STUDENTS' ASSUMPTIONS DURING PROBLEM SOLVING

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ABSTRACT

This paper analyses the solving process of two undergraduate students on a non-routine mathematical problem. By comparing these students' work, it can be observed that their processes suggest different approaches in relation to the way a solution was sought. One student followed a process in which his principal activities were centred on *discovering* those key ideas that would allow him to tackle the problem. The other seemed to be more focused on *inventing* a way of dealing with the situation and building a solution to the problem. Since in order to invent a solution some useful facts have to be first stated or discovered, it may be speculated that a process which aim is to *invent* is more flexible than a process which aim is to *discover*.

Key Words: Mathematical Problem Solving, Approaches

1. Introduction

By analysing students' problem solving activities and the (externalised) reasons behind their courses of action it may be possible to gain some insight into their assumptions about the nature of the solutions they are trying to achieve. It may be speculated that a student whose aim is to *discover* a solution believes that the solution is something that already exists and that his or her duty is to uncover it. On the other hand, a student whose aim is to *invent* a solution either does not believe that a solution is *out there* or believes that if this is the case s/he still can create her/his own solution. The objective of this paper is to discuss these different approaches and their implications to problem solving.

2. Approaches to Mathematics and Problem Solving

In relation to teaching mathematics, what is meant by "mathematics" (i.e., the view held towards mathematics) affects the way in which mathematical problems are presented and the way in which problem solving is conducted (Shoenfeld, 1992; Goldin, 1998). For instance, the definition-theorem-approach to mathematics is a paradigm that has affected mathematics education by focusing attention to the logico-deductive activities carried out by the student (Davis and Hersh, 1986). Furthermore, studies related to what students believe is expected from them when doing mathematics suggest that they hold different views of what "doing mathematics" is about, and that this, in turn, affects their achievement (e.g., Alcock and Simpson, 2001; Hazzan, 2001).

In the case of problem solving, it may be speculated that a solvers' idea about the nature of mathematics may affect the way in which a solution is sought. For example, if a student holds a Platonist view of mathematics (see Hersh, 1997), his or her approach may suggest an attempt to discover those key entities required to solve the problem. On the other hand, if a student holds a view of mathematics as a human construction, his or her approach may be better defined by an attempt to "build" a solution.

The purpose of this paper is *not* to show that the solving processes analysed here fall into a "Platonist" or into a "Constructivist" approach. The scope is more limited in the sense that the aim is to discuss two observed approaches in relation to solving a mathematical, non-routine problem. These approaches suggest different assumptions about the nature of the solution that is expected to be found, and this will also be discussed.

3. Methodology

The written work analysed here belongs to two students from a group that took part in a ten-week, problem-solving course. The course participants were all doing undergraduate degrees in maths, physics or computer sciences. The course was structured with the objective of introducing students to vocabulary and concepts that could help them reflect on their own solving processes and share their experiences (based on Mason, Burton and Stacey, 1985).

During the course, students were required to solve problems and encouraged to develop a rubric for recording their ideas and experiences. As a final assignment, they had to choose one between two problems and were required to submit a script of their process. The solving processes discussed here correspond to this final assignment.

Students' solving processes (corresponding to the final assignment) were coded and the activities identified suggested two categories. On one side, some students' processes suggested that their objective was to *discover* a solution to the problem. Alternatively, other processes suggested that the main objective was to *create* a solution to the problem. An analysis of Martin and Kyle's solving processes is shown here in order to discuss these two categories identified. (Real names have been changed to ensure anonymity.)

The problem. The problem that the students had to solve was stated in the following way:

> These rectangles [see Figure 1] are made from 'dominoes' (2 by 1 rectangles). Each of the large rectangles has a "fault line" (a straight line joining opposite sides). What fault-free rectangles can be made?

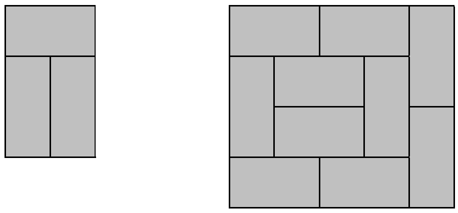


Figure 1

The problem was open in the sense that it allowed students to decide whether they wanted to approach it by assuming that "no fault-free rectangles" could be made, or the opposite. Another characteristic of this problem was its geometric nature and thus the fact that students worked (as expected) with geometric representations. In general, Fault-Free rectangles (FFR henceforth) *can* be built if their dimensions are 5 by 6 or larger, and not equal to 6 by 6.

