

A GRAPHICAL EXPLORATION OF THE CONCEPTS OF EIGENVALUE AND EIGENVECTORS IN \mathbb{R}^2 AND \mathbb{R}^3

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ABSTRACT

We study a graphical approach to the concepts of eigenvalue and eigenvector of a square matrix. This approach is based on an interactive computational environment created with *Cabri Geometry II*. The environment allows a simultaneous display of: a) a square matrix A of size 2×2 or 3×3 , b) an arbitrary vector \mathbf{v} in \mathbb{R}^2 or \mathbb{R}^3 and its image under the linear transformation defined by A and c) the graph of the characteristic polynomial $P(\lambda)$ associated with A . The entries A as well as the coordinates of vector \mathbf{v} can be directly (this is, on-screen) manipulated, so providing a method for a graphical analysis of eigenvalues and eigenvectors of a given matrix. This exploration is guided by the graph of $P(\lambda)$

Key Words: Teaching with Technology, Eigenvalues, Eigenvectors

1. Introduction

The concepts of eigenvalues and eigenvectors constitute an important topic for a first year course in Linear Algebra and it is, at the same time, a source of difficulties for students. Some of these difficulties could be related with the diversity of mathematical objects in which these notions rest and in the almost exclusive use of algebraic symbols in the process of teaching.

In this work we present a computational environment that uses graphical and numerical representations of a dynamic nature, designed with the purpose of facilitating the conversion among different representations of the same mathematical object. In the context of the theory of R. Duval [1] on Semiotic Representation of Registers, this conversion is a cognitive activity, needed to achieve a conceptual apprehension of the mathematical concepts.

The idea to create a computational environment based on dynamical representations has been taken from Sierpiska [2], but the representations here used are different, because the purpose of this work is considerably more modest regarding the level of abstraction with which the concepts are discussed.

We present the activities designed with Cabri Geometry II [3] to explore the concepts of eigenvalue and eigenvector of a square matrix and we describe the instructions to construct the files that are used. In the elaboration of these files we have tried to reduce to the minimum level the requisites on software needed to interact with them.

2. Definitions and Calculations

Even though the notions of eigenvalue and eigenvector can be defined in a more general way, in a first course in Linear Algebra the definitions can be presented as follows: If T is a linear transformation from \mathbb{R}^n to \mathbb{R}^n , defined by $T(\mathbf{v})=A\mathbf{v}$, where A is a matrix of size $n \times n$, then it is said that a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq \mathbf{0}$, is a *eigenvector* of T if there exists a real number λ such that $T(\mathbf{v})= \lambda\mathbf{v}$. The number λ is called an *eigenvalue* of T . As it is usual, in this paper we will refer to λ as the eigenvalue of matrix A and to \mathbf{v} as its corresponding eigenvector.

From these definitions it is possible to go directly to the calculation of eigenvalues and eigenvectors of the linear transformation. But the procedure of algebraic calculations is not simple at all. The diversity of mathematical objects and algorithms involved sometimes makes it difficult for the student their reproduction, and this induces to memorization. Many of the errors made by students when performing the calculations could be explained in terms of the reduced attention that the teaching activity dedicates to the comprehension of the significance of the concepts, symbols and operations that are involved in these calculations.

Figure 1 shows, in a schematic form, the concepts and operations that are involved in the calculation of eigenvalues and eigenvectors. (See [4])

