academic tasks: Prestructural – the task is not addressed appropriately, the student hasn't understood the point; Unistructural – one or a few aspects of the task are picked up and used (understanding as nominal); Multi-structural – several aspects of the task are learned but are treated separately (understanding as knowing about); Relational – the components are integrated into a coherent whole, with each part contributing to the overall meaning (understanding as appreciating relationships); Extended abstract – the integrated whole at the relational level is reconceptualized at a higher level of abstraction, which enables generalization to a new topic or area, or is turned reflexively on oneself (understanding as transfer and as involving metacognition) (Biggs & Collis, 1991).

3. The Goals of the Present Study

One of the main objectives of this study is to define the reasoning and the difficulties experienced by students in identifying the concept of function through its symbolic and graphical representations. This study is motivated by practical concerns and theoretical needs. The practical concerns focus on the difficulties experienced by students in grasping the concept of functions. By taking into account different systems of representations, we can identify specific variables related to cognitive contents, and, in this way, organize didactical approaches to promote the students' articulation of different representations in a meaningful manner. The theoretical needs come from the lack of a systematic theoretical framework of representations capable of supporting the kinds of understandings, which are necessary for university students to identify and use the concept of functions. Both practical and theoretical concerns are interwoven in understanding the relations between the multiple representations of functions.

Specifically, the purpose of the study was twofold:

(a) To define the level of difficulty in identifying the concept of function through its graphical and symbolic representations, and

(b) To trace the developmental trend (if any) in the student's ability to identify mathematical functions in different modes of representation.

4. Method

Participants

The participants in this study were all first-year students in the department of mathematics at the University of Cyprus (N=38). These students were attending a freshman calculus course. There were 13 male and 25 female students, who graduated from lyceums where the emphasis was on mathematics and physics and succeeded in the university entrance examinations. They attended a one-year calculus course during their final year at the lyceum and graduated with very high marks in mathematics.

Instrumentation

The instrument used in this study to collect information of students' understanding of function representations was a questionnaire, which consisted of two parts involving 20 tasks in total. The first part included 9 relations and the students were asked to indicate whether or not the relations could describe one or more functions (see Table 1). The second part involved 11 graphs and students were asked to decide which of these graphs resulted from functions of the form y=f(x) (see Table 2). In both parts students were asked to justify their answers by writing their explanations.

5. Results

The Difficulty Level and the Developmental Trend

In order to search for a possible developmental trend and difficulty levels in the identification of functions among freshmen, we analyzed the data using latent class analysis. Tables 1 and 2 summarize the "difficulty level" of each of the tasks of symbolic and graphical representations of functions, respectively.

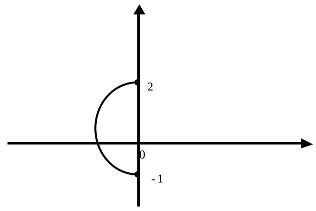
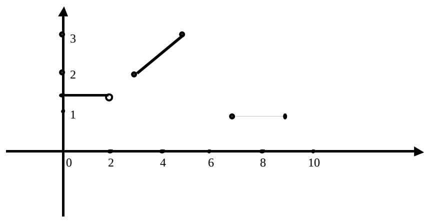
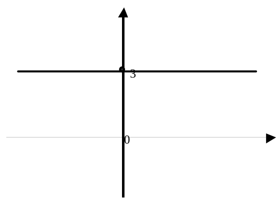
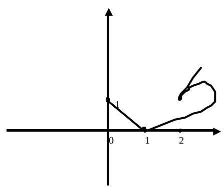
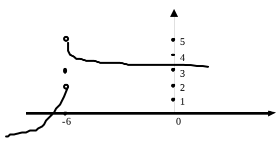
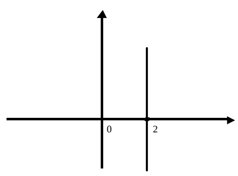
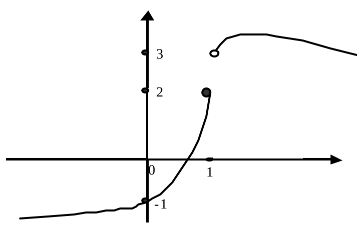
Table 1: The Difficulty Level of the Functions Represented by Symbolic Forms

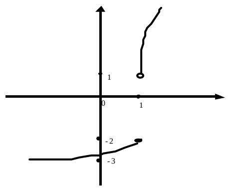
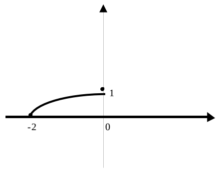
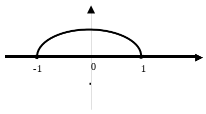
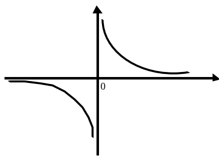
Situation	Relations	Mean	Std. Deviation
a*	$x^2+y^2=3$	0.2105	0.4132
b	$y=\int_0^1 \sqrt{x^3+x+1}dx$	0.2895	0.4596
c	$a^2-b=0$	0.5000	0.5067
d	$f(y)=e^y$	0.6053	0.4954
e	$x^4=3y$	0.6842	0.4711
f	$a=\sqrt{2}$	0.8158	0.3929
g	$f(x)=3$	0.9211	0.2733
h	$y=x^2$	0.9211	0.2733
i	$s=3t$	0.9474	0.2263

* For each situation, the subjects were asked to indicate whether the symbolic representation corresponded or not to a function.

