AN ANALYSIS OF THE PRODUCTION OF MEANING FOR THE NOTION OF BASIS IN LINEAR ALGEBRA[1]

Amarildo M. DA SILVA
Mathematics Department
Federal University of Juiz de Fora, Brazil

Romulo LINS
Mathematics Department/Postgraduate Program in Mathematics Education
UNESP-Rio Claro, Brazil

ABSTRACT

In this study we have investigated the production of meaning for the notion of basis in Linear Algebra, supported by the Theoretical Model of Semantic Fields, proposed by R. Lins (2001). It was conducted in three parts: (1) a historical-critical study, based on secondary sources, in which the key question was 'in which semantic fields were operating the mathematicians who constituted the notion of basis in the historical process of emergence of the elementary notions of Linear Algebra?'; (ii) an analysis of Linear Algebra textbooks, to investigate meanings which could be produced for the notion of basis from their reading; and, (iii) interviews with students of a first course on Linear Algebra (undergraduate mathematics degree), aiming at eliciting the meanings actually produced by them while engaged in solving proposed problems. The study allowed us to identify several and distinct meanings for the notion of basis being produced, coming from our many 'informants'; it had as a general objective to gather information which could help us and other professors a better reading of the classrom dynamics in a Linear Algebra course.

KEYWORDS: meaning production, basis, linear algebra

[1] This research was partially supported by PICDT-MEC through UFJF and by CNPq grant 350823/93-6.

Introduction

In this study we investigate the production of meaning for the notion of basis in Linear Algebra. It was conducted in three parts: (i) a historical-critical study; (ii) an analysis of textbooks on Linear Algebra; and, (iii) a case study with undergraduate students (mathematics degree) taking a first course on Linear Algebra. We adopted as a framework the Theoretical Model of Semantic Fields (TMSF) as proposed by R. Lins .

The central notion taken from the TMSF was that of 'meaning', characterised as "that which a person can and actually says about an object in a given activity" (see, for instance, Lins, 2001).

In the meaning production process some statements are locally taken as true without the need of further justification; a set of those 'local stipulations' taking part in a given meaning production process we call 'kernel'. Finally, we will call 'Semantic Field' to an activity of producing meaning in relation to a certain kernel; we will say, for instance, that a person is operating on this or that semantic field.

Just to illustrate, a local stipulation in a given activity (a child solving an arithmetical problem) could be related to whole-part relations, for instance, that ‘if a whole has two parts and one is removed, the other part is left’, although in another situation it could be necessary to justify this statement. Another general example is that one might use a mathematical result without thinking of why it is true, but in other situations it might be necessary to consider that.

Kernels are, then, sets of (locally) absolute truths that one uses in the process of, say, solving a problem, and semantic fields are characterised by the fact that whatever meaning being produced in a given activity, the person is ‘using’ those local stipulations as the firm ground to produce new statements, to justify them. For instance, one can speak of a semantic field of whole-part relation or a semantic field of a scale-balance, among others, when someone is producing meaning for a linear equation.

The historical-critical study

In this section we present what was found while we were trying to elicit the semantic fields in which mathematicians of the past were operating as the notions of basis was being constituted, during the emergence of the basic concepts of Linear Algebra.

We examined the work of mathematicians looking for the objects they constituted and for what justified what they were saying about them or doing with them (we call this the 'logic of the operations'). We included authors who did not constitute the notion of basis but were of interest because of some ideas present in their work, related to what we wanted to elicit. We took Crowe (1967), Dorier (1990) and Granger (1974) as central references.

Our first informant was L. Euler (1707-83). He speaks of objects such as functions, homogeneous and non-homogeneous linear differential equations, general and particular solutions of differential equations. Dorier (op. cit) and Bourbaki (1976) suggest he knew, for instance, that the general solution of a homogeneous linear differential equation of order n is a linear combination of n particular solutions. But he did not give evidence of having constituted the notion of linear independence of a set of solutions and that suggests he did not constitute the notion of basis for the set of all solutions.

Our second informant was F. G. Frobenius (1849-1917). Our reading of his work on a general theory of solving systems of linear equations led us to think that he constituded the notion of basis for the solutions of a homogeneous system, as he seems to use as local stipulations, in that respect, the notions of linear combinations of solutions, linear independence of solutions and considered the maximum number of independent solutions.

From W. R. Hamilton (1805-65) we were interested on his work on quaternions. He operated with them both geometrically and algebraically, and they were understood as linear combinations of the four units (1, i, j, k) using coefficients in **R**. Around a kernel which had objects such as complex numbers, vectors, ordered pairs, triples and quadruples of real numbers, Hamilton developed a notion of basis for the quaternions.

