LEARNING TO TEACH ALGEBRA:
An Italian experience with reference to technology

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ABSTRACT

The Ministry of Education (MPI) and the Italian Mathematical Union (UMI) have carried out a project for in-service teacher training in algebra. It consists on three stages. First a group of 20 selected teachers attended a series of lectures. The lectures were videotaped. On the ground of these lectures the teachers produced didactic materials (forms, references, etc) recorded in a CD. The final products (videotape and CD) are sent to the schools all over the country to be shown to mathematics teachers. Four lecturers developed the subject (the teaching and learning of algebra) according to the following streams:

- general educational issues based on international literature in the teaching and learning of algebra
- algebra and information technology
- a new approach to algebra through number theory
- history and algebra.

The present work reports the technological part of this teaching project.

The use of a computer algebra system can improve the teaching of algebra, helping the teacher in several ways.

However several difficulties can show up in the classroom use of CAS: elementary “pencil and paper” algebra rules and procedures are not always the same as the tasks performed by a machine, some problems arise in the relationship between algebra and graphics.

The aim of the lessons and of material produced is:

1. to give the teacher a good knowledge of a CAS (Derive);
2. to explain the problems that can arise in the classroom use of symbolic math;
3. to give a hint for the solution of these problems.

Keywords: Algebra Teaching, Computer Algebra Systems.

1. Introduction

The Ministry of Education (MPI) and the Italian Mathematical Union (UMI) have carried out a project for in-service teacher training in algebra. It consists of three stages.

In the first one, a group of 20 selected teachers attended a series of lectures. Four lecturers developed the subject (the teaching and learning of algebra), according to the following streams:

- general educational issues based on international literature in the teaching and learning of algebra;
- algebra and information technology;
- a new approach to algebra through number theory;
- history and algebra.

The lectures were videotaped. On the ground of these lectures the teachers produced didactic materials (forms, references, etc) recorded in a CD. The final products (videotape and CD) are sent to the schools all over the country to be shown to mathematics teachers.

The present work reports the technological part of this project and describes, in the next section, how the lecturer organised the course, with the aim of explaining the fundamental ideas of computer algebra to the teachers; then some of the problems that arise in the use of computer algebra are presented; in the final section some material prepared by the teachers is described.

2. The fundamentals of a CAS

The use of a Computer Algebra System (CAS) is now quite common in the classroom; among the different types of software commercially available, Derive is the best known and widely used in Italian schools.

However the use of Derive, in most cases, is limited to the graphing of functions and the applications to analytic geometric and calculus. One of the aims of this project is to give a starting hint for a more widespread use of the software, stressing its computing and algebraic capabilities; the graphical part is not considered, being a common background for the teachers involved in the project.

It should be emphasised that the lectures do not want to be a comprehensive guide to Derive, but only an introduction. Based on this material, the teacher is invited to build his/her personal teaching material, more closely related to his/her project and tailored on the students background.

All in-service teachers in Italy in the late eighties were involved in the PNI (Piano Nazionale per l'Informatica), where Pascal was taught together with its applications to Mathematics; an important feature of this project was the necessity of emphasising what is different and new in Derive, compared to a Pascal-type programming environment; these differences are very noticeable when we use numbers.

For this reason it is not surprising that a strong emphasis was put on the Exact mode used by Derive in treating rational numbers and radicals; this feature, together with the use of "arbitrary precision" integers, opens up a new field of applications to prime numbers, to the production and verification of numerical conjectures; all these applications are impossible or very limited using the Integer or LongInteger type of Pascal.

The second topic is algebra of polynomials; here we must notice that Derive has different levels of simplification (the basic simplification, the EXPAND and the FACTOR command). The FACTOR command enables us to factor a polynomial at different levels according to the algebraic field we are considering. The treatment of algebraic functions and the operations on polynomials is also considered, introducing the QUOTIENT, REMAINDER and POLY_GCD functions.

The following step is very important: the use of variables and the definition of functions. Since Derive is a functional language, the user must be aware that the definition of a function is a "natural step" in Derive. Here again one of the main aims was to convince the audience, familiar with Pascal, to change to this new way of working: a bottom-up approach was used in showing how we can start with a simple function and, using composition, obtain a more complex one.