Table 1 shows that situations s=3t, $y=x^2$, and f(x)=3 were the easiest symbolic functions identified by freshmen ($\overline{X}_{(s=3t)}$ =0.95, SD=0.23; $\overline{X}_{(y=x^2)}$=0.92, SD=0.27; $\overline{X}_{(f(x)=3)}$ =0.92, SD =0.27), while situations a and b were the hardest for students to determine whether the relation was a function or not ($\overline{X}_a$=0.21, SD=0.41; $\overline{X}_b$= 0.29, SD=0.46). Situation c was correctly answered by half of the students, while the situation h, which is equivalent to c, was correctly identified as a function by more than 92% of the students.

Table2: The Difficulty Level of the Functions Represented by Graphical Forms

Situation	Graphs presented to students	Mean	Std. Deviation
A*		.2368	,4309
B		0.4474	0.5039
C		0.5000	0.5067
D		0.6842	0.4711
E		0.7105	0.4596
F		0.7632	0.4309
G		0.7895	0.4132

H		0.8421	0.3695
I		0.8684	0.3426
K		0.8947	0.3110
L		0.9211	0.2733

* For each situation, the subjects were asked to indicate whether the graphical representation corresponded or not to a function.

Table 2 shows the difficulty level of the tasks given in graphical forms. The graph depicted in situation L was correctly identified as a function by 92% of the students ($\overline{X}$ =0.92, SD=0.27). Situation A was the hardest task for students since only 24% of them answered it correctly. Situations B and C were also difficult for students, while situations I, and K were answered correctly by the great majority of the students (87%, and 89%, respectively).

Multivariate analysis of data showed that there were statistically significant differences among the situations in symbolic and graphical forms. Students identified functions from symbolic representations more easily than functions from graphical representations, confirming, to an extent, Vinner's (1992) results. The presence of a consistent trend in the difficulty level across translations seems to support the assumption for the existence of a specific developmental pattern. Thus, on the basis of the respective frequency quartiles, the students were ranked to success; four classes were defined: low achievers--Class 1 (n=9), below average achievers --Class 2 (n=9), above average achievers --Class 3 (n=11), and high achievers --Class 4 (n=9).

Table 3 shows the tasks successfully performed by more than 50% of the students in each class. The data included in Table 3 indicate that there is a developmental trend in students' abilities to complete the assigned tasks because success on any translation by more than 50% of the students in each class was associated with such success by more than 50% of the students in subsequent classes.

Table 3: The Developmental Trend of Students' Abilities to Identify Functions

	Class 1	Class 2	Class 3	Class 4
Level 1	*i(89%), h(89%), g(89%), f(78%), L(78%), K(89%), I(89%), H(66%), G(89%), F(66%), E(78%)	i(100%), h(100%), g(89%), f(78%), L(89%), K(89%), I(100%), H(78%), G(78%), F(78%), E(78%)	i(100%), h(100%), g(90%), f(90%), L(100%), K(89%), I(89%), H(90%), G(78%), F(78%), E(78%)	i(100%), h(100%), g(100%), f(90%), L(100%), K(100%), I(89%), H(100%), G(89%), F(78%), E(89%)
Level 2			d(72%), e(72%) D(82%)	d(77%), e(90%) D(78%)
Level 3				c(77%), b(66%) C(56%), B(56%), A(52%)

* The small and capital letters refer to situations shown in Table 1 and 2 respectively. The numbers in parentheses indicate the percentages of students' successful answers in each situation.

Cognitive Developmental Levels

The findings seem to support the hypothesis that there are at least three cognitive developmental levels, which characterize students' thinking in the identification and discrimination among symbolic and graphical representations of functions. Class 1 and Class 2 students seem to successfully perform the same tasks; however, students in Class 2 responded with greater facility as shown by the percentages of successful answers shown in Table 3. The fact that students were unable to successfully perform a higher level task unless they could perform tasks of the preceding level seems to provide compelling evidence that the levels, as identified, may generate a hierarchy of thinking. We claim that the three levels of thinking used in identifying functions from their symbolic or graphical representations correspond to the three of the five levels of cognitive thinking identified by Biggs and Collis (1991), ie., the unistructural, multistructural, and relational levels. In what follows, the hypothetical levels and the major characteristics of each developmental level are described in relation to Biggs and Collis' thinking levels. To this end, we used students` written explanations, which were provided during the completion of the questionnaire.