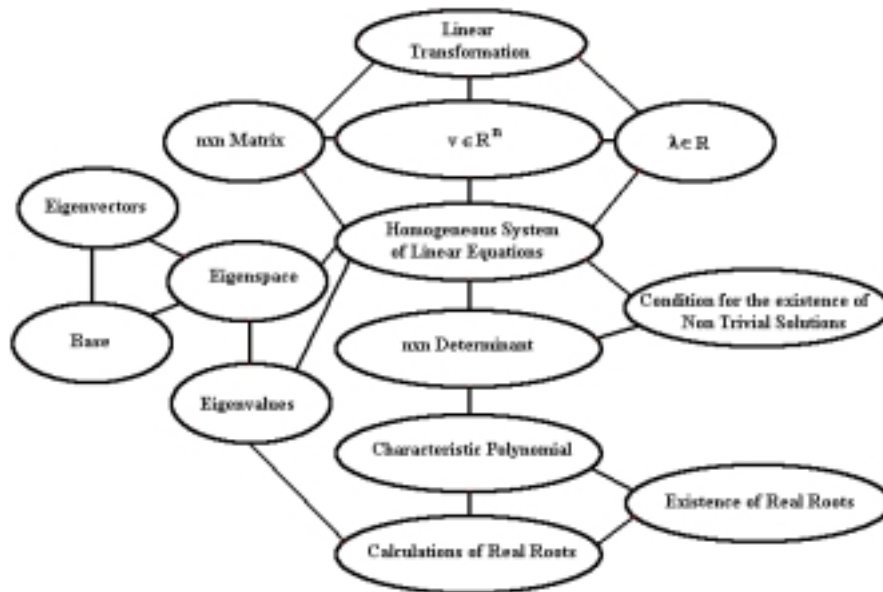


Figure 1

This procedure can be summarized as follows:

1. Set the vector equation $A\mathbf{v} = \lambda\mathbf{v}$, or, equivalently, $(A - \lambda I)\mathbf{v} = \mathbf{0}$, where I denotes the identity matrix of size $n \times n$;
2. Write the vector equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ as an homogeneous system of linear equations;
3. Calculate and set equal to zero the determinant of the matrix $A - \lambda I$ to obtain the characteristic polynomial, $P(\lambda)$;
4. Calculate, or find approximated solutions to, the real solutions of $P(\lambda)$, in case they exist. Each real root of $P(\lambda)$ is a real eigenvalue;
5. Find the set of non trivial solutions (subspace S_i) for the homogeneous system of equations obtained in Step 2, for each real eigenvalue λ_i ;
6. Determine a base for each subspace S_i . The vectors that constitute a base for S_i are the eigenvectors corresponding to λ_i .

3. The Context of the Teaching Activities

The teaching approach described in this work is a part of a more general project, which pursues a reformulation of the teaching of the Linear Algebra course that is offered at the Universidad de Sonora, to science and engineering student. This reformulation combines the work in the classroom with the activities in a computational environment created with Cabri. The design of activities with Cabri is oriented by the two following general principles:

1. The computational environment shall allow the interaction of the student with the representations provided by the computer to the extent of permitting the student to perform modifications, with the objectives of detecting behavioral patterns and of formulating conjectures on the represented objects and their characteristics;
2. A first graphical approach to the mathematical concepts can be useful to create a more concrete base of significance, before examining these concepts in a more abstract level, and the manipulation

performed by the student on graphical-dynamical representations can help in the construction in this base of signification.

In the course on Linear Algebra, the immediate background to the topic discussed in this article is *linear transformations*, which more important properties are also explored with a computational environment similar to the one described here.

4. The Computational Environment

The environment that has been created with Cabri Geometry II allows the exploration of the concepts of eigenvalue and eigenvector for square matrices of sizes 2×2 and 3×3 and works with three simultaneous on-screen representations, namely:

1. The graphical representation of \mathbf{v} and of $T(\mathbf{v})$, where \mathbf{v} is a vector that can be directly manipulated, which, in turn, modifies vector $T(\mathbf{v})$;
2. The matrix A , whose entries can be varied, so modifying vector $T(\mathbf{v})$ and the characteristic polynomial $P(\lambda)$;
3. The graphical representation of $P(\lambda)$, which can be manipulated, not directly, but through the entries of matrix A .

These representations allow the student to perform explorations at two levels:

1. A vector \mathbf{v} will be an eigenvector of A when \mathbf{v} and $T(\mathbf{v})$ are collinear. Then the objective consists in “dragging” vector \mathbf{v} until this happens. Once such a vector \mathbf{v} has been found, the student can calculate the magnitude (i.e., the absolute value) of the corresponding eigenvalue by dividing $\|T(\mathbf{v})\|$ by $\|\mathbf{v}\|$. Additionally, he/she can move \mathbf{v} without changing its direction to verify that this motion does not alter the calculated eigenvalue. At this level the graph of the characteristic polynomial remains fixed and its real roots coincide with the found eigenvalues.

2. By varying the entries of matrix A the student can look for eigenvalues and eigenvectors of other matrices. This allows the exploration of the behavior of eigenvalues and eigenvectors for some interesting matrices, as are, for example, diagonal, symmetric, triangular, singular, etc.