The complex field plays an important role in CAS; in this first step the basic operations on complex numbers are presented; but this it is not sufficient, since several options for computations in complex fields can affect also elementary algebra; for instance the choice of a BRANCH for complex non single-valued functions may give results in the computation of a root. Then it is necessary to go deeper in the subject and to study the choice of a branch for roots and logarithms.

Algebraic equations and inequalities are an important topic in high school algebra; here again the use of a CAS may be very useful; it is interesting to see how Derive can solve algebraically third and fourth order equations and that the results provided not always are easily understandable, again for a problem of representation of complex numbers.

Derive can solve equations algebraically and numerically, while using Pascal we can solve them only numerically; this capability opens up a new interesting application: how to teach the student the limitations of algebraic procedures and how to make him/her aware of the opportunity of switching from the algebraic solution to the approximate solution.

Vectors and matrices are fundamental tools in a CAS, that can help in performing boring tasks such as inverting a matrix or reducing it; moreover, they are a fundamental tool for programming, since some iterative programming structures are implemented in Derive by the definition of a vector; then the study of vectors and matrices is also a preliminary step to programming.

Derive (at least in Version 4, used in the course) has a limited programming environment; but its peculiarity of being a functional language may be very interesting in teaching; SUM, PRODUCT and VECTOR were introduced first; then a deep presentation of functional iteration (both on scalar and vector) follows. The last topic is recursion and the relation between the recursive and the iterative definition of a function and the related problems of computing efficiency.

3. The problem of using a CAS

The presentation of Derive described in the last section was mainly technical; it aim was to provide the teacher with the necessary background about the features of Derive, in order to use CAS in the classroom, knowing "what it happens inside"; however during the presentation several issues arose about the didactical problems of the use of CAS in the classroom.

When speaking about problems due to the classroom use of a CAS the general trend is to tackle the subject in a too wide sense, blaming the CAS for many difficulties that are not typical of CAS but that exist for all types of software.

Then we must concentrate on problems that really depend on the interaction between numerical, algebraic or graphical results produced by the CAS and the same results obtained with paper and pencil; we describe some of them.

1. While the student can easily master algebra of polynomials and the results are generally predictable and understandable by the students, the opposite happens when we consider algebraic functions; here it is difficult to understand the effect of the three different levels of simplification.

2. The SOLVE function for the solution of equations involving rational functions does not control the compatibility of solutions; then we have non acceptable solutions, as shown in lines #1 and #2 of Figure 1. Moreover, Derive seems to work in extended real numbers; the results shown in lines #3 and #4 are easily explained using limits, but are very hard to understand for a 15 years old student of elementary algebra.

#1: $$\text{SOLVE}\left(x^2 - 4 + \frac{x-1}{x^2 - 3\cdot x + 2} = \frac{1}{x-2},\ x,\ \text{Real}\right)$$

#2: $$x = -2 \vee x = 2$$

#3: $$\text{SOLVE}\left(\frac{x+1}{x^2+1} = \frac{x+2}{x^2+2},\ x,\ \text{Real}\right)$$

#4: $$x = \pm\infty \vee x = 1 \vee x = 0$$

Figure 1

3. The use of complex functions in internal computations of Derive can create some problems; while algebraic computations can be controlled by carefully choosing the branches and the domain of variables, the graphs drawn by Derive may be quite different from the graphs that a student in basic algebra can expect.

Two examples are shown in Figure 2 and in Figure 3. In the first one, the function $y = \sqrt{x-1}\sqrt{x+1}$ is plotted, in the second example the function $y = |\ln x|$.

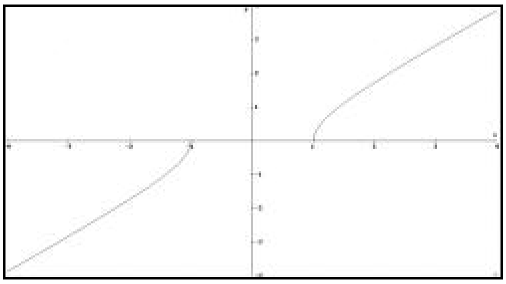

Figure 2

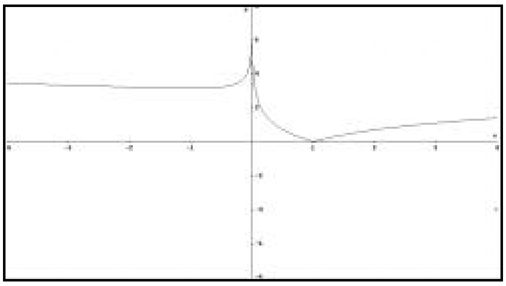

Figure 3

The problem of "wrong graphs" is important in teaching; as a matter of fact the use of computer in this context does not help the teacher and may cause false problems, generated by the use of a more advanced tool (complex numbers) by Derive, while the student is taught to work in the real domain; it is necessary to force Derive to work only with reals.