Level 1: At this level, students identify some kinds of function representations but are then distracted or misled by an irrelevant aspect. Thus, students attempted to identify mathematical functions from a given symbolic or graphical form but their approaches were not always systematic. Students recognized functions from symbolic relations only if the relations were expressed in terms of the dependent variable as in situations a, f and h. Students also identified the symbolic representations of functions when the relations included symbols that are commonly used in their textbooks or during instruction. For instance, students at this level identified functions when x and t were used to denote the independent variables, and y and s are used for the dependent variables. However, level 1 students did not always provide correct answers when the above symbols had a different role in the relations as shown in case e ($x^4=3y$), where the relation was solved in terms of the independent variable.

Students at level 1 identified functions from graphs when the graphs depicted functions with interval domain or the union of successive intervals as in situations H and L. Situation F is the only graph where it was correctly recognized by students that it did not represent a function, because it depicted an extreme situation where x =2 corresponds to real numbers. In most cases, students were not able to reach a final decision or to provide a consistent answer. For example, although the tasks in situations h and c (see Table 1) were equivalent, very few of the students at this level performed successfully both of these tasks, probably because they were distracted by the context of the relationship or the symbols involved. In the same way, students' responses in identifying functions from graphs were inconsistent (see Table 2).

Level 1 appeared to be a period of transition that is characterized by the students' naïve and often inflexible attempts to identify functions from their symbolic and graphical forms. Their thinking was more indicative of what Biggs and Collis (1991) termed as the unistructural level in the sense that one aspect of the function concept is usually pursued. For example, many of the students incorrectly identified situation D (see Table 2) as a function, focusing their attention on the left side of the graph and ignoring the right part, which probably confused them. The unistructural nature of students' thinking at this level was also exemplified by their responses to situation b. Most of them identified it as a function but they could not recognize that it was a constant function and thus students proceeded to define the domain as $(-\infty, +\infty)$.

Level 2: In contrast to level 1, students exhibiting level 2 thinking, when faced with representational situations of functions, demonstrated a readiness to recognize and discriminate symbolic and graphical functions in a consistent way. The characteristic of this level is that students improved their ability to identify the functions involving the symbolic and graphical modes with the exception of the functions in situations c, b, C, B, and A (see Table 3). Students at this level recognized more than one relevant feature of function representations and attempted to explain their reasoning in a way that integrates their knowledge about the concept. Students at this level identified functions even in the cases where the symbols played a different role in the relations as in situation d or the relations were solved in terms of the dependent or the independent variable as in situation e. Level 2 students identified not only the graphs that level 1 students did but they also identified that "strange" graphs such as situation D did not represent a function.

Students assessed at Level 2 appeared to exhibit characteristics of the multistructural level within the symbolic and graphical forms (Biggs & Collis, 1991). The following extracts from students' written answers indicate how students' reasoning at levels 1 and 2 differed with respect to the justifications they provided for their responses. Level 1 students (unistructural level) who thought the equation $x^2+y^2=3$ could be described by one or more functions gave the following reasons for their responses, suggesting that they had focused on one aspect of the problem: "This is a circle with radius 3", "It's a function since you can express the equation as $y=\sqrt{3-x^2}$ ". On the other hand, students at level 2 provided answers that suggested that they had concentrated on more than one aspect of the concept of function (multistructural level): "It can describe a function if you restrict domain", "You can solve for y and look at only the + or the – square root. Thus, you will have two different functions", "The circle can be broken into two half circles".

Level 3: Students exhibiting Level 3 thinking made precise connections between the graphical and symbolic representations of mathematical functions. This was evidenced by the consistency of students' answers in the identification of functions in the symbolic and graphical forms. The fact that students at this level successfully performed most tasks indicates that their thinking is consistent with the characteristics of the relational level. That is, they integrate the concept of functions with its multiple representations into a meaningful structure and are able to generate abstractions in mathematical relationships (Biggs & Collis, 1991). However, situation a was not correctly answered even by the students at this level, implying that there is another level, the extended abstract level, which was not considered in the present study.

6. Conclusions

Representations enable students to interpret situations and to comprehend the relations embedded in problems. Thus, we consider representations to be extremely important with respect to cognitive processes in developing mathematical concepts. The main contribution of the present study was the identification of hierarchical levels among the graphical and symbolic representations of mathematical functions. An association was verified between the students' ability to identify various representations of the mathematical functions. Specifically, it was found that representations that could be identified as functions by low achievers were identified with greater ease by students in higher achievement classes, whereas the mathematical functions in some situations could only be performed by top students.

The present study is a first attempt to develop a framework for describing and probably predicting first year university students' thinking in the identification of mathematical functions from their symbolic and graphical forms. This framework recognizes developmental levels and is in agreement with neo-Piagetian theories that postulate the existence of sub stages or levels that reflect the structural complexity of students' thinking (Biggs & Collis, 1991). The analysis revealed that students exhibit three developmental levels. Students exhibiting level 1 tend to adopt a narrow perspective in identifying mathematical relationships as functions. They do not provide complete and consistent answers. There is a tendency to overlook the data in the given representations, that is, to focus on one aspect, rather than on the elements of the concept of function in combination. Students who demonstrate level 2 thinking recognize functions by combining more than one aspects of the concept and tend to provide systematic justifications for their reasoning. However, they lack the ability to consistently relate the symbolic and graphical forms of functions, which is the characteristic feature of Level 3.

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