GEOMETRY: BACK TO THE FUTURE?

Luiz Carlos GUIMARÃES
Instituto de Matemática
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.
e-mail: lcg@labma.ufrj.br

Elizabeth BELFORT
Instituto de Matemática
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.
e-mail: beth@dmm.im.ufrj.br

Franck BELLEMAIN
Centro de Informática
Universidade Federal de Pernambuco, Recife, Brazil.
e-mail: fb@cin.ufpe.br

ABSTRACT

In this paper we consider the use of current technology to help us address fundamental difficulties in comprehending geometrical thought in a geometry course designed for Mathematics teachers. Our conception of geometry involves a full spectrum of activities, from concrete exploration and experimentation, through conjecturing, problem-solving, and on to formal proof. However, the full range of this spectrum requires very different qualities of thinking that seem to make it very difficult to implement in a mathematical curriculum. This was apparent in the work of Archimedes, who conceived of formulae for areas and volumes using a 'mechanical method' that relied on thought experiment, but published many of his results only in terms of 'proof by exhaustion', which severed the link between creative conception and formal presentation. This dichotomy has continued throughout history, with our present school curriculum seemingly oscillating between a focus on formal Euclidean proof on the one hand and a more empirical study of 'space and shape' on the other. Our approach faces both ends of this dilemma by using dynamic geometry softwares and a study of selected historical struggles to develop formal proof and to help students to reflect on the dialectic process between exploratory work with figures and proof elaboration. We outline our theoretical perspective, consider the issues involved, and report on the progress of different types of student who attended the course.

Introduction

Geometry seems to be firmly back in the curricula of undergraduate mathematics in an increasing number of leading Mathematics departments, as witnessed by a growing offer of new textbooks (Hartshorne's *Geometry: Euclid and Beyond*, is a particularly beautiful example). A somewhat related phenomena is the mounting stream of documents stressing the need for a greater emphasis on geometry in the school mathematics curriculum (see for instance NCTM, 2000 and Oldknow & al., 2001).

While in some countries, as it is the case of France, geometry has always taken an important place in the curriculum and is considered as a fundamental subject, in others (including Brazil), Dieudonne's war cry of "down with Euclid", back in the sixties, seems to have echoed for far too long and with particularly zealous fervour. In this paper we discuss ways of providing, both in undergraduate and continuing education teacher training programs, effective means for adequately preparing the teachers who must undertake the task of teaching geometry in secondary schools.

Our conception of geometry involves a full spectrum of activities, from concrete exploration and experimentation, through conjecturing, problem-solving, and on to formal proof. The importance of geometry in the curriculum goes beyond it's recognised content: it's acknowledged as fundamental for the development of the understanding of mathematics and science in general. *«Elle est un objet d'enseignement incontournable tant du point de vue de l'étude des situations spatiales que du point de vue de la constitution de la rationalité scientifique.»* (Bkouche, préface of the book Géométrie (Carral, 1995)). It is regarded as a good opportunity for the student to evolve from observation skills to hypothetical-deductive skills (Rauscher, 1993).

On the other hand, our experiences show that the majority of teachers working in Brazilian secondary schools has a less than adequate grounding in geometry. This assumption is confirmed in two separate ways:

-the knowledge of geometric fundamentals of samples of teachers attending continuing education courses at our institutions, whenever put to the test, has proved far from adequate. There seems to be a direct bearing in their ability to deal with basic notions, and with the concept of proof.

-the mediocre performance, in geometry related questions, by students in all university entrance examinations. As a consequence, our troubles in first year courses are not very different from what we see reported in dozens of accounts from other countries. For instance, if we consider the Linear Algebra courses, we know for sure that our students are not getting the grounding in spatial geometry that was to be desired before entering the University, and that has an influence in their initial difficulty to deal with the spanning of subspaces and the geometry of linear transformations.

A vicious circle is in place, whereby less than adequately prepared students start teacher training courses in Mathematics, finish their courses with a less than adequate proficiency, and go into the profession feeling less than secure about their own mathematical ability. To us, a good place to try to break this stalemate is a geometry course rich in opportunities to deal with deep questions which have a bearing in the school curriculum, specially as regards mathematical proof.