The work of H. Grassman (1809-77) was not understood by his peers, like Gauss and Moebius, at the time they were presented (Dorier, 1990). He worked with objects like extensive, derivable and elementary magnitudes, units (primitive, relative and absolute) and systems of units. From those he developed notions of linear dependence and independenc, linear combinations (for instance, speaking of derivable magnitudes), dimension, real vector spaces and of basis.

G. Peano (1858-1932) took to himself the task of making Grassman's work comprehensible to more people but, in doing so, he made original contributions. In particular he shows that if the product of three vectors, in the form of a determinant, is different from zero, then any vector can be written as a linear combination of those three. Peano, like Grassman, constituted the notion of a basis, and gave, for the first time an axiomatic presentation of vector spaces. Dorier thinks his understanding of dimension of a vector space is not completely clear.

The analysis, very briefly presented here, of the work of those authors, showed that, operating on different semantic fields, those mathematicians constituted objects that we could identify as precursors of our notion of basis. But it also indicated the extent to which the meanings produced for basis in each case were strongly related to the overall construction of each mathematician, to the problems they were trying to solve and the ideas they were trying to clarify/organise. It is in this sense that we say that the production of meaning can be only understood inside specific activities and not in an ideal, general, sense; that was also taken into account when we examined textbooks and when we interviewed students.

The study of textbooks

Our selection of authors here did not follow any specific principle. We simply looked at a considerable number of textbooks on Linear Algebra, and collected different definitions or characterisations for basis.

The question guiding the analysis here was "what statements can be made about 'basis of a finite-dimensional vector space' following each of those definitions or characterisations?"

We will consider here only two characterisations found in textbooks:

1) a basis for a vector space is an ordered set of vectors that both is linearly independent and generates the whole set of vectors

2) a basis for a vector space is a linearly independent set of vectors such that the number of vectors in it is equal to the dimension of that space.

It is worth noticing that both characterisations are possible, given that different authors organise the presentation of ideas differently.

To illustrate how assuming one of those meanings for basis could affect the thinking about the same problem, we present an example. Two students are presented with the question:

> "Consider $\mathbf{R}^3$ with the usual vector space structure, and A={u=(1,0,0), v=(0,1,-1), w=(0,0,2)}. Is A a basis for $\mathbf{R}^3$?"

Student 1 says 'yes' and offers this justification:

> "the set A is linearly independent because none of the vectors can be written as a linear combination of the other two. Also, A generates the whole space."

Student 2 says 'yes' and offers this justification:

> "$\mathbf{R}^3$ is a vector space and dim $\mathbf{R}^3 = 3$. As the vectors in A are [...] linearly independent, A is a basis for $\mathbf{R}^3$"

In this case it is pretty obvius who tought which way, but one consequence might remain hidden: student 2 depends, at this point, of some way of determining the dimension of the space s/he is working with. In many situations this leads students to some form of naturalised notion of 'space' with consequences we discuss elsewhere.

Also, from the statements of each student it is clear that the local stipulations are different in each case, that is, their thinking relate to different sets of objects. For the professor, it is crucial that s/he be able to read such processes and be aware of the implications of choices s/he makes in preparing a course and teaching it. We believe the TMSF offers a framework which supports that reading and usefully guides course preparation and teaching.

With students

What meanings would be produced for the notion of basis by a student taking a first course on Linear Algebra? That was the question guiding the case study.

Among the students who volunteered to be interviewed (undergaraduate mathematics degree), we chose two; let's call them Mark and Eli. The introductory course they were taking consisted of: matrices, systems of linear equations, determinants and finite-dimension vector spaces.

Four tasks were designed, and presented to them on four different sessions.

In our analysis we focused on: (i) the objects they were thinking with/about; (ii) the local stipulations being taken; and, (iii) the logic of the operations, that is, how those local stipulations and objects were supporting what they were saying.

We will discuss one of the tasks:

> "Consider the plane p given by $x - 2y + z = 0$ in $\mathbf{R}^3$. Find two basis for p "

Mark writes $x = 2y - z$ and considering the vector (2y-z, y, z) he he obtains the vectors (2,1,0) and (-1,0,1). From that he concludes that they form a basis for p and that the dimension is 2. Then he isolates y and repeats the process. He said that,

> [MARK] "From the start I realised it was not the space $\mathbf{R}^3$, because of the text of the question. Because there it says that p is a plane. So, if it is a plane, there are only two vectors [sic] and the dimension is 2. So, to be $\mathbf{R}^3$ the dimension must be 3. From start I knew it was a subspace of $\mathbf{R}^3$. So it's enough to find two vectors."