To do this some programming capabilities are required (as described in [Boieri, 1996]), since it is necessary to change the definitions of some built-in functions.

4. The teaching material

At the end of the course described in the last sections, the teachers worked at the production of teaching material. Here we give a brief review of a part of this material, with some remarks; we selected the teaching units more directly and closely related to algebra; for each subject we show the age of the students that are supposed to use it in the Italian school system.

1. Using numbers and variables in Derive; how to write an algebraic expression in Derive; the relation between handwriting of an expression and "linear writing" of Derive (age 14-15).

This unit is very important; the student is used to read and to write arithmetic and algebraic expressions in the standard multilevel form. Derive requires linear writing and the user must use correctly parentheses. The number of these parentheses can be quite large, even for simple expressions, and this can be very confusing for student.

To solve this problem, the unit emphasises how to transform an algebraic expression in a tree. Derive has a very useful tool for the interpretation of an algebraic expression (and then for the construction of the related tree): using the mouse or the keyboard arrows, the user can explore an expression at the different levels.

At the end of the unit the student could be able to switch from one to another way of writing an expression: standard textbook form, linear Derive form and tree.

2. How to check a numerical conjecture (age 14-15).

   Computer algebra allows working with integers of "arbitrary length" in exact mode; this technical feature can be used in teaching, opening up a new field of application: the verification or the refutation of conjectures. Some conjectures can be formulated using an elementary language and a student of 14-15 can tackle very interesting problems, starting from the easy ones, such as the sum of the first n integers, and ending with some conjectures about prime numbers.

3. How to solve an equation and a linear system using a step by step procedure (age 14-15).

   In this unit first and second degree equations are considered, together with equations involving rational functions or radicals, that can be reduced to first or second order equation.

   In Section 3 we pointed out some of the technical problems arising in the solution of algebraic equations in Derive. After a thorough analysis of these problems, the teachers agreed about the necessity of avoiding the "black box" approach, i.e. using the SOLVE function. They agreed about the opportunity of using a step by step procedure of solution, assisted by Derive, working on complete expressions or on subexpressions.

   When an equation must be solved, the student is stimulated to write down his/her solving strategy and then to use Derive in order to perform the steps.

   The "added value" of the use of Derive is not its computing power but the necessity for the student of organising very precisely the questions to be posed to Derive and the answers received by it.

   The problems of incorrect graphs arising from the use of complex numbers in internal computations were considered by the teachers too advanced to be presented to the students in high school; in this case the set of correctly defined functions is used as a "black box".

4. Graphing a function with Derive; the search for zeroes of a polynomial equation (age 15-16).

   This unit is closely related with Unit 3; after having solved first and second degree equations, we move to higher order polynomial equations.

   Using Derive, we can show to the student that it is possible to solve third and fourth equations (as pointed out in Section 2); some carefully chosen examples can show how Derive either computes a " readable" solution or computes an ugly expression of several lines; in any case we get a solution. With fifth order equations, sometimes we get a solution; sometimes Derive is unable to give an answer.

   These examples can motivate a presentation of some results about the solvability of algebraic equations, about formulas for third and fourth order equations and about non-solvability of equations of fifth or higher degree in the general case.

Moreover they can be the starting point for a discussion and for a study that can be continued throughout high school classes: the relation between algebraic and approximate solution of equations.

Derive is an ideal tool for this study, since it offers a purely algebraic microworld (the Exact mode) and an approximate computing microworld (the Approximate mode) in the same piece of software; we can try first an algebraic solution of the given equation, then move to an approximate one, when necessary.

We want to emphasise that this is a starting point; indeed, the approximate methods of solution of an equation involve the concept of sequence (bisection method) and some calculus (Newton method).

The teachers agreed about the opportunity of introducing the possibility of solving approximately in the first two years; it is reasonable to start using approximate methods as "black boxes", moving to a higher knowledge and to a wider use of them in the last three years of high school.

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