5. The Teaching Activities

The main activities performed by students at first level are related with the search of eigenvalues and eigenvectors. In the case of square matrices of size 2×2 the student has to rotate vector \mathbf{v} , around the origin, in search of a vector \mathbf{v}_1 that is collinear with $T(\mathbf{v}_1)$. Once vector \mathbf{v}_1 has been found the student is asked to drag it, keeping its direction unchanged, to conclude that, in the direction of \mathbf{v}_1 , there are infinitely many vectors that are collinear with $T(\mathbf{v}_1)$. All of these vectors are multiples of \mathbf{v}_1 and, hence, it makes sense to choose one of them, \mathbf{v}_1 say, as a base for the subspace to which they belong.

Since \mathbf{v} and $T(\mathbf{v})$ are collinear for every vector in the direction \mathbf{v}_1 , then for every \mathbf{v}_1 there is a real number λ such that $T(\mathbf{v}_1) = \lambda \mathbf{v}_1$. The student can now take any of these vectors and with the “Distance and Length” tool from the Cabri menu can calculate the norms of \mathbf{v}_1 and $T(\mathbf{v}_1)$. Number λ is then the quotient between the norm of $T(\mathbf{v}_1)$ and the norm of \mathbf{v}_1 . This quotient has to be taken with positive sign

if \mathbf{v}_1 and $T(\mathbf{v}_1)$ have the same direction and with negative sign if they have opposite directions. Figure 2 shows the eigenvector $\mathbf{v}_1=c(1,1)$, $c \neq 0$ corresponding to the eigenvalue $\lambda=2$ of matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

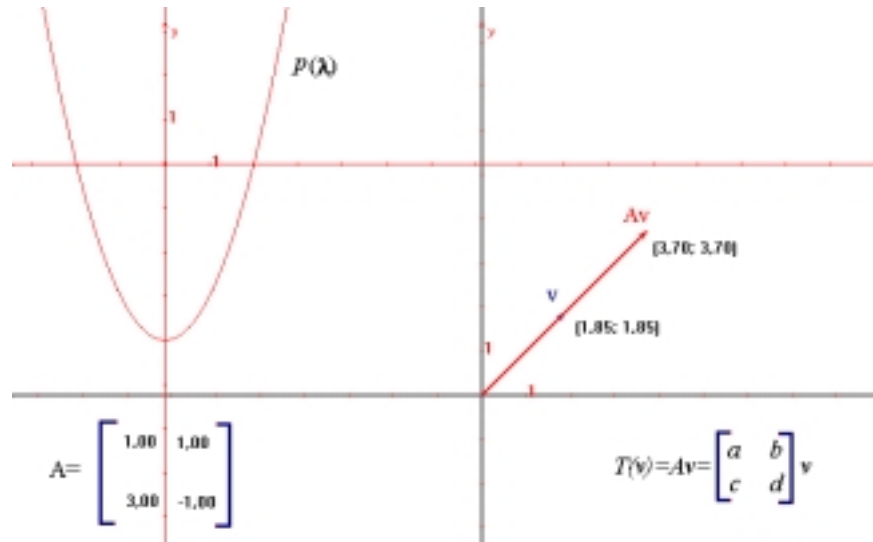


Figure 2. An approximated eigenvector in \mathbb{R}^2

The calculations can be repeated for several of the vectors found in the direction of \mathbf{v}_1 in order to conclude that the number λ is the same for all such vectors.

The construction in \mathbb{R}^3 is manipulated in a slightly different way as compared with the one in \mathbb{R}^2 . These differences are related with the problem of representing a three-dimensional vector on a two dimensional screen. In \mathbb{R}^2 the graphical search for collinearity between vectors \mathbf{v} and $T(\mathbf{v})$ is exhaustive, since in rotating vector \mathbf{v} around the origin, the whole plane is “swept”. In \mathbb{R}^3 , however, the search is not that simple. According to the design of the construction, students have to combine two ways of moving vector \mathbf{v} in order to detect collinearity: one of them on the point P and the other on the point \mathbf{v} . Point \mathbf{v} is the end point of the vector and point P is its orthogonal projection on XY plane. See Figure 3 for an approximated eigenvector of

$A = \begin{bmatrix} -1.43 & 1.51 & 0.55 \\ 2.14 & -1.00 & 1.12 \\ 1.11 & 1.49 & -1.09 \end{bmatrix}$