He is clearly working with the assumption that the dimension of that subspace is 2, but because he does not verify whether the two vector are linearly independent, we understood that the dimension 2 was simply a property of a naturalised plane, a plane like the surface of a wall; it is simply too usual that everybody knows that the surface of a wall is bi-dimensional. That is also supported by his statement that 'there are only two vectors': those would correspond to the usual representaion of a plane, in analytical geometry, as a system of two Cartesian axis.

Eli has a different solution. She says,

> [ELI] "To be a basis, a set has to be LI and generate the space. For the plane p we have the generic vector (2y-z, y, z)."

From there she finds the generators by taking y=0 and then z=0 and verifies they are LI. When discussing her solution with the interviewer, Eli says that the set "has to generate $\mathbf{R}^3$". When the interviewer asks her about this, she says she got confused and that,

> [ELI] "[…] a difficulty, also, that I think I have, we have. That thing of using numbers in an equation to see what it generates in terms of, like, plane, solid, straight line. Like, sometimes we even know, but when we have to imagine, like…"

Her difficulty seems to be associated with not identifying directly and immediately from the equation what the subspace 'is'. Maybe this is the reason why she does not think like Mark, with the 'natural' dimension 2. In any case, our point here is that her thinking was different from Mark's and that means that the objects they were thinking with were different and that the meaning of basis for each of them was different.

As we have said before, in a classroom situation the professor must be aware of those processes and be able to handle them if teaching is to be effective.

After an exchange with Eli, Mark realises that,

> [MARK] "[…] I forgot to verify whether one vector is independent of the other [sic]. Because if they are dependent they won't generate a plane […] they'll generate a straight line…"

But as the conversation continued he returned to his idea that it was enough to know that p was a plane. In terms of the TMSF, the interpretatioon is that in that specific activity 'linear independence' was not constituted into an object nor was a local stipulation. The interaction with Eli shows that Mark *could* produce meaning for it in that context, but also that *actually* that object did not belong properly to his thinking in that situation.

Mark thought with: equations, variables in an equation, generic vector, vector as a directed line (he uses drawings), (natural) dimension, subspace, $\mathbf{R}^3$. We think there is the suggestion here that naturalised objects (dimension and vector, here) are more likely to become (unnoticed) local stipulations than notions which are unfamiliar (linear independence, in this case).

Eli thought with: generic vector, equation, subspace, set, generate, linear independence. No evident naturalised notion seems to be centrally present in her thinking. Differently from Mark, who sees the equation as being the plane, for her it is more likely a relationship between the variables.

What each does in the course of solving the problem is based on what those objects *are* (for them); this is what we referred to as the logic of the operations (on the objects). Eli's plane, for instance, did not have the Cartesian axis attached to it (in this activity), so she has to think with linear independence; because the role of the axis is to provide a system of coordinates, it is clear that if one has the Cartesian diagram of the plane in mind it does not even make sense to have axis that are not 'independent'.

Just to make a relevant point . One could be strongly tempted here to say that Eli is thinking algebraically while Mark is thinking geometrically. As a local description it might look useful, but from the point of view of the TMSF it is misleading, as the static nature of such description (referring to *states*) makes it insufficient for the reading and understanding of *processes*.

Final remarks

Overall, our study highlighted a broad set of meanings that can be produced for the notion of basis in Linear Algebra, working with a varied group of informants. Those meanings reach from the ones found in textbooks, through the ones found in the work of mathematicians of the past, to the ones we found in the thinking of students.

By no means we wanted to produce a 'catalog' of meanings; what we wanted was to highlight the fact that it is not sufficient, from the point of view of mathematics education, to treat present-day definitions and characterisations as the (true) essence of something that is also to be (sometimes implicitly and many times incorrectly) found in the past and in students. We suggest that the complexity of meaning production can be only dealt with properly in mathematics education if we make *processes* our central object of study and understanding.

One of the students in our study said:

> "It's as the name already says, to be a basis [foundation, stepping stone] for you to know about whatever a person asks you, or an exercise, right?"

Particularly in the mathematical education of future teachers, we think it is necessary to raise the awareness of the existence of those processes and to help them to develop ways of dealing with such situations. And it does not seem plausible that normal courses on mathematical subjects (the same taken by future researchers in mathematics) are adequate. On the contrary, the results of our current research project suggest that the education of future teachers would benefit from a different approach in the classrooms.

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