ALGEBRA IN THE TRANSITION FROM HIGH SCHOOL TO UNIVERSITY

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ABSTRACT

In this paper we communicate different aspects relative to the design and implementation of a didactical proposal for the teaching of algebra in a first university course (*Elements of Algebra*) at the Universidad Nacional de la Patagonia Austral (UNPA) for students that require a solid formation in mathematics.

One of authors of this proposal is the teacher in charge of Elements of Algebra at UNPA. The other one develops her work at the University of Buenos Aires. To implement the changes planned in the class and in the kind of tasks the students will be required to solve, we decided it was essential to include a stage of work with the rest of the teachers of *Elements of Algebra.*

For the elaboration of a proposal we have considered various dimensions of analysis:

- A theoretical frame about the teaching and learning of mathematics, taking into account the theorization of G. Brousseau, G. Vergnaud and Y. Chevallard, among others.
- A reflection about the meaning of algebraic symbols in their use as a tool to solve problems, considering for this theoretical elements furnished by A. Arcavi and J. Drouhard, among others.
- A critical analysis of the selection of problems that previously formed part of the practical work, keeping those which could allow a work centered in the construction of the sense of mathematical objects and the particular methods of algebra.
- The knowledge of the characteristics of the population of students to whom this would be directed.

As a result of this work, a **new exercise booklet** was produced for a part of the course. Then, all the subject teachers attended a **workshop to discuss the problems proposed in the booklet** and some questions relative to its implementation. Finally, **changes have been made to the course and some episodes were registered** and analyzed**.**

The purpose of this paper is to explain briefly the four dimensions of analysis considered when elaborating the practical work and to describe and analyze the three instances of work mentioned in the previous paragraph. We will also try to show –from certain aspects of the effective realization- the difficulties that appear when a change is introduced, that requires the reformulation of the personal relationship of each student with the study of mathematics, as well as repositioning students and teachers in their roles in the classroom.

The students, the teachers and the mathematical activity are in the center of our study interests.

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[1] Teacher of Elements of Algebra at the Universidad Nacional de la Patagonia Austral - Province of Santa Cruz, Patagonia, Argentina -with a formation on Didactics of Mathematics: centered on Didactics of Algebra, she formulated and implemented a methodological change in the subject *Elements of Algebra* of the first year for the careers of System's Analyst and Mathematics Teacher. This implementation is the result of a project of pedagogical innovation that was selected in a Pilot Convocation by the Secretary of Superior Education of the Ministry of Culture of the Nation in November, 2000, which allowed this teacher to do an assistantship of two months at the Centro de Formación e Investigación en Enseñanza de las Ciencias of the University of Buenos Aires.

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1. Description of the problem

Characteristics of the population of students and representations of the university teachers

In general, the students that have finished high school have acquired skills in algebraic operations not linked to the situations in which they can be used. At the beginning of the course Elements of Algebra (annual) at UNPA, they find themselves facing a "different kind of mathematics" marked by the presence of a new transversal and fundamental element: proof. This is a rupture in the passage from one level to the other. There is a lack of balance between what the student knows and how what he knows is used, because there are familiar objects, but they do not "function" as they did in High School. For example, all the knowledge acquired about operations with polynomials is not sufficient to develop strategies that allows them to formulate and proof general statements about numbers.

After a short period of time at University the students have the impression that they were not taught anything in High School. And this impression is "somehow confirmed" by the university teachers. In general, the teachers of Elements of Algebra come to the conclusion that the remarkable failure of our students (usually, less than 10% of the students pass) "is due to" a deficient previous formation, which they reduce to "absence of some algorithms and lack of a study habit".

These teachers apparently identify the work in mathematics in University as heavily linked to language and the formal manipulation of the rules of the language to prove. They end up insisting more on the proving procedures than in the sense of the objects and the practices. The work is finally reduced to the acquisition of the rules of treatment of the formal language, showing a rigid and finished mathematical task. The structure of mathematics is lost.