Geometry in the School Curriculum: the role of Dynamical Geometry Softwares

The recent change in the curriculum concerning geometry is that, maybe with the help of dynamic geometry software[1], geometry is taught nowadays with a bigger emphasis on experimental approaches. «*Les diverses activités de géométrie habitueront les élèves à expérimenter et à conjecturer, et permettront progressivement de s'entraîner à des justifications au moyen de courtes séquences déductives* » (Programmes de cinquième, BO, p. 24). Carral (1995) also says «*La géométrie élémentaire doit être considérée comme une science physique et son apprentissage doit se faire comme une science expérimentale…* ».

But even in countries like France, where geometry has always been an important part of mathematical teaching, it has also always been one of the hardest to teach, particularly when proof is concerned. Trouble with the concept of proof seems to be a feature not exclusive of Brazilian or French schools. As we write this we can find the following text, in the web site of the British Association of Teachers of Mathematics, as part of an apparent endorsement for a book soon to be distributed by ATM: "*This book argues the case for the use of proof based on 'seeing is believing'. Using a 'Tracing', on top of a 'Diagram', we can often show clearly the truth of an assertion. In other words we can prove it.*"

In general, the curriculum oscillates between more figure exploration/less formal geometry teaching and less figures/more proof elaboration. The dialectic process between exploratory work with figures and proof elaboration, which can be seen in the historical evolution of geometry, seems to give curriculum formulators a hard time.

Maybe more than just knowledge, we want the student to develop competencies in knowledge construction. Particularly, we want him to acquire skills in exploring figures, elaborating and experimenting with conjectures, , problem solving and proof formulation. But this set of skills, which seems so natural to the scientifically trained, does not come so naturally to the students. The concrete object does not have the same signification and is not explored in the same way by the mathematician and by the student: the way the concrete object is used strongly depends on the previous knowledge of who is using it. Even more important is that teaching based on the exploration of the concrete object makes the none evident assumption that the interaction with the concrete will effectively produce the construction of the desired knowledge (Balacheff, 1999).

In the case of geometry, the concrete object is often a diagram, and to understand the differences between the student and the teacher in it's exploration it, researchers in maths education often consider two different objects (Parzysz, 1988; Arsac, 1989; Laborde et Capponi, 1994; Balacheff, 1999):

- a concrete object, the drawing, which is a material representation of the figure,
- a formal object, called the figure, which corresponds to the class of drawings representing the same set of specifications.

One of the difficulties in the geometry classroom is that the student may be thinking in terms of the drawing instead of the geometrical object, whereas the teacher is using the abstract representation of the geometrical objects, the figure. Helping the student to read a figure in a geometrical way, and to use it as a tool to conjecture or understand proof, and not as the proof itself is part of the job of teaching geometry.

[1] As the authors include people involved in the development of two different dynamical geometry packages, the reference here is to the genre, as opposed to a particular implementation which, we feel, only strengthens the argument.

From this point of view, dynamic geometry softwares can have a specific contribution. They can provide new representations of geometrical objects which, in some ways, concretise the formal figure. We take as one of our assumptions that these softwares can provide new ways to learn geometry, and by way of consequence, new ways to teach the subject. Their use in programs of improvement of mathematical preparation of teachers of Mathematics provides us also with the opportunity to discuss with them how to integrate mathematical softwares in their teaching toolkit.

A course of Geometry for teachers: the choice of a historical reference

The course we are experimenting with Mathematics teachers (both at graduate and undergraduate level) discusses the contents and notes of the "Elementos de Geometria", by A. M. Legendre (1809), and some of descendants of this book. Incursions into more modern treatments and contemporary results are made when appropriate.

There were many different reasons for this choice, of which we mention only a few:

- the text was written by a mature mathematician, at a time close enough in history that we have a fair idea of what was known by him at the time of writing. Some results were new then, as the proof by Lambert that π is an irrational number (the proof that π is transcendent took a while longer, even though Legendre sounds convinced that this is so in his notes). The question on the ruler and compass constructible polygons was settled by Gauss a few years after the first edition was published (in 1794), and this information is included in the editions afterwards.
- and, of course, there is proposition XIX of book I, where Legendre tries to prove, unaided by the fifth postulate of Euclid, that the sum of the internal angles of a plane triangle equals two right angles. Throughout different editions he changes the proofs, each one of them beautiful, and each resorting to a hidden postulate (discussed by him later in a note). The use of apparently correct proofs to exercise the critical judgement of mathematics students was an established habit in soviet schools (see Bradis, Minkovskii & Kharcheva (1999) and references therein).