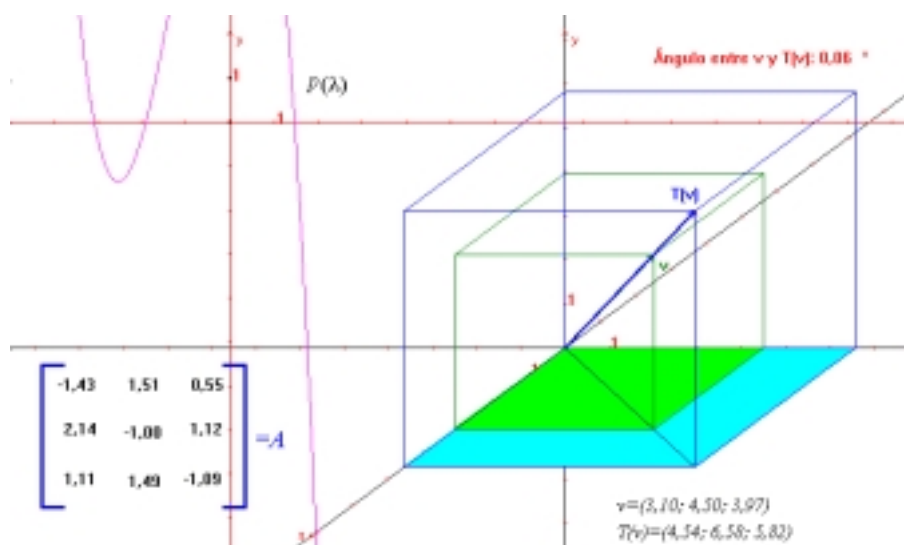


Figure 3. An approximated eigenvector in \mathbb{R}^3

Since vectors \mathbf{v} and $T(\mathbf{v})$ in \mathbb{R}^3 may seem to be collinear without actually being, the environment shows on-screen the measure of the angle θ between \mathbf{v} and $T(\mathbf{v})$ as a guide that orients the search. With this resource, the aim is to find a vector \mathbf{v} such that the angle θ is either 0° or 180° . Given that Cabri does not provide a direct way for calculating the norm of a vector in \mathbb{R}^3 , the environment also includes the coordinates of \mathbf{v} and $T(\mathbf{v})$ in order to facilitate the calculation of $\|\mathbf{v}\|$ and $\|T(\mathbf{v})\|$. With these quantities, the student can obtain the absolute value of the corresponding eigenvalue.

The exploration activities so far performed by students belong to what we have called Level 1 Explorations. Once familiarized with the calculations of eigenvalues and eigenvectors in this environment the student is asked to answer some questions, the answer of which requires changing the level of exploration activities. These questions are of the following type:

- If the characteristic polynomial of a matrix A has no real roots, how many eigenvectors can A have?
- What is the relation among the eigenvectors of a symmetric matrix?
- How many eigenvalues does a singular matrix have?
- How many eigenvectors does a singular matrix have?
- If a matrix A is diagonal, how are the entries of A related to its eigenvalues?
- If the characteristic polynomial of a matrix A has a real root of multiplicity two, how many eigenvectors can A have?
- If the diagonal entries of a diagonal matrix A are all the same, how many eigenvectors does A have?

In order to answer these questions has to try several matrix entries and pay attention to the graphical behavior of $P(\lambda)$. All the questions involve activities belonging to Level 2 and they have turned out to be more difficult to answer as compared with those formulated at Level 1.

6. Conclusions

Students have been able to successfully perform the activities proposed in this environment and they have shown interest in the topic. The different answers given by students have generated a fruitful discussion about the meaning that the topics under study acquire in this environment. However, the supervision of the professor has turned out to be very important to give the activity the proper orientation, particularly on those areas where difficulties have been detected.

Some of these difficulties are the following:

1. Students have had problems to identify negative eigenvalues, partly because in this case the collinearity of \mathbf{v} and $T(\mathbf{v})$ is not always clear in the environment and partly because sometimes they find it difficult to identify the effect of multiplying a vector times a negative number.
2. They have found it difficult to explain what happens with eigenvalues and eigenvectors for singular matrices and they have requested for help in this case to identify on-screen the eigenvector that corresponds to a zero eigenvalue.
3. In the last question, sometimes it has been hard for students to get the conclusion that the eigen-subspace has dimension 2.

These difficulties have been directly observed during the teaching development. A more detailed analysis on the achievements and difficulties to move from one register of semiotic representation to another that shall be based on the written answers given by students is in progress. Given the diversity of representations and registers that the teaching design involves, this analysis has not been simple at all.