In previous years we observed that some students could repeat a procedure of a demonstration, that is, they recognized a procedure and could apply it in another proof, which did not mean that they understood what they were doing, they were only doing what they were asked to.

Taking this problematic into account we decided to redesign de course of Elements of Algebra.

2. Theoretical Frame

- **Global problem: teaching and learning mathematics**

As we have said, our conception about the teaching and learning of mathematics is nourished by theoretical elements of the Theory of Didactical Situations of G. Brousseau, the Anthropological Theory of Y. Chevallard and the Theory of Conceptual Fields of G. Vergnaud.

To summarize their most important characteristics we will transcribe some paragraphs from the Curricular Design of the city of Buenos Aires[3].

> "There are many ways of knowing a mathematical concept, which depend on everything a person has had a chance to do in relation to that concept. This is a fundamental starting point to reflect on teaching.
>
> The set of practices that a student uses for a mathematical concept will construct the sense of that concept for that student.

[3] P. Sadovsky. Pre- Curricular Design for the General Basic Education. General Framework. Formative Sense of Mathematics in School. Secretary of Education. Government of the city of Buenos Aires. 1999.

We assume a position according to which the process of reconstruction of a mathematical concept begins with the set of intellectual activities that are used for a problem whose solution cannot be found with the knowledge available."

Problem solving must not be the only kind of work done in class. To make the work in class fertile, it is fundamental to include moments of reflection about what has been done, articulation of different strategies, discussions about the economy of certain procedures, confrontations of the perspectives of the students… This has to do with creating a space for debates, for establishing conjectures, to validate them.

- **Local problematic: the teaching and learning of Algebra**

There are many researches about the problem of the teaching and learning of Algebra. We have mainly considered the contributions of researchers Abraham Arcavi (1994/95), Anna Sfard (1991), Josep Gascón (2000), Brigitte Grugeon (1997), J.F. Nicaud (1994), Jean Philippe Drouhard et al. (GECO 1997), Mabel Panizza, Patricia Sadovsky and Carmen Sessa (1995).

We would like to highlight some considerations of J.F. Nicaud (1994) about the treatment of algebraic expressions. He says that the mathematical objects that represent algebraic expressions are partial semantic models, where calculations can be done or transformations performed over formal expressions can be justified, that is, algebraic calculations can be meaningful. He defines three semantic levels:

- **First level** (evaluation level): significance is given to an algebraic expression by replacing values in the variables and doing the corresponding calculations.
- **Second level** (treatment level): an expression is transformed into equivalent ones. It implies the knowledge of the transformations laws and how to justify them. This justification is based on the fact that an expression and its transformed coincide for every evaluation.
- **Third level** (level of resolution of problems): strategies are known that permit the choice of the necessities to solve a particular problem, giving sense to the calculations.

We believe that a freshman student at UNPA has not achieved the third semantic level in the treatment of algebraic expressions, that is, he "does not know" how to organize his activity to arrive at a conclusion.

We could say that the student is a "formal automaton" as described by GECO (1997), that is, a student that, when manipulating algebraic expressions of elementary algebra, does not take into account that by transforming an expression he must obtain another equivalent to it. In this case, the question of the validation of the result is not posed in terms of the equivalence of the writings obtained, but above all in terms of the conformity with rules and proceedings (for example, "what is subtracting passes adding").

It is on these aspects that we will center our proposal.

3. Changes proposed

According to what we have said so far, we have given priority in our proposal to the dimension of algebra as a tool for validation. We will there distinguish various levels:

- Algebra as a tool for generalizing numerical properties.
- Algebra as a tool for calculation, to find a result or to validate assertions.

- Algebra as a model for intra and extra-mathematical situations.

We can identify different dimensions in the changes proposed, even though they are naturally closely related.

i) Changes in the problems of practical work number 2 of the course, which deals with the field of real numbers**.**

In the previous version of this practical work students were asked to prove properties of real numbers based on the axioms. Our experience of many years reveals that this work had a great impact on the students: they could not grasp the logic of the task and felt they had no resources to do the required demonstrations. As a result of this, on one hand, they lost their self-esteem, and on the other, and according to the difficulties they encountered, the students were not sure of the use of this kind of task for their mathematical development. This causes a lack of confidence, which constitutes another ingredient of the atmosphere of the class and it does not contribute to the learning environment we wish to install.