Legendre's proofs are a more subtle challenge than the geometrical examples in the last mentioned reference, where the absurdity of the statement can be made immediately apparent by a carefully drawn figure. Contrast this with Legendre's proposition XIX, where the statement refers to a result known as correct, the writer has the authority of a classical master and, if we try to check every step of the proof with a (Euclidean) plane geometry computer package, the software will (have to) confirm the truth of every statement. The history of the birth of hyperbolic geometry alone would justify von Neuman's ([1961], quoted in Artmann, 1999) careful choice of words, before saying "*that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure*". Mathematicians had to know what geometry they wanted before they were able to devise models for hyperbolic geometry. It was almost irresistible to succumb to the authority of "the nature of the straight line", even for mathematicians of the calibre of Legendre.

The different attempts by Legendre to prove that the sum of the internal angles of a plane triangle equals two right angles, or to prove a postulate equivalent to the parallel postulate, provide good illustrations of the dialectic relation between figures and proofs. In his case, the figure cannot provide the geometrical information he needs, as this depends on knowledge not available to him when exploring the figure. Instead of helping, the figure he uses implicitly suggests information he then proceeds to use, and destroys his argument as a mathematical proof.

The role of Dynamical Geometry Softwares as a Tool for searching for solutions

Among the modern treatments covered in some detail during the course, we include transformations. In particular inversions, so we can construct the Poincaré model of the hyperbolic plane and make the flaws in Legendre's argument adequately clear. The treatment at this point is greatly aided by resorting to dynamic geometry software to aid in the visualisation of the meaning of theorem statements and proofs.

The guiding principles we try to enforce when using Dynamical geometry softwares in the course can be summed up in the words of Archimedes, contained in the foreword to his "The Method of Mechanical Theorems": "*... a certain special method, by means of which you will be enabled to recognise certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding the proofs of these same theorems. For some things, which first became clear to me by the mechanical method, were* afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. *For it is easier to supply a proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge*." [our italics] (Dijksterhuis, 1987, pp 312, 313).

The frustration felt by many commentators of the classical Greek texts on geometry, with the "*tantalising ... absence of any indication of the steps by which they worked their way to the discovery of their great theorems*" (Heath, introductory note to The Method, pg. 6) must be paralleled by many readers of geometry textbooks. That dynamic geometry softwares can be a nearly ideal tool to engage students in activities leading to formulation of useful conjectures has been said by many. Taken to extremes, the deformation of the usage seems to lead some to propose to do away with mathematical proofs altogether (for a sober discussion, see the introduction in de Villiers, 1999). Instead of adding to this, we present two examples. The first comes straight from Legendre's text (the appendix to book IV), but we use the English wording of Wentworth (1938), a formerly popular American textbook which went through tens of editions:

"Theorem: *Of all polygons with all sides given but one, the maximum can be inscribed in the semicircle which has the undetermined side for its diameter*"

As stated, the method of proof seems almost mandatory: suppose at least one vertex is not on a semicircle, and check that pulling it to comply will increase the area. Notice the format of the statement though, which models the majority of variational problems which cropped into geometry textbooks after Legendre. The reader is *given* the solution, and asked to *prove* that it is the right one.

That is also the case of our next example but, this time, instead of fully reproducing the statement in F.G.-M.(1920, pg. 768), we shall withhold part of the information:

"*Given that E is a fixed point in the interior of the convex angle ĐA, find the smallest segment BC, joining the two sides of angle ĐA, which passes through E.*"[2]

There is no tool in geometry comparable to the use of the derivative to *find* the conditions which must be satisfied by segment *BC*, and that is why the original statement goes something like "prove that the segment with the property so and so is the right one". In this case, though, the textbook tells us how the original author found the right segment: Newton proposed this problem

[2] The segment BC is often called Philo's line.

after having solved (through calculus) the more general case where, instead of being in an angle, *E* is a point in the region between two given curves.

But the lack of calculus is no reason to postpone the investigation of this type of problems. Dynamic geometry softwares can aid us in two crucial steps towards the right solution:

- disproving wrong conjectures we make along the way. The segment orthogonal to the angle bisector is a frequent guess in our course when students are trying to solve this last example, as is the segment tangent to the inscribed circle passing through *E*.
- helping to make apparent the features common to the solution of particular cases (*E* in the angle bisector, or *E* in one of the sides of the triangle, for instance).