Appendix: Instructions to construct the Cabri File to work in \mathbb{R}^2

In this appendix the Cabri instructions to generate the environment in \mathbb{R}^2 are given. The construction in \mathbb{R}^3 is similar.

1. With the “Show Axes” tool ask Cabri to plot a set Cartesian coordinates. Refer to this system as System I.
2. Draw an arbitrary “Vector” on the origin of coordinates.
3. With the “Label” tool denote with \mathbf{v} the end point of the vector.
4. With the “Numerical Edit” tool write four numbers in a 2x2 matrix look-like configuration. These numbers will represent the matrix A associated with the linear transformation $T(\mathbf{v})=A\mathbf{v}$. Figure 4 shows matrix $A = \begin{bmatrix} 1.5 & 1.8 \\ 2.2 & 1.2 \end{bmatrix}$, with $a=1.5$, $b=1.8$, $c=2.2$ and $d=1.2$.

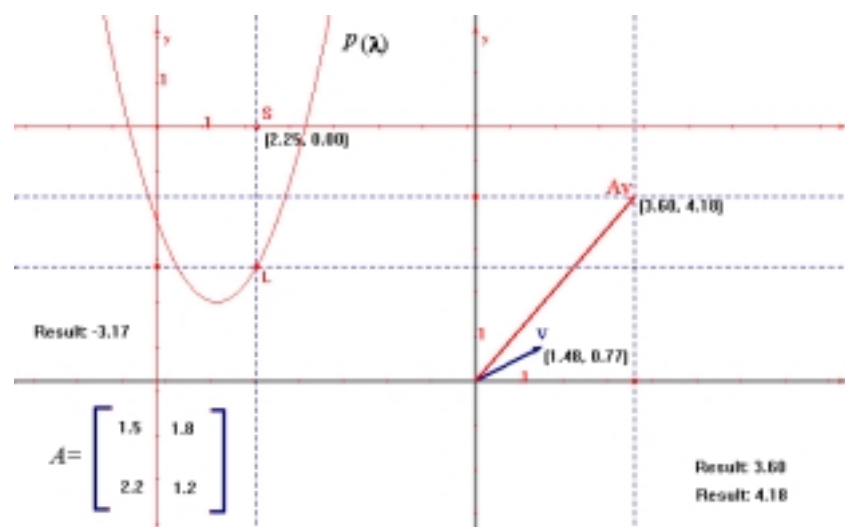


Figure 4: A detailed aspect of the construction in \mathbb{R}^2

5. Use the “Calculate” tool to compute the coordinates of vector $A\mathbf{v}$ and drag each of the corresponding coordinates from the calculator to the Cabri screen.
6. With the “Measurement Transfer” tool project the first coordinate of vector $A\mathbf{v}$ to the X axis and the second coordinate to the Y axis. Mark with a “Point” the corresponding projections.
7. Draw a “Perpendicular Line” to X axis through the point determined on this axis in Step 6 and then draw a “Perpendicular Line” to the Y axis through the corresponding point.
8. Draw the “Intersection Point” between the two perpendicular lines obtained in Step 7 and call $A\mathbf{v}$ this point.
9. Draw a vector from the origin of coordinates to point $A\mathbf{v}$. If (x, y) denote the coordinates of vector \mathbf{v} then the coordinates of $A\mathbf{v}$ will be $(1.5x+1.8y, 2.2x+1.2y)$.
10. Using the “New Axes” tool draw another system of coordinates. Refer to this system as System II. Fix the origin of System II sufficiently far away from the origin of System I.
11. Use the “Point on Object” tool to select an arbitrary point S on the X axis of System II. With the “Equations and Coordinates” tool ask Cabri for the coordinates of this point in System II. Draw a “Perpendicular Line” to X axis (of System II) passing through S.
12. Use the entries of matrix A (see Step 4) to calculate the number $P(\lambda)=(a-\lambda)(d-\lambda)-cb$, taking λ as the abscissa of point S.
13. “Drag” the number obtained in Step 12 from the calculator to the Cabri screen.
14. Use the “Measurement and Transfer” tool to project $P(\lambda)$ into the Y axis of System II. Draw a “Perpendicular Line” to Y axis through the point determined by $P(\lambda)$ on the Y axis.
15. Draw the “Intersection Point” between the perpendicular lines obtained in Steps 11 and 14. Call L such point.
16. Ask Cabri for the “Locus” of L when S moves along X axis of System II. This Locus is the graph of the characteristic polynomial $P(\lambda)$ of A . See Figure 4.

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