4. General view of Martin and Kyle's solution process

In general terms, Martin's process may be described by the exploration of three main ideas. Firstly, he analysed the idea of "blocking and extending" in order to explain *why* faults appeared and how they could be blocked in order to build FFRs. Secondly, he looked at the possible "building blocks" of rectangles made with 2 x 1 dominoes. In relation to this, first he tried to justify that, since the basic structures that compose rectangles made with dominoes are inevitably faulty, FFRs cannot be built. However, he later found a FFR and decided to use the argument of the basic structures to describe the composition of FFRs. Finally, Martin's third idea was that of building a set of FFRs made of 3 x 2, 4 x 2 and 1 x1 "dominoes". From the latter approach, he expected to build a fault-free structure which "dominoes" could then be easily split and transformed into 2 x 1 dominoes.

In Kyle's case, his process is better described by three stages rather than by three basic ideas. Kyle's first stage appeared to consist of a process of systematic specialisation aimed at building a FFR; the product of this stage was, actually, a FFR. The second stage can be defined as the analysis of the newly-built FFR and the development of a way of extending it by 2 units in either direction (horizontally or vertically) and keeping it faultless. Since the FFR found by Kyle had dimensions 6 x 5, he showed that, by his method of extension, he could build any FFR with dimensions (6+2n) x (5+2n) with n = 1, 2, 3, … Finally, during the third stage of his process, Kyle aimed at answering the question of whether FFR with even by even dimensions could be made at all. In order to do this, he tried to build an even by even FFR by combining his initial systematic specialisation with the new idea of systematically increasing the dimensions of a rectangle. The result of this last approach is shown in the next section.

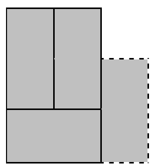
It may be said that Martin's general approach was to look for potential key ideas for tackling the problem and then to verify whether these ideas were useful or not. On the other side, Kyle's key ideas seemed to have emerged from a process of experimenting with particular aspects of the problem and trying to make use of the results of this analysis. In Kyle's case, the results from one stage usually constituted the key ideas for the next stage. As for Martin, the different ideas explored were more or less independent.

5. Martin and Kyle's different approaches

Martin's solution process

As said before, Martin's solving process seemed to be guided by the exploration of three different ideas. The first idea was that of finding a way of "blocking [faults] and extending [the size of the rectangle]" in order to build FFRs. It is interesting to note that, in developing this concept, Martin seemed to be trying to develop a "systematic approach" that would allow him to control the situation and solve the problem. The following passage is from Martin's written description of his solving process:

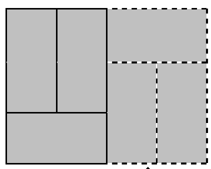
> AHA! Will try a systematic approach of 'blocking' faults.
>
> INTRODUCE *concept of block*. Given a rectangle with [a] fault line a block is a single tile added to stop the fault. E.g.,



← block

Figure 2

> INTRODUCE *concept of extension*. Once a block is made, the shape is extended to create a new rectangle by adding tiles, e.g.,



←extension

↑

block

Figure 3

Conjecture 1. Method of blocking and extending will produce a fault-free rectangle.

Martin devised and presented his method of "blocking and extending" as one, which would allow him to systematically build a FFR. It may be said that he based this method on two simple ideas: (1) the idea that faults must be blocked and (2) the idea that this leads to the need of increasing the size of the rectangle. However, the fact that he was not being able to build FFRs using this method, suggested him that they could not be built at all. As Martin put it: “repeated failure put idea of ‘no solution’ in my head”. In this way, he decided to modify his method and, later, to abandon it.