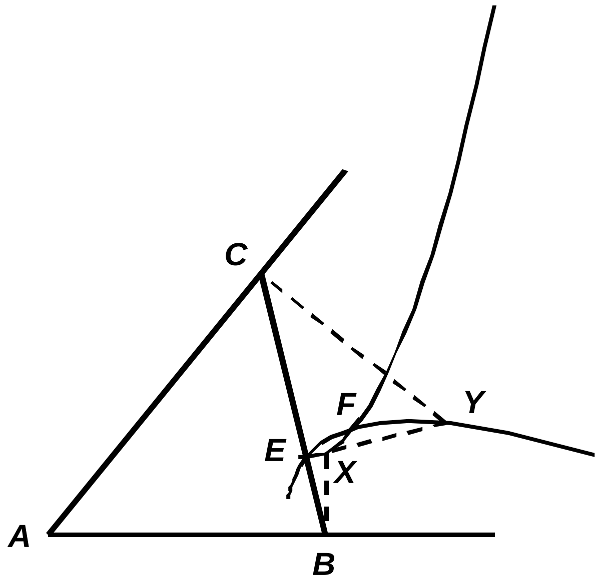

Figure 1: the smallest segment BC, joining the two sides of the angle, which passes through E is obtained when points *X* and *Y* coincide with F.

In this last example, the software might even steer us into a way to *construct* the right segment, which we would not get with the proof alone. As it turns out, to construct the right segment we have to construct first the point *F* in Figure 1. *B* is then obtained projecting *F* on the side of the angle. *F* is obtained as the intersection of two parabolas, the loci of points *X* and *Y* in the figure, point *Y* being the intersection of the perpendicular through *E* to segment *BC* with the perpendicular through *C* to the side *AC* of the angle. Point *X* is similarly constructed from the side *AB*. Point *F* it is not constructible with ruler and compass, but a paper folding construction is possible, and that in itself leads to interesting discussions. [3]

An Informal Evaluation of Outcomes of the Course

In assembling the course materials there were several issues to be resolved that themselves constituted worthwhile parallel teaching experiments. Not the least of those was the preparation of a new edition, both in paper and electronic, of Legendre's translation into Portuguese, published originally in 1809. All the retyping and language adaptation was executed with and by future mathematics teachers. The reflections sparkled in them by this work were so rewarding that we are

[3] See also Cuoco and Goldenberg (1997) for different examples and a complementary viewpoint.

now starting work of the same kind with other groups and other books. But a description of this would take us away from the main object of this paper.

Instead of this, let's report on the subsequent careers of two groups of students who took part on preliminary versions of the course. One group is formed by undergraduate students and the other is formed by three teachers: one had just then graduated, and the remaining two are experienced teachers, coming back to the university after fifteen years or more of practice, for a graduate course.

The three teachers will have completed their M.Sc. degrees by the time this is read. The two more experienced teachers, who work in highly respected schools in Rio de Janeiro, have decided to prepare their master dissertations as texts other teachers could use to complement their views on axiomatic approaches to geometry. One of the dissertations discusses the axiomatic required to treat paper folding as a mathematical object. It also includes a comprehensive research on published theorems that can be derived using this approach. The other dissertation studies the hyperbolic versions of traditional Euclidean geometry theorems.

The remaining teacher, the one who had just graduated, also decided for geometry as a theme for his dissertation. He studied the generalisation of results of Steiner on the Simson line when the triangle is inscribed in a conic, instead of in a circle. Projective geometry arguments are used throughout the work to generalise the intended results.

As for the undergraduate students, four of them (out of a group of twelve) have decided to write their final undergraduate essays in Geometry. One of them is tackling the solution of maxima and minima problems by geometrical methods, starting with Legendre's fourth book and its appendix. The other three are now involved in the editorial project mentioned above.

Conclusions

To speed up the process of educating teachers with the needed expertise in geometry, we propose to take advantage of the momentum provided by a stronger contemporary stimulus over the educational system: the need to incorporate ICT technologies into the school curriculum. That a mathematics teacher must have access to adequate preparation to cope successfully is true in the case of ICT as well as for geometry. We propose to deal with both needs in a single program, dedicated to prepare teachers to integrate ICT into their classroom through the device of placing them on an environment where they use ICT to learn geometry. In this work we endeavoured to present a strong case for two assumptions we made when starting this project:

- the benefits of using geometry softwares as an integral tool in undergraduate and continuing education geometry courses;
- the benefits, both cultural and mathematical, of revisiting, through the viewpoint of dynamic geometry, classical results in the geometry literature.

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