In the reformulation we considered another scheme, pointing to the acquisition of "symbol sense" (Arcavi, 1994).

We will show as an example some of the exercises proposed. (Exercise 3 is inspired in an activity narrated in GECO (1997).

3. $(a + b)^2 =$ …………………

a) Complete the right hand side with an algebraic expression so that the equality will always be true.

b) Complete the right hand side with an algebraic expression so that the equality will always false.

c) Complete the right hand side with an algebraic expression so that the equality will sometimes be true and sometimes false.

Give an example where it is true and another where it is false.

d) Describe the set of solutions of question c).

4. a) Invent two "formulas" that are always true.

b) Invent two "formulas" that are always false.

c) Invent two "formulas" that sometimes are true and sometimes false. For each of the "formulas" that you invented, give examples of values for which they are false and values for which they are true. Describe all the solutions.

These exercises were thought according to various purposes:

- To allow the student a personal work of creating expressions according to different requirements.
- To be able to consider the conditions a), b) and c) as possible for every equality between two algebraic expressions, breaking the dichotomy right/wrong.
- To make the algebraic rules manipulated by students and their solidity explicit to the teacher.
- To allow in a class discussion of the resolutions, a circulation of the different methods used and the knowledge of each student.

We knew that this task would be a challenge for the teachers because unexpected answers would force them to use their own algebraic abilities.

The dynamics of the class during the discussion of these problems was far from the one for the traditional problems of the kind “prove that…”, that were solved showing the “correct proof“.

For these two problems, as in others of the worksheet, we were also trying to recuperate conceptions, concepts and terminology seen in High School. The words “formula”, “algebraic expression”, “identity”, “equation”, “function”, that generally coexist in the body of knowledge of students in an isolated way, are re-captured trying to enrich their senses.

Another example of the proposed changes is the following:

In the practical work students were asked to prove that if the product of two numbers is 0, then at least one of them is 0. This property was evident to the students, but impossible for then to prove. On the other hand, it was not available when, in another exercise, they had a product equated to 0. All this shows us how “useless” this exercise was at this stage. Instead, we proposed the following exercise:

Let a and b be two real numbers.

a) Find all the solutions of a.b = 10. Can you describe them all?

b) Find five solutions of a.b = 3. Can you describe them all?

c) Find all the solutions of a.b = 0. Can you describe them all?

with the idea that only after certain manipulation, that could eventually include the graphic representation of the solution, the teacher would pose the question: “How can we justify that, if a.b = 0 then a = 0 or b = 0”?

We will finally mention exercise 16:

16. Are there two real positive numbers a and b that verify that $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$?

Justify your answer.

This exercise appeared in the previous version of the worksheet, in the last place. Almost no one solved it, since they needed a strong guidance from the teacher to do so, resulting of little value for the cognitive advance of the students. Our hypothesis was that, in the new practical work the problem would be tried by the students, as a result of all the previous work, and so we decided to leave it.

In summary, the exercises that form the new practical work give sense to the mathematical activity of the student, favoring his independence and a posterior reflection that allows a discussion with his peers and teacher about the work being done.

We were worried about the student just trying to remember algorithms instead of using his common sense and creativity.

We were also interested in enriching the field of activities that the student recognizes as relative to the mathematical work, incorporating the following:

- establishing conjectures,
- validating results,
- finding counterexamples to invalidate a possible result,
- determining the domain of validity of a formula,
- analyzing different strategies of solution for one problem,

- explaining his solution to others,
- listening to and debating the solutions of his peers.

ii) Changes in the job of the teacher

As we pointed out in the description of the problem, most of the teachers of this course were centered mainly in the learning of algebraic manipulation of the rules of language to demonstrate. And, even though they failed at it, they blamed this failure to the lack of knowledge of the students.