Martin’s second approach was to look for the basic building blocks of rectangles made with dominoes. He believed, at a point, that it was not possible to build FFR from dominoes and aimed at proving this by showing that all rectangles contain sub-blocks that are faulty. For doing this, he first conjectured that all large rectangles contain either two vertical dominoes arranged side by side, or two vertical dominoes arranged one on top of the other, or both. In his solution process, he wrote the following:

AHA! Must test all ways of sticking

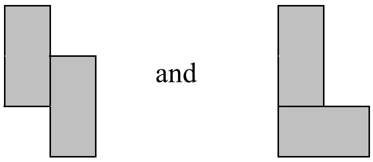


Figure 4

together (to each other and self). We must prove all ways of doing so necessarily imply.

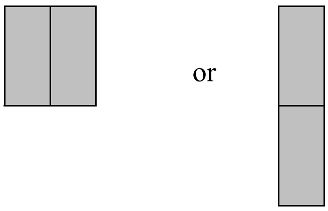


Figure 5

to complete rectangle. E.g.,

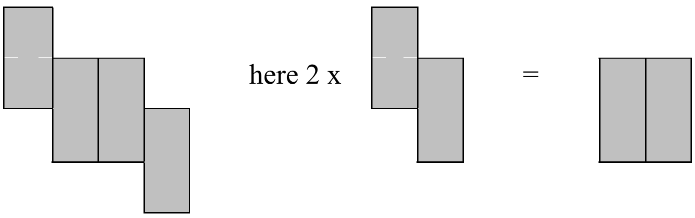


Figure 6

In this second approach, Martin tried to justify that every rectangle made up with dominoes would contain a basic (faulty) combination, and, therefore, that FFR could not be built. He found a “convincing” argument to the first part of this conjecture but, “with much disappointment and frustration”, ended up producing a FFR. So he found himself in the position of having to modify his approach once more and to look for new ideas. The following figure shows Martin’s FFR:

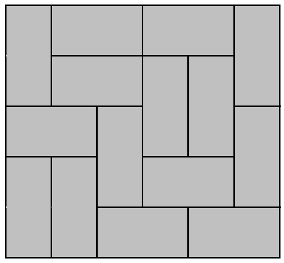
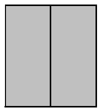


Figure 7

However, Martin did not abandon the idea of basic building blocks completely. Instead, he modified his approach by writing a new conjecture:

Conjecture 4 Faultless rectangles with



must have

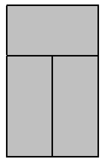


Figure 8

Conjecture 4 was modified several times, but the idea of distilling the basic blocks contained in rectangles and FFRs remained. Martin provided an argument to show that it is inevitable to use these basic blocks and concluded that the 3-domino shape shown in Conjecture 4 is "necessary for a faultless rectangle".

So in his first and second approaches, it can be seen how Martin seemed to be looking for ideas that could be useful for dealing with the problem. The idea of building FFRs systematically through a simple but "justified" method (as in his first approach) did not prove to be very effective. Then, the idea of distilling the basic building blocks in rectangles made with dominoes (and later in FFR) did not seem to provide much information as to "what fault-free rectangles can be made". Nonetheless, at this point, he had already accumulated two pieces of information: (1) that it is possible to build FFRs using dominoes and (2) that these FFRs contain 2 and a 3-domino, basic structures.

The third of the basic ideas explored by Martin was the following:

> AHA! Relax original question, allow any size and ratio for rectangles to create a fault free rectangle (with view to dividing up larger rectangles to 2 x1). Specialise randomly:

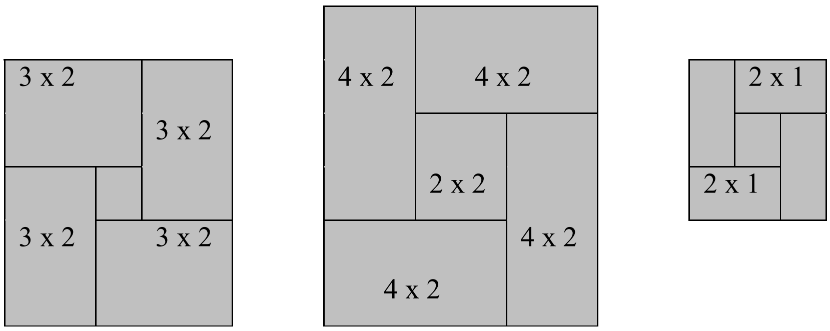


Figure 9

The idea behind this approach was that once a FFR rectangle was built with these 3 x2, 4 x 2, etc. rectangles, then these so-called "larger rectangles" could be split and transformed into 2 x 1 rectangles (dominoes). This did not turn out to be true in practice. However, by comparing these new FFR (made up of larger "dominoes") to his previously built FFR, he found that they had some basic structures in common. The result was that this took him back to the 3-domino, basic structure mentioned above. Namely,