Due to the exercises we presented and the kind of work we proposed, that always included a moment of reflection and discussion, an important change in the teacher's position was needed.

The activities we showed in the last paragraph of i) were also new to the teacher as activities for a mathematics class.

iii) Changes in the "job" of the students

We have enumerated different activities – inventing formulas, debating, establishing conjectures, explaining to others, listening to and debating other solutions – that imply a strong change with regards to what the students used to do in class – struggle alone to solve an exercise and then listening to the solution given by the teacher-.

Besides this, we asked them to work in groups to give them the chance of a first and more "private" moment of discussion, without the intervention of the teacher. We programmed that a possible teacher's intervention in the groups did not have to be an "evaluation" of what the students were producing, but one that would help them deepen their work and contribute to justify what they were doing. (This point would be, without any doubts, a big rupture for the teacher with respect to his traditional role).

At last, we planned that the groups had to present (in writing) the solutions to some problems that had not been discussed in class. This was to make them pay attention to the written formulation and it also gave information to the teacher about the advances of the group work.

The teacher corrected the exercises, making a mark in case of a mistake or something imprecise, without saying what the mistake was, without saying "the right answer".

The corrected exercises return to the group to be re-written and only when the students could not solve the problem, the teacher would intervene.

The periodicity of the assignments would also let the students and the teacher have an idea of the *evolution* of the work.

4. Difficulties and achievements in the implementation

To carry out the design previously reported, we had many meetings with the teachers of the course: we presented and analyzed the new proposal for practical work number 2 before it was used with the students.

Even though the teachers were asking for a change –according to the high failure rate- these changes we proposed were institutionally too far from what is considered as a "university mathematics course". At the same time, they alternated between trying to understand the object of each exercise and the type of class dynamics we proposed, and its rejection for considering it more appropriate for "High School".

As the new work was implemented, we had meetings to discuss what had happened and to plan the work in class.

Even though there were important changes in the kind of work of the students, the teachers found it difficult to manage the moments of collective recuperation of the personal work and to take advantage of the different solutions that would arise.

For example, teachers were very uneasy about exercise 3 because they could not anticipate what the students would do. In a class, a student suggested the following to complete a “false equality”

$$(a + b)^2 = (a + b + 1)^2$$

The teacher said in one of our meetings “I screwed up and I accepted it as correct”. This teacher had no problem the following class to go back to this exercise and find, with the students, the set of solutions of the equation, but his words revealed that his new job was less “secure and comfortable” than the job he was used to do.

From the point of view of the student’s work, we can say that it improved significantly. As we anticipated, they tried to solve exercise 16 and obtained different solutions that allowed a fruitful debate. This was a confirmation that the problems could somehow work in interaction with the “knowledge” of the students. This “knowledge” does not only include objects and procedures but also topics related to the kind of practices developed before. Our students showed that the work they had done up to that point “backed them up” to try exercise 16 without problems.

5. Final comments and future perspectives

It was clear to us that the greatest difficulties in the implementation were on the side of the teaching team.

The mathematical formation of the teachers is an important variable to take into account, because what is understood as mathematical activity is conditioning for what is considered that teaching mathematics is, and algebra in particular.

But what seems more important is that the institutional requirements do not prevent the teacher from listening to the students and work *from* their knowledge. Much more work has to be done to obtain this.

Teaching to prove with an increasing level of formality is not an exclusive task of the course Elements of Algebra: it is a long process that needs coordinated teaching actions (in periods that are longer than a course).

The starting point of this process is given by the state of knowledge of the students. In this sense, the criticisms we had received saying that the kind of work we proposed was more proper for High School is not pertinent because this type of work is actually absent from Argentinean High Schools.

Without any doubts, part of the work that we propose in the course of Elements of Algebra could be taught in High School[4] and when this happens, we will have to think in another kind of practice for the first year of University.

As for now, we think that the proposal made is realistic in its objective of improving the quality of the mathematical work of the students and of helping them in their start at University.

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