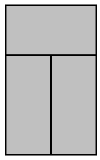


Figure 10

The reason why Martin's aim can be described as that of discovering lies in his overall approach. Martin's approach suggests that he was looking for a key idea (or ideas) in order to solve the problem. By comparing Martin's solving process to Kyle's (discussed in detail below), it may be said that the latter followed an approach that is better described by the way ideas were developed and transformed.

Kyle's solution process

Kyle's first approach was to "specialise systematically". At the beginning of his process he wrote: "[I] don't know the nub of the question yet! Specialise to understand what the question really wants first. Specialise systematically". In this way, he began by trying to build FFRs with 2, 3, 4, 5 and 6 dominoes. But he was not able to build any FFR and thus declared himself "STUCK! Not sure if it can be done". At this point, he decided to try to transform his last (faulty) rectangle into a FFR by adding dominoes to it (i.e., without restrictions on the number of dominoes used). The result was encouraging as he was able to build a FFR with dimensions 5 by 6.

A second important stage in Kyle's process was to analyse the way in which he had produced his first FFR and to devise a method for increasing its dimensions. As a result from the analysis, he distilled a list of important steps:

- Started off with basic 6-domino shape
- Eliminated horizontal fault by adding [blocks] "1", "2", "3" [see figure below]
- Added "4" to counteract vertical fault
- Swapped "5", "6" to vertical to counteract horizontal fault
- Built around to make it complete
-

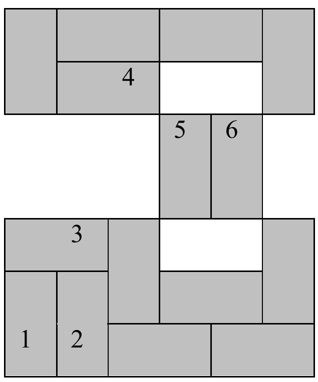


Figure 11

It may be said that it was the fourth step (swapping 2 horizontal dominoes for two vertical ones) the one that formed the basis for Kyle's method of extending his 6 x 5 FFR by two units in either direction, i.e., horizontally or vertically. This method consisted of taking a FFR and selecting a pair of, say, horizontal dominoes (if the size is to be increased vertically) placed one of top of the other. Then, all the dominoes on the level of the top domino or above (from the pair selected) had to be removed. The next step was to swap the selected pair of dominoes by a pair vertical ones and to add a full row of vertical dominoes to the top. Finally, the structure had to be "capped" again in order to return it to a rectangular shape. This would then produce a FFR larger than the previous one by at least two units (as layers of vertical rectangles—which increase the height by 2—can be added indefinitely). In order to justify why this method worked, Kyle added that:

> This new shape creates no horizontal faults and any vertical faults will be irrelevant because the existing rectangle (the "bottom part") cancels these out.

Using this method, Kyle showed that his FFR with dimensions 6 x 5 could be used to construct any N x M FFR with N = 6, 8, 10, … and M = 5, 7, 9, … As said above, Kyle based this method on the idea that, once a FFR is constructed, its dimensions could be increased systematically by "opening" the figure, inserting dominoes as required and, finally, "recapping" the figure in order to return it to a rectangular shape. The only condition was to make sure that, after the splitting, faults were "immediately taken care of".

The last stage in Kyle's solution was his sub-process of trying to decide whether N x M rectangles of even dimensions could be built. So far, he had only been able to construct FFRs with even by odd dimensions. Furthermore, his method for increasing their size only allowed him to add 2n (n = 1, 2, 3, …) units (vertically, horizontally or to both dimensions) to already-built FFRs. So he suspected that even by even rectangles could not be made. For verifying this, he decided to begin by using the same strategy he had used before and that yielded his first even by odd FFR. I.e., he began with a faulty rectangle and added dominoes to it hoping that this would eventually lead to a FFR. Also, in order to increase the possibilities of this FFR having even by even dimensions, he decided to begin with 4 dominoes instead of 3. In Kyle's words:

> TRY… and find an n x m rectangle which is fault free [n, m even].
> Earlier method: start with a basic rectangle and extend it. Previously started with a 3-domino rectangle so start with a 4-domino, this in the hope of getting n, m even.

After experimenting with this approach and not being able to construct any FFR, Kyle concluded that:

> Staring with 2 dominoes together, it is impossible to cancel 1 fault at a time without producing an even by odd rectangle each time infinitely or until you find a fault-free rectangle.

But, apparently, he was not convinced of his own arguments and thus continued trying to build an even by even FFR by experimenting and, at the same time, trying to use the information he had already accumulated on the problem. He was not successful in this attempt and closed his solving process by providing an unclear argument to the following conjecture:

> Starting with an even by odd dimensioned rectangle, and performing a series of iterations, it is impossible to get an even by odd rectangle as a result.

As said before Kyle followed a solving process that can be described in terms of the strategies he developed for building and extending FFRs. Also, each strategy was built on the result of either previous strategies or previous sub-processes (e.g., the idea of building even by even FFRs using the concept of extension defined in a previous strategy). Comparing Kyle's solution process to Martin's, it may be said that Kyle's aim is better described as that of using emergent ideas to *build* a solution. Martin proceeded by developing key ideas and then testing them (hoping, maybe, to find one on which to base his solution); Kyle on the other side, noted useful ideas as he worked with particular aspects of the problem and then transformed these ideas into a strategy.

6. Conclusion

The analysis of Martin and Kyle's solving processes shows that, for dealing with the same problem, these students followed different routes. The nature of the differences between their processes suggests that their assumptions about the solution they were attempting to find may be different. Martin's approach could be described—from a researcher's point of view—as that of assuming that ideas are "out there" while Kyle's approach as that of creating solutions.

For Hersh (1997) a mathematician assumes the role of a Platonist when he works as if he believed that mathematical entities cannot be created and that they "exist whether we know them or not" (p. 73). On the other side, when a mathematician works as if mathematics is not discovered but created, s/he is, according to the author, working as a formalist or an intuitionist. But, in spite of this apparently clear-cut distinction, the author suggests that mathematicians may actually adopt these two roles at different times:

> When several mathematicians solve a well-stated problem, their answers are identical. They all *discover* that answer. But when they create theories to fulfill some need, their theories aren't identical. They *create* different theories. (Hersh, 1997, p. 74)

In the case of the students, trying to give an account of the assumptions held during problem solving is an activity that can help us gain insight into their understanding about mathematics. This can be done, as Hersh indirectly suggests, by looking at what students do (and say, and write) during problem solving. In terms of validity, qualitative research methods such as grounded theory (see Glaser and Strauss, 1967) provide means for producing valid results. In this respect, even though an account of a student's assumptions will inevitably be a researcher's construction, it ca also be a scientifically justified one.

REFERENCES

Alcock, L. and Simpson, A. (2001). Cognitive Loops and Failure in University Mathematics. *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, 25, 2, pp. 33 –40.

Glaser, B. and Strauss, A. (1967). *The discovery of Grounded Theory: Strategies for Qualitative Research*, Aldine: Chicago.

Goldin, G. (1998). Representational Systems, Learning, and Problem Solving in Mathematics. *Journal of Mathematical Behavior*, 17, 2, pp. 137 –165.

Hazzan, O. (2001). Reducing Abstraction: The Case of Constructing an Operation Table for a Group. *Journal of Mathematical Behavior*, 20, 2 pp. 163 –172.

Hersh, R. (1997). *What is Mathematics, Really?*, Random House: London.

Mason, J., Burton, L. and Stacey, K. (1985). *Thinking Mathematically*, Addison-Wesley: Wokingham.

Schoenfeld, A. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics. In Grows, D. (Ed). *Handbook of Research on Mathematics Teaching and Learning*, Macmillan